1. (20 points) Consider $V$ the subset of $P_2$ defined by

$$V = \left\{ p(t) : \int_0^1 p(t) \, dt = 0 \right\}$$

a. Show that $V$ is a subspace of $P_2$.

b. Find a basis for $V$.

2. (30 points) Let $V$ be the set of $3 \times 3$ matrices $A$ such that the vector

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

is in the kernel of $A$. Is $V$ a subspace of $\mathbb{R}^{3 \times 3}$?

3. (20 points) Find the $QR$ factorization of the matrix

$$A = \begin{bmatrix} 4 & 25 \\ 0 & 0 \\ 3 & -25 \end{bmatrix}$$

4. (30 points) Consider the linear system

$$\begin{cases} x + y - z & = 2 \\ x + 2y + z & = 3 \\ x + y + (k^2 - 5)z & = k \end{cases}$$

where $k$ is an arbitrary constant.

a. For which value(s) of $k$ does this system have a unique solution? Find the solution.

b. For which value(s) of $k$ does this system have infinitely many solutions? Find all the solutions.

c. For which value(s) of $k$ is the system inconsistent?
5. (20 points) Consider the transformation \( T(f(t)) = t(f'(t)) \) from \( P_2 \) to \( P_2 \).

a. Show that the transformation \( T \) is linear.

b. Find the kernel and the nullity of the transformation \( T \).

c. Use part (b) to find the rank of the transformation \( T \).

d. Is the transformation \( T \) an isomorphism?

6. (20 points) Consider the matrix
\[
A = \begin{bmatrix}
-3 & 0 & 4 \\
0 & -1 & 0 \\
-2 & 7 & 3
\end{bmatrix}
\]

a. Find all real eigenvalues of \( A \) with their algebraic multiplicities.

b. Find an eigenvector corresponding to the eigenvalue \( \lambda = 1 \)

7. (40 points) Consider a linear transformation \( T \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \). We are told that the matrix of \( T \) with respect to the basis \( \begin{bmatrix} 3 \\ 5 \end{bmatrix} \), \( \begin{bmatrix} 5 \\ 8 \end{bmatrix} \) is \( \begin{bmatrix} 1 & 9 \\ 9 & 7 \end{bmatrix} \).

Find the standard matrix of \( T \).

8. (20 points) Consider two distinct numbers, \( a \) and \( b \). We define the function
\[
f(t) = \det \begin{bmatrix}
1 & 1 & 1 \\
1 & b & t \\
1 & b^2 & t^2
\end{bmatrix}
\]

a. Show that \( f(t) \) is a quadratic function. What is the coefficient of \( t^2 \)?

b. Explain why \( f(a) = f(b) = 0 \). Conclude that \( f(t) = k(t-a)(t-b) \), for some constant \( k \). Find \( k \), using your work in part (a).

c. For which values of \( t \) is the matrix is invertible?