MATH 350 EXAM #2 Spring 2005

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are not permitted.

1. (20 points) Suppose that $f : [a\ ,\ b] \longrightarrow [a\ ,\ b]$ is continuous. Prove that

$$\exists\ c \in [a\ ,\ b], \quad f(c)\ =\ c$$

Note that c is called a fixed point.

2. (20 points) Decide which of the following limits exist and which do not. Prove that your answer is correct.

   a. $\displaystyle\lim_{x\to 0} \frac{|x|}{x}$

   b. $\displaystyle\lim_{x\to 0^+} x^{\alpha}\cos\frac{1}{x}$; where $\alpha \in \mathbf{R}$

   c. $\displaystyle\lim_{x\to \frac{\pi}{2}^-} \frac{\tan x}{x}$

   d. $\displaystyle\lim_{x\to -\infty} x^2 \sin x$

3. (20 points) State whether each of the following statements are **TRUE** or **FALSE**. You do need to show your work when your answer is **FALSE** only.

   a. $f : [a\ ,\ b] \longrightarrow \mathbf{R}$ such that $f(a)f(b) < 0$. Then, there is *necessarily* $c \in (a\ ,\ b)$ such that $f(c) = 0$.

   b. A uniform continuous function f on a bounded interval I is *necessarily* bounded.

   c. Let I be an interval and (x_n) be a convergent sequence in I. If $f : I \longrightarrow \mathbf{R}$ is a function, then $(f(x_n))$ is *necessarily* a convergent sequence.

   d. A polynomial of degree n ($n \geq 0$) is uniformly continuous on any bounded interval.

HEY, THERE'S MORE—TURN THE PAGE OVER!

4. (20 points) Let E be a nonempty subset of $\mathbf{R}$ and $f : E \to \mathbf{R}$ is uniformly continuous. Assume (x_n) is Cauchy. Prove that $(f(x_n))$ is Cauchy. What happens if f is continuous only?

5. (20 points) Let I be a bounded interval and $f : I \longrightarrow \mathbf{R}$. Prove that if f is uniformly continuous on I, then f is bounded on I. What happens if I is unbounded?

6. **(Bonus Question.)**(10 points) Suppose that $f : \mathbf{R} \to \mathbf{R}$ is continuous and satisfies

$$\forall x, y \in \mathbf{Q}\ , \quad f(x+y) \;=\; f(x) + f(y)$$

Prove that

$$\exists\, a \in \mathbf{R} \text{ such that } \forall x \in \mathbf{R} \quad f(x) \;=\; ax$$