1. (15 points) Suppose that \( x_n \in \mathbb{N} \) for \( n \in \mathbb{N} \). If \( (x_n) \) is Cauchy, prove that there are numbers \( a \) and \( N \) such that \( x_n = a \); \( \forall n \geq N \)

2. (15 points) Decide which of the following limits exist and which do not. Prove that your answer is correct.

   a. \( \lim_{x \to 1^+} \frac{1}{\log x} \)
   
   b. \( \lim_{x \to 0} \cos \frac{1}{x} \)
   
   c. \( \lim_{x \to 0} \frac{\sin x \log(1 + x^2)}{x \tan x} \)

   (You may assume that \( \lim_{u \to 0} \frac{\log(1 + u)}{u} = 1 \))

3. (20 points) State whether each of the following statements are TRUE or FALSE. You do need to show your work when your answer is FALSE only.

   a. \( f : [0, 1] \rightarrow [0, 1] \). Then, there is necessarily \( x_0 \in [0, 1] \) such that \( f(x_0) = x_0 \). 
   
   b. Let \( I \) be an interval and \( (x_n) \) be a Cauchy sequence in \( I \). If \( f : I \rightarrow \mathbb{R} \) is continuous, then \( (f(x_n)) \) is necessarily Cauchy.
   
   c. A continuous function \( f \) on a bounded interval \( I \) is necessarily bounded.
   
   d. Let \( I \) be an interval and \( (x_n) \) be a convergent sequence in \( I \). If \( f : I \rightarrow \mathbb{R} \) is continuous, then \( (f(x_n)) \) is necessarily a convergent sequence.
   
   e. A polynomial of degree \( n \) \((n \geq 0)\) is uniformly continuous on \( \mathbb{R} \).

HEY, THERE’S MORE—TURN THE PAGE OVER!
4. (20 points) Suppose that $f : [a , b] \to \mathbb{R}$ is continuous and $f(a) \neq f(b)$. Prove that if $p$ and $q$ are two positive real numbers, then

$$\exists c \in (a , b), \quad pf(a) + qf(b) = (p + q)f(c)$$

5. (30 points) Suppose that $f : [a , \infty) \to \mathbb{R}$ is continuous and there is $l$ such that $\lim_{x \to +\infty} f(x) = l$. Prove that $f$ is bounded on $[a , \infty)$.

6. (Bonus Question.) (10 points) Assume that $f : [0 , 1] \to \mathbb{R}$ is continuous. Prove that

$$f(q) = 0 \quad \forall q \in \mathbb{Q} \cap [0 , 1] \iff f(x) = 0 \quad \forall x \in [0 , 1]$$