1. (20 points) Suppose that \( f : [a, b] \rightarrow [a, b] \) is continuous. Prove that

\[ \exists c \in [a, b], \quad f(c) = c \]

Note that \( c \) is called a fixed point.

2. (20 points) Decide which of the following limits exist and which do not. Prove that your answer is correct.

a. \( \lim_{x \to 0} \frac{|x|}{x} \)

b. \( \lim_{x \to 0^+} x^\alpha \cos \frac{1}{x}; \) where \( \alpha \in \mathbb{R} \)

c. \( \lim_{x \to \frac{\pi}{2}^-} \frac{\tan x}{x} \)

d. \( \lim_{x \to -\infty} x^2 \sin x \)

3. (20 points) State whether each of the following statements are TRUE or FALSE. You do need to show your work when your answer is FALSE only.

a. \( f : [a, b] \rightarrow \mathbb{R} \) such that \( f(a)f(b) < 0 \). Then, there is necessarily \( c \in (a, b) \) such that \( f(c) = 0 \).

b. A uniform continuous function \( f \) on a bounded interval \( I \) is necessarily bounded.

c. Let \( I \) be an interval and \( (x_n) \) be a convergent sequence in \( I \). If \( f : I \rightarrow \mathbb{R} \) is a function, then \( (f(x_n)) \) is necessarily a convergent sequence.

d. A polynomial of degree \( n \) \( (n \geq 0) \) is uniformly continuous on any bounded interval.

Hey, there’s more—turn the page over!
4. (20 points) Let $E$ be a nonempty subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ is uniformly continuous. Assume $(x_n)$ is Cauchy. Prove that $(f(x_n))$ is Cauchy. What happens if $f$ is continuous only?

5. (20 points) Let $I$ be a bounded interval and $f : I \to \mathbb{R}$. Prove that if $f$ is uniformly continuous on $I$, then $f$ is bounded on $I$. What happens if $I$ is unbounded?

6. (Bonus Question.) (10 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and satisfies

\[ \forall x, y \in \mathbb{Q}, \quad f(x + y) = f(x) + f(y) \]

Prove that

\[ \exists a \in \mathbb{R} \text{ such that } \forall x \in \mathbb{R}, \quad f(x) = ax \]