ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) your instructor's name, and (4) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. *A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit.* Textbooks, classnotes, crib sheets, or calculators are not permitted.

1. (30 points) Let $A \in \mathbf{R}^{n\times n}$ be symmetric and positive definite, $b \in \mathbf{R}^n$, and J a cost function defined by

$$J(x) \;=\; \frac{1}{2}x^tAx - x^tb \quad ; \quad \forall x \in \mathbf{R}^n$$

   If $\{u^{(1)}, u^{(2)}, ..., u^{(n)}\}$ is a set of vectors that is orthonormal in the usual sense, and if these vectors are used as the directions of search to minimize J, will the solution be obtained after n steps?

2. (40 points) Using Q as in the Gauss-Seidel method, prove that if A ($n \times n$ matrix) is strictly diagonally dominant, then $||I - Q^{-1}A||_\infty < 1$. Conclude!

3. (30 points) State whether each of the following statements are **TRUE** or **FALSE**. You do not need to show your work.

   a. If $F : [a\ ,\ b] \longrightarrow [a\ ,\ b]$, then F must have a fixed point.

   b. $F(x) \;=\; \tan^{-1} x$ is contractive on an arbitrary closed interval.

   c. If $F : \mathbf{R} \longrightarrow \mathbf{R}$ continuous, then F does have a fixed point.

   d. $F(x) \;=\; \sqrt{x}$ is contractive on $\left[0\ ,\ \frac{1}{2}\right]$.

   e. $F(x) \;=\; 3 - x^2$ has a fixed point on $\left[-\frac{1}{4}\ ,\ +\frac{1}{4}\right]$.

HEY, THERE'S MORE—TURN THE PAGE OVER!

4. (30 points) Let x_0, x_1, ... , x_n be distincts points in $\mathbf{R}$ and q a polynomial of degree smaller than n . Prove that

$$\sum_{i=0}^{n} \frac{q(x_i)}{(x_i - x_0)...(x_i - x_{i-1})(x_i - x_{i+1})...(x_i - x_n)} \;=\; 0$$

5. (30 points) Count the number of multiplications and/or divisions needed to invert a unit $n \times n$ lower triangular matrix.

6. (40 points) Let A be a normal matrix that is $AA^* \;=\; A^*A$.

   a. Suppose that x and y are eigenvectors of A corresponding to different eigenvalues. Prove that $x^*y \;=\; 0$.

   b. Prove that A and A^* have the same eigenvectors.