1. (20 points) Find an orthonormal basis of the kernel of the matrix

\[ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \]

2. (30 points) Consider the linear system

\[
\begin{align*}
x + y - z &= 2 \\
x + 2y + z &= 3 \\
x + y + (k^2 - 5)z &= k
\end{align*}
\]

where \( k \) is an arbitrary constant.

a. For which value(s) of \( k \) this system is inconsistent?

b. For which value(s) of \( k \) does this system have one solution? Find the solution.

c. For which value(s) of \( k \) does this system have infinitely many solutions? Find all the solutions.

3. (30 points) Find the \( QR \) factorization of the matrix

\[ A = \begin{bmatrix} 4 & 25 \\ 0 & 0 \\ 3 & -25 \end{bmatrix} \]

4. (40 points) Consider the transformation \( T(f(t)) = t(f'(t)) \) from \( P_2 \) to \( P_2 \).

a. Show that the transformation \( T \) is linear.

b. Find the kernel and the nullity of the transformation \( T \).

c. Use part (b) to find the rank of the transformation \( T \).

d. Is the transformation \( T \) an isomorphism?

HEY, THERE’S MORE—TURN THE PAGE OVER!
5. (20 points) Consider the matrix
\[ A = \begin{bmatrix} 1 & k \\
1 & 1 \end{bmatrix} \]
where \( k \) is an arbitrary constant.

a. For which values of \( k \) does the matrix \( A \) have two distinct real eigenvalues?

b. For which values of \( k \) does the matrix \( A \) have no real eigenvalue?

6. (30 points) Consider a linear transformation \( T \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). We are told that the matrix of \( T \) with respect to the basis \( \begin{bmatrix} 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ 8 \end{bmatrix} \) is \( \begin{bmatrix} 1 & 9 \\ 9 & 7 \end{bmatrix} \).

Find the standard matrix of \( T \).

7. (30 points) Consider two distinct numbers, \( a \) and \( b \). We define the function
\[ f(t) = \det \begin{bmatrix} 1 & 1 & 1 \\
a & b & 1 \\
a^2 & b^2 & t^2 \end{bmatrix} \]

a. Show that \( f(t) \) is a quadratic function. What is the coefficient of \( t^2 \)?

b. Explain why \( f(a) = f(b) = 0 \). Conclude that \( f(t) = k(t - a)(t - b) \), for some constant \( k \). Find \( k \), using your work in part (a).

c. For which values of \( t \) is the matrix is invertible?