1. (30 points)
   a. Consider \( y'(x) = f(x, y(x)) \) where \( f \) is continuously differentiable. Find \( y^{(2)}(x) \).
   b. Determine the Lipschitz constants with respect to \( y \) for the following functions:
      \[
      f(x, y) = \frac{2y}{x}; \quad x \geq 1
      \]
      \[
      g(x, y) = \tan^{-1} y
      \]
      \[
      h(x, y) = \frac{(x^3 - 2)^7}{17x^2 + 4}
      \]
   c. Let \((x_0, y_0)\) and \((x_0 + \xi, y_0 + \eta)\) be given points, and assume \( f(x, y) \) is \( n + 1 \) times continuously differentiable for all \((x, y)\) in some neighborhood of \((x_0, y_0)\), \((x_0 + \xi, y_0 + \eta)\). Derive the Taylor’s series of \( f \) at \((x_0, y_0)\).

2. (30 points) Let \( y(x) \) be the solution of the following initial value problem (IVP)
   \[
   (IVP) \begin{cases}
   y'(x) = f(x, y(x)) \\
y(x_0) = \tilde{y}_0
   \end{cases}
   \]
   and let \((y_n)_{n \in \mathbb{N}}\) be a sequence obtained with Euler’s algorithm (EA)
   \[
   (EA) \begin{cases}
y_{n+1} = y_n + hf(x_n, y_n) \\
y_0 = \tilde{y}_0
   \end{cases}
   \]
   Assume that
   i. \( y(x) \) the solution of (IVP) has a bounded second derivative on \([a, b]\).
   ii. \( \frac{\partial f}{\partial y}(x, y) \leq 0 \); \( x_0 \leq x \leq b \) and \( -\infty < y < +\infty \).

Show that, for all sufficiently small \( h \), we have
\[
|y(x_n) - y_n| \leq \frac{h}{2} (x_n - x_0) \max_{x_0 \leq x_n \leq b} |y^{(2)}(\xi_n)|
\]
with \( \xi_n \) between \( y(x_n) \) and \( y_n \).

HEY, THERE’S MORE—TURN THE PAGE OVER!
3. (40 points) Consider the following multistep method

\[ y_{n+1} = 3y_n - 2y_{n-1} + \frac{h}{2} [f(x_n, y_n) - 3f(x_{n-1}, y_{n-1})] ; \quad n \geq 0 \]

[MM] (3)
y_0, y_1 \text{ are given.}

a. Is this method consistent?
b. Is this method stable? Does the sequence \((y_n)_{n \in \mathbb{N}}\) converge?
c. Find the order of this method.
d. Deduce an explicit expression of the truncation error \(T_n(y)\) assuming that \(y\) is three times continuously differentiable.
e. Consider solving the following initial problem (IVP)

\[ \begin{cases} y'(x) = 0 \\ y(x_0) = 0 \end{cases} \]

(IVP) (4)
i. Find the solution \(y(x)\) of IVP.
j. Using \(y_0 = y_1 = 0\), find the numerical solution \(y_n ; n \geq 0\).
k. Perturb the initial data \(z_0 = \frac{\epsilon}{2}, z_1 = \epsilon\) for some \(\epsilon \neq 0\). Find the corresponding numerical solution \(z_n ; n \geq 0\).
l. Evaluate \(\max_{0 \leq n \leq N(h)} |y_n - z_n|\) and find the limit when \(h \to 0\). Conclude.