ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are not permitted.

1. (15 points) Suppose that $x_n \in \mathbf{N}$ for $n \in \mathbf{N}$. If (x_n) is Cauchy, prove that there are numbers a and N such that
$$x_n = a \; ; \quad \forall n \geq N$$

2. (15 points) Decide which of the following limits exist and which do not. Prove that your answer is correct.

   a. $\lim_{x \to 1^+} \frac{1}{\log x}$

   b. $\lim_{x \to 0} \cos \frac{1}{x}$

   c. $\lim_{x \to 0} \frac{\sin x \log(1 + x^2)}{x \tan x}$
   (You may assume that $\lim_{u \to 0} \frac{\log(1 + u)}{u} = 1$

3. (20 points) State whether each of the following statements are **TRUE** or **FALSE**. You do need to show your work when your answer is **FALSE** only.

   a. $f : [0 \, , \, 1] \longrightarrow [0 \, , \, 1]$. Then, there is *necessarily* $x_0 \in [0 \, , \, 1]$ such that $f(x_0) = x_0$.

   b. Let I be an interval and (x_n) be a Cauchy sequence in I. If $f : I \longrightarrow \mathbf{R}$ is continuous, then $(f(x_n))$ is *necessarily* Cauchy.

   c. A continuous function f on a bounded interval I is *necessarily* bounded.

   d. Let I be an interval and (x_n) be a convergent sequence in I. If $f : I \longrightarrow \mathbf{R}$ is continuous, then $(f(x_n))$ is *necessarily* a convergent sequence.

   e. A polynomial of degree n $(n \geq 0)$ is uniformly continuous on $\mathbf{R}$.

HEY, THERE'S MORE—TURN THE PAGE OVER!

4. (20 points) Suppose that $f : [a\ ,\ b] \longrightarrow \mathbf{R}$ is continuous and $f(a) \neq f(b)$. Prove that if p and q are two positive real numbers, then

$$\exists\, c \in (a\ ,\ b)\,, \quad pf(a) + qf(b) \;=\; (p+q)f(c)$$

5. (30 points) Suppose that $f\ :\ [a\ ,\ \infty) \to \mathbf{R}$ is continuous and there is l such that $\lim_{x \to +\infty} f(x) = l$. Prove that f is bounded on $[a\ ,\ \infty)$.

6. **(Bonus Question.)**(10 points) Assume that $f : [0\ ,\ 1] \longrightarrow \mathbf{R}$ is continuous. Prove that

$$f(q) \;=\; 0 \quad \forall q \in \mathbf{Q} \bigcap [0\ ,\ 1] \iff f(x) \;=\; 0 \quad \forall x \in [0\ ,\ 1]$$