MATH 582B EXAM #2 Spring 2004

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, or crib sheets, are not permitted.

1. (40 points) For a fixed positive number h, let $P_2(x)$ be the Lagrange polynomial interpolating $f(x)$ at $x = 0, h, 2h$.

   a. Use P_2 to derive a numerical integration formula $I_h(f)$ for $I(f) = \int_0^{3h} f(x)\,dx$.
   b. Find the sign of $\pi(t) = t(t-1)(t-2)$ for $t \in [0\ ,\ 3]$.
   c. Assume that $f \in \mathcal{C}^3\left([0\ ,\ 3h]\right)$. Find $E_h(f) = I(f) - I_h(f)$

2. (10 points) The *midpoint rule* over the interval $[x_{i+1}\ ,\ x_{i-1}]$ is given by

$$\int_{x_{i-1}}^{x_{i+1}} f(x)\,dx = (x_{i+1} - x_{i-1})\, f(x_i)$$

   Determine the composite midpoint rule over the interval $[a\ ,\ b]$ with uniform spacing of $h = \dfrac{a-b}{n}$ such that $x_i = a + i\,h$ for $i = 0, 1, 2, ..., n$ (n even).

3. (50 points)

   A. Consider a sequence of polynomials $\{\phi_n\}_{n\in\mathbf{N}}$ satisfying the following four properties:

   i. the degree of ϕ_n is n for all $n \in \mathbf{N}$
   ii. $\int_{-1}^{1} \phi_n(x)\phi_m(x)\,dx = 0 \quad \text{if} \quad n \neq m$
   iii. $\int_{-1}^{1} (\phi_n(x))^2\,dx = 1\ ; \quad \forall n \in \mathbf{N}$
   iv. the coefficient of x^n in $\phi_n(x)$ is positive

   (a) Prove that the $\phi_1, \phi_2, ..., \phi_n$ are linearly independent for all $n \in \mathbf{N}$.
   (b) Prove that the sequence $\{\phi_n\}_{n\in\mathbf{N}}$ is unique.
   (c) Construct ϕ_0, ϕ_1, and ϕ_2.

HEY, THERE'S MORE—TURN THE PAGE OVER!

B. We denote by $x_1 < x_2 < ... < x_n$ the zeros of $\phi_n(x)$ in the interval $[-1\ ,\ 1]$. We recall that for $f \in \mathcal{C}^{2n}([-1\ ,\ 1])$, we have

$$f(x) \;=\; H_n(x) + \epsilon_n(x)$$

where $H_n(x)$ is the Hermite interpolation polynomial of f at the nodes $x_1, x_2, ..., x_n$ defined by

$$H_n(x) \;=\; \sum_{j=1}^{n} f(x_j)h_j(x) + \sum_{j=1}^{n} f'(x_j)\tilde{h}_j(x)$$

with

$$h_j(x) \;=\; \left[1 - L_j'(x_j)(x - x_j)\right](L_j(x))^2$$

$$\tilde{h}_j(x) = (x - x_j)(L_j(x))^2$$

$$L_j(x) = \frac{\Pi_n(x)}{(x - x_j)\Pi_n'(x_j)}$$

$$\Pi_n(x) \;=\; (x - x_1)(x - x_2)...(x - x_n)$$

and

$$\epsilon_n(x) \;=\; (\Pi_n(x))^2 f[x_1, x_1, x_2, x_2, ..., x_n, x_n, x]$$

(d) Show that $\displaystyle\int_{-1}^{1} h_j(x)\,dx \;>\; 0\;;\quad j = 1, 2, ..., n.$

(e) Show that $\displaystyle\int_{-1}^{1} \tilde{h}_j(x)\,dx \;=\; 0\;;\quad j = 1, 2, ..., n.$

C. We recall that the Gaussian quadrature for the integral $I(f) = \displaystyle\int_{-1}^{1} f(x)\,dx$ is given by

$$I_n(f) \;=\; \sum_{j=1}^{n} \omega_j\, f(x_j)$$

(f) Find the $I_2(f)$

(g) evaluate $I(f) - I_2(f)$

D. Consider the following integral

$$I = \int_0^{\pi} e^x \cos x\,dx$$

(h) Compute I.

(i) Use the Gaussian quadrature formula $I_2(f)$, obtained in part(f), to evaluate I and compute the error $I - I_2(f)$.