MATH 582B EXAM #3 Spring 2004

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are not permitted.

1. (30 points)

   a. Consider $y'(x) = f(x\ ,\ y(x))$ where f is continuously differentiable. Find $y^{(2)}(x)$.

   b. Determine the Lipschitz constants with respect to y for the following functions

$$f(x\ ,\ y) = \frac{2y}{x}\ ; \quad x \geq 1$$

$$g(x\ ,\ y) = \tan^{-1} y$$

$$h(x\ ,\ y) = \frac{(x^3-2)^{27}}{17x^2+4}$$

   c. Let $(x_0\ ,\ y_0)$ and $(x_0+\xi\ ,\ y_0+\eta)$ be given points, and assume $f(x\ ,\ y)$ is $n+1$ times continuously differentiable for all $(x\ ,\ y)$ in some neighborhood of $(x_0\ ,\ y_0)$, $(x_0+\xi\ ,\ y_0+\eta)$. Derive the Taylor's series of f at $(x_0\ ,\ y_0)$.

2. (30points) Let $y(x)$ be the solution of the following initial value problem (IVP)

$$\text{(IVP)} \quad \begin{cases} y'(x) = f(x\ ,\ y(x)) \\ y(x_0) = \tilde{y}_0 \end{cases} \tag{1}$$

and let $(y_n)_{n\in\mathbf{N}}$ b ea sequence obtained with Euler's algorithm (EA)

$$\text{(EA)} \quad \begin{cases} y_{n+1} = y_n + hf(x_n\ ,\ y_n) \\ y_0 = \tilde{y}_0 \end{cases} \tag{2}$$

Assume that

   i. $y(x)$ the solution of (IVP) has a bounded second derivative on $[a\ ,\ b]$.

   ii. $\dfrac{\partial f}{\partial y}(x,y) \leq 0\ ; \quad x_0 \leq x \leq b$ and $-\infty < y < +\infty$.

Show that, for all sufficiently small h, we have

$$|y(x_n) - y_n| \leq \frac{h}{2}(x_n - x_0) \max_{x_0 \leq x_n \leq b} |y^{(2)}(\xi_n)|$$

with ξ_n between $y(x_n)$ and y_n.

HEY, THERE'S MORE—TURN THE PAGE OVER!

3. (40 points) Consider the following multistep method

$$(\text{MM}) \quad \begin{cases} y_{n+1} = 3y_n - 2y_{n-1} + \dfrac{h}{2}\left[f(x_n\ ,\ y_n) - 3f(x_{n-1}\ ,\ y_{n-1})\right]\ ; \quad n \geq 0 \\ y_0\ ,\ y_1 \text{ are given.} \end{cases} \tag{3}$$

a. Is this method consistent?

b. Is this method stable? Does the sequence $(y_n)_{n\in\mathbf{N}}$ converge?

c. Find the order of this method.

d. Deduce an explicit expression of the truncation error $T_n(y)$ assuming that y is three times continuously differentiable.

e. Consider solving the following initial problem (IVP)

$$(\text{IVP}) \quad \begin{cases} y'(x) = 0 \\ y(x_0) = 0 \end{cases} \tag{4}$$

i. Find the solution $y(x)$ of IVP.

ii. Using $y_0 = y_1 = 0$, find the numerical solution $y_n\ ;\ n \geq 0$.

iii. Perturb the initial data $z_0 = \dfrac{\epsilon}{2}$, $z_1 = \epsilon$ for some $\epsilon \neq 0$. Find the corresponding numerical solution $z_n\ ;\ n \geq 0$.

iv. Evaluate $\max\limits_{0\leq n\leq N(h)} |y_n - z_n|$ and find the limit when $h \to 0$. Conclude.