ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor’s name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and BOX IN YOUR FINAL ANSWERS. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are not permitted.

1. (20 points) Let \( I \) be an open interval that contains 0 and \( f : I \to \mathbb{R} \). If there exists an \( \alpha > 1 \) such that \( |f(x)| \leq |x|^\alpha \) for all \( x \in I \), prove that \( f \) is differentiable at 0. What happens when \( \alpha = 1 \)?

2. (20 points) Evaluate the following limits
   a. \( \lim_{x \to \frac{\pi}{2}} (\pi - 2x) \tan x \)
   b. \( \lim_{x \to 0} \frac{\sin \frac{1}{x}}{e^{\frac{1}{x}} + 1} \)
   c. \( \lim_{x \to \infty} x^2 \left( e^{\frac{1}{x}} - e^{\frac{1}{x+1}} \right) \)
   d. \( \lim_{x \to a} \frac{x^x - a^x}{x - a} \); \( a > 0 \)

3. (30 points)
   a. Show that \( 1 + x \leq e^x \); \( \forall x \in \mathbb{R} \)
   b. For \( k \in [0, 1) \), consider the following sequence
      \[ u_n = (1 + k)(1 + k^2) \cdots (1 + k^n) \]; \( \forall n \in \mathbb{N} \)
      Prove that \( (u_n) \) is a convergent sequence.

4. (30 points) Find all \( C^2 \) functions \( f : \mathbb{R} \to \mathbb{R} \) such that
   \[ \exists \theta \in [0, 1], \forall x \in \mathbb{R} \text{ and } \forall h \in \mathbb{R} : f(x + h) = f(x) + hf'(x + \theta h) \]

5. (Bonus Question.) (20 points) For \( n \in \mathbb{N} \), find the \( n \)-th derivative of
   \[ f(x) = \frac{x^2 + 1}{(x + 1)^3} \]