1. (20 points) Solve the following linear systems twice. First, use Gaussian elimination and give the factorization $A = LU$. Second, use Gaussian elimination with scaled row pivoting and determine the factorization of the form $PA = LU$

$$\begin{bmatrix} -1 & 1 & -4 \\ 2 & 2 & 0 \\ 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

2. (15 points) Let $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Determine whether these expressions define norms on $\mathbb{R}^n$

a. $\max |x_2|, |x_3|, ..., |x_n|$

b. $\sum_{i=1}^n |x_i|^3$

c. $\left\{ \sum_{i=1}^n |x_i|^{\frac{3}{2}} \right\}^2$

3. (20 points)

a. Prove that if $A$ is a symmetric matrix whose leading principal minors are nonsingular, then $A$ has a factorization $LDL^T$ in which $L$ is a unit lower triangular matrix and $D$ is diagonal.

b. Prove that if $A$ is invertible and has an $LU$ decomposition, then all principal minors of $A$ are nonsingular.
4. (30 points) Consider $A$ and $B \in \mathbb{R}^{n \times n}$. We assume that $A$ is invertible and $||B - A|| < ||A^{-1}||^{-1}$ where $|| \cdot ||$ is a matrix norm subordinate to a vector norm. Prove that:

a. $B$ is invertible

b. $B^{-1} = A^{-1} \sum_{k=0}^{\infty} (I - BA^{-1})^k$

c. $||A^{-1} - B^{-1}|| \leq ||A^{-1}|| \frac{||I - A^{-1}B||}{1 - ||I - A^{-1}B||}$

5. (15 points) Count the number of multiplications and/or divisions needed to invert a unit $n \times n$ lower triangular matrix.