MATH 350 EXAM #1 Fall 2004

(SAMPLE)

1. Find the limit of each of the following sequences. If a limit does not exist, write "**DNE**" for your answer.

a. $x_n = \cos(n\pi)$

b. $y_n = \dfrac{n}{2^n}$

c. $z_n = \dfrac{\arctan(ne^n + n^2)}{n \ln n}$

d. $t_n = \dfrac{1}{\sqrt{n+1} - \sqrt{n}}$

e. $u_n = \dfrac{n}{n^2+1} + \dfrac{n}{n^2+2} + \dfrac{n}{n^2+3} + \cdots + \dfrac{n}{n^2+n}$

2. State whether each of the following statements are **TRUE** or **FALSE**. You do need to show your work when your answer is **FALSE** only.

a. A subsequence of a bounded sequence *must* converges.

b. A convergent sequence has *at most* one limit.

c. Let (x_n) be a divergent sequence. Then, the limits of any two subsequences (if they exist) *must* be different.

d. A convergent sequence is *bounded* by its limit.

e. A bounded sequence has *at most* one convergent subsequence.

f. Let (x_n) (resp. (y_n)) be a sequence that converges to an extended real number x (resp. y) as $n \to \infty$. Then, we have

$$\lim_{n\to\infty} (x_n + y_n) = x + y \quad \text{and} \quad \lim_{n\to\infty} (x_n y_n) = xy$$

3. Let (x_n) be a sequence of real numbers defined by

$$x_n = \sum_{k=0}^{n} \frac{1}{\left(\begin{array}{c} n \\ k \end{array}\right)}$$

where $\left(\begin{array}{c} n \\ k \end{array}\right)$ are the binomial coefficients

$$\left(\begin{array}{c} n \\ k \end{array}\right) = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

Assume that

$$\sum_{k=2}^{n-2} \frac{1}{\binom{n}{k}} \leq \frac{2}{n}$$

Then, prove the convergence of the sequence (x_n) and find the limit.

4. Let a be a positive number and consider the sequence (x_n) defined by

$$\begin{cases} x_1 & > & 0 \\ x_{n+1} & = & \frac{1}{2}\left(x_n + \frac{a^2}{x_n}\right) \end{cases}$$

   a. Show that $x_n \geq a$; $\forall n \geq 2$

   b. Show that (x_n) is a decreasing sequence.

   c. Deduce that (x_n) is a convergent sequence and find its limit.

5. Let (x_n) be a sequence defined by

$$\begin{cases} x_1 & = & \frac{1}{2} \\ x_{n+1} & = & {x_n}^2 + \frac{3}{16} \end{cases}$$

   a. Show that $x_n \geq 0$; $\forall n \in \mathbf{N}$

   b. Show that (x_n) is a decreasing sequence.

   c. Deduce that (x_n) is a convergent sequence and find its limit.

6. Let (x_n) and (y_n) be two sequences defined by

$$\begin{cases} x_n & = & \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \\ y_n & = & x_n + \frac{1}{n!} \end{cases}$$

   Prove that (x_n) and (y_n) converge to the same limit.

7. Consider a and b two real numbers such that $0 < a < b$. Let (x_n) and (y_n) be two sequences defined by

$$x_1 = a \quad \text{and} \quad y_1 = b$$

   and for $n \geq 2$

$$\begin{cases} x_{n+1} & = & \sqrt{x_n y_n} \\ y_{n+1} & = & \frac{x_n + y_n}{2} \end{cases}$$

a. Prove that $0 < x_n < y_n$; $\forall n \in \mathbf{N}$

b. Show that (x_n) is an increasing sequence.

c. Show that (y_n) is an decreasing sequence.

c. Show that (x_n) and (y_n) converge to the same limit.

8. Let (x_n) and (y_n) be two sequences such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Consider the sequence (z_n) defined by

$$z_n = \sup(x_n, y_n) \ ; \quad \forall n \in \mathbf{N}$$

Prove that (z_n) is a convergent sequence and find its limit.

9. Consider (x_n) a sequence of real numbers defined by

$$\begin{cases} x_1 \in \mathbf{R} \\ x_{n+1} = \dfrac{x_n - 6}{x_n - 4} \end{cases}$$

a. Prove the existence of two real numbers α and β $(\alpha < \beta)$ such that

$$\text{If } x_1 = \alpha, \quad \text{then} \quad x_n = \alpha \ ; \quad \forall n \in \mathbf{N}$$

and

$$\text{If } x_1 = \beta, \quad \text{then} \quad x_n = \beta \ ; \quad \forall n \in \mathbf{N}$$

b. Assume that $x_1 \neq \alpha$ and $x_1 \neq \beta$. Consider the sequence (y_n) defined by

$$y_n = \frac{x_n - \alpha}{x_n - \beta} \ ; \quad \forall n \in \mathbf{N}$$

i. Find a relation between y_n and y_{n+1} and then establish the convergence of the sequence (y_n).

ii. Deduce the limit of the sequence (x_n).