1. (15 points) Find a basis for the space of all lower triangular $2 \times 2$ matrices.

2. (25 points) Consider $V$ the subset of $P_2$ defined by

$$V = \left\{ p(t) : \int_0^1 p(t) \, dt = 0 \right\}$$

a. Show that $V$ is a subspace of $P_2$.

b. Find a basis for $V$.

3. (30 points) Consider the linear transformation

$$T(M) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}^{-1} M \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

from $U^{2 \times 2}$ to $U^{2 \times 2}$, where $U^{2 \times 2}$ is the space of upper triangular $2 \times 2$ matrices.

a. Show that $T$ is linear.

b. Find the matrix of $T$ with respect to the basis $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

4. (30 points) Consider the transformation $T(f(t)) = t(f'(t))$ from $P_2$ to $P_2$.

a. Show that the transformation $T$ is linear.

b. Find the kernel and the nullity of the transformation $T$.

c. Use part (b) to find the rank of the transformation $T$.

d. Is the transformation $T$ an isomorphism?