MATH 581 EXAM #3 Fall 2005

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, and (5) a grading table. You must work all the problems on the exam. Show ALL your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are NOT permitted.

1. (20 points) Let A be a symmetric positive definite matrix and b a vector in $\mathbf{R}^n$. Prove the following equivalence:

$$A\,x^* \;=\; b \quad\Longleftrightarrow\quad J(x^*) \leq J(x)\ ; \quad \forall x \in \mathbf{R}^n$$

Where

$$J(x) \;=\; x^T\,A\,x - 2\,x^T\,b$$

2. (60 points) Let A be a symmetric positive definite matrix and b a vector in $\mathbf{R}^n$. Let (x^k) be the iterates of the steepest descent algorithm for solving $A\,x = b$.

   a. Describe the algorithm.

   b. Count the number of elementary operations needed at each iteration.

   c. Prove the convergence of the algorithm.

   d. We want to find the rate of convergence of the algorithm.

      i. Prove that

$$||x - x^{k+1}||_A = \left(1 - \frac{||r^k||_2^2}{||r^k||_A^2} \cdot \frac{||r^k||_2^2}{||r^k||_{A^{-1}}^2}\right) ||x - x^k||_A$$

      where $r^k = A\,x^k - b$

      ii. Prove the existence of a positive constant C such that

$$||x - x^{k+1}||_A \;\leq\; C\,||x - x^k||_A$$

      (Hint: use the following Kantorovich inequality:

$$\frac{(B\,x\ ,\ x)\,(B^{-1}\,x\ ,\ x)}{(x\ ,\ x)^2} \;\leq\; \frac{(\lambda_{max} + \lambda_{min})^2}{4\,\lambda_{max}\,\lambda_{min}}\ ; \quad \forall x \neq 0$$

      where B is a symmetric positive definite matrix and $\lambda_{max}, \lambda_{min}$ its largest and smallest eigenvalues).

      iii. When does the method converge faster?

3. (20 points) Consider the following Householder transformation

$$H_u \;=\; I - 2\,u\,u^T \;; \quad \forall u \in \mathbf{R}^n$$

a. Prove that if $||u||_2 = 1$, then H_u is an orthogonal matrix.

b. Prove that for any $a \in \mathbf{R}^n$, there is $u \in \mathbf{R}^n$ and $\alpha \in \mathbf{R}$ such that

$$H_u\, a \;=\; \alpha\, e^{(1)}$$

where $e^{(1)}$ is the first vector of the standard basis of $\mathbf{R}^n$.

c. Describe the QR factorization process when using successive Householder transformations.

4. **(BONUS QUESTION)** (20 points) Prove the Kantorovich inequality.