MATH 262 FINAL Fall 2004

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are not permitted.

1. (20 points) Find an *orthonormal* basis of the plane

$$x_1 + x_2 + x_3 = 0$$

2. (30 points) Consider the linear system

$$\begin{cases} x + 2y + 3z = a \\ x + 3y + 8z = b \\ x + 2y + 2z = c \end{cases}$$

where a, b, and c are arbitrary constants.

   a. For which value(s) of a, b, and c this system is *inconsistent*?

   b. For which value(s) of a, b, and c does this system have one solution? Find the solution.

   c. For which value(s) of a, b, and c does this system have infinitely many solutions? Find all the solutions.

3. (20 points) State whether each of the following statements are **TRUE** or **FALSE**. You do **NOT** need to show your work.

   a. If A is a $n \times m$ matrix and $\vec{v}$ is a vector in $\mathbf{R}^n$, then $A\vec{v}$ is a vector in $\mathbf{R}^m$.

   b. The inverse of $n \times m$ matrix is $m \times n$ matrix.

   c. The image of a $m \times n$ matrix is a subspace of $\mathbf{R}^n$.

   d. If V and W are subspaces of $\mathbf{R}^n$, then their union $V \bigcup W$ must be a subspace of $\mathbf{R}^n$ as well.

   e. If A is an $m \times n$ matrix, then $dim(KerA) + rank(A) = m$.

   f. If vectors $\vec{v_1}, \vec{v_2}, ..., \vec{v_m}$ are orthonormal in $\mathbf{R}^n$, then they must form a basis.

   g. If $AB = 0$ for $n \times n$ matrices A and B, then $BA = 0$.

   h. If $A^2 = A$ for an invertible $n \times n$ matrix A, then A must be I_n.

   i. If A is $n \times m$ matrix and B is $p \times m$ matrix, then AB is $n \times p$ matrix.

   j. Let A and B be two invertible matrices. Then, $(ABA^{-1})^3 = AB^3A^{-1}$.

HEY, THERE'S MORE—TURN THE PAGE OVER!

4. (20 points) Find the orthogonal projection of $\begin{bmatrix} 49 \\ 49 \\ 49 \end{bmatrix}$ onto the subspace of $\mathbf{R}^3$ spanned by $\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$.

5. (40 points) Consider the linear transformation

$$T(M) = M \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} M$$

from $\mathbf{U}^{2\times 2}$ to $\mathbf{U}^{2\times 2}$, where $\mathbf{U}^{2\times 2}$ is the space of upper triangular 2×2 matrices.

a. Show that the transformation T is linear.

b. Find the matrix of T with respect to the basis $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

c. Use part (b) to find the image, the kernel, the rank, and the nullity of the transformation T.

d. Is the transformation T an isomorphism?

6. (20 points) Perform the Gram-Schmidt process on the following sequences of vectors

$$\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

7. (20 points) Consider V the subset of P_2 defined by

$$V = \left\{ p(t) : \int_0^1 p(t)\, dt = 0 \right\}$$

a. Show that V is a subspace of P_2.

b. Find a basis for V.

8. (30 points) Consider two subspaces V and W in $\mathbf{R}^n$. Prove that their intersection $V \bigcap W$ must be a subspace of $\mathbf{R}^n$ as well.