1. Find the limit of each of the following sequences. If a limit does not exist, write “DNE” for your answer.

   a. \( x_n = \cos(n\pi) \)
   
   b. \( y_n = \frac{n}{2^n} \)
   
   c. \( z_n = \frac{\arctan(ne^n + n^2)}{n \ln n} \)
   
   d. \( t_n = \frac{1}{\sqrt{n+1} - \sqrt{n}} \)
   
   e. \( u_n = \frac{n}{n^2 + 1} + \frac{n}{n^2 + 2} + \frac{n}{n^2 + 3} + \cdots + \frac{n}{n^2 + n} \)

2. State whether each of the following statements are TRUE or FALSE. You do need to show your work when your answer is FALSE only.

   a. A subsequence of a bounded sequence must converge.
   
   b. A convergent sequence has at most one limit.
   
   c. Let \((x_n)\) be a divergent sequence. Then, the limits of any two subsequences (if they exist) must be different.
   
   d. A convergent sequence is bounded by its limit.
   
   e. A bounded sequence has at most one convergent subsequence.
   
   f. Let \((x_n)\) (resp. \((y_n)\)) be a sequence that converges to an extended real number \(x\) (resp. \(y\)) as \(n \to \infty\). Then, we have
   \[
   \lim_{n \to \infty} (x_n + y_n) = x + y \quad \text{and} \quad \lim_{n \to \infty} (x_n y_n) = xy
   \]

3. Let \((x_n)\) be a sequence of real numbers defined by
   \[
x_n = \sum_{k=0}^{n} \frac{1}{\binom{n}{k}}
   \]

   where \(\binom{n}{k}\) are the binomial coefficients
   \[
   \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!}
   \]
Assume that
\[ \sum_{k=2}^{n-2} \frac{1}{n \choose k} \leq \frac{2}{n} \]

Then, prove the convergence of the sequence \((x_n)\) and find the limit.

4. Let \(a\) be a positive number and consider the sequence \((x_n)\) defined by
\[
\begin{align*}
x_1 &> 0 \\
x_{n+1} &= \frac{1}{2} \left( x_n + \frac{a^2}{x_n} \right)
\end{align*}
\]
   a. Show that \(x_n \geq a\); \(\forall n \geq 2\)
   b. Show that \((x_n)\) is a decreasing sequence.
   c. Deduce that \((x_n)\) is a convergent sequence and find its limit.

5. Let \((x_n)\) be a sequence defined by
\[
\begin{align*}
x_1 &= \frac{1}{2} \\
x_{n+1} &= x_n^2 + \frac{3}{16}
\end{align*}
\]
   a. Show that \(x_n \geq 0\); \(\forall n \in \mathbb{N}\)
   b. Show that \((x_n)\) is a decreasing sequence.
   c. Deduce that \((x_n)\) is a convergent sequence and find its limit.

6. Let \((x_n)\) and \((y_n)\) be two sequences defined by
\[
\begin{align*}
x_n &= \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} \\
y_n &= x_n + \frac{1}{n!}
\end{align*}
\]
Prove that \((x_n)\) and \((y_n)\) converge to the same limit.

7. Consider \(a\) and \(b\) two real numbers such that \(0 < a < b\). Let \((x_n)\) and \((y_n)\) be two sequences defined by
\[
\begin{align*}
x_1 &= a \quad \text{and} \quad y_1 = b \\
\end{align*}
\]
and for \(n \geq 2\)
\[
\begin{align*}
x_{n+1} &= \sqrt{x_n y_n} \\
y_{n+1} &= \frac{x_n + y_n}{2}
\end{align*}
\]
a. Prove that $0 < x_n < y_n$ ; $\forall n \in \mathbb{N}$

b. Show that $(x_n)$ is an increasing sequence.

c. Show that $(y_n)$ is an decreasing sequence.

c. Show that $(x_n)$ and $(y_n)$ converge to the same limit.

8. Let $(x_n)$ and $(y_n)$ be two sequences such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$. Consider the sequence $(z_n)$ defined by

$$z_n = \sup(x_n, y_n) ; \forall n \in \mathbb{N}$$

Prove that $(z_n)$ is a convergent sequence and find its limit.

9. Consider $(x_n)$ a sequence of real numbers defined by

$$\begin{cases} 
  x_1 \in \mathbb{R} \\
  x_{n+1} = \frac{x_n - 6}{x_n - 4} 
\end{cases}$$

a. Prove the existence of two real numbers $\alpha$ and $\beta$ ($\alpha < \beta$) such that

If $x_1 = \alpha$, then $x_n = \alpha$ ; $\forall n \in \mathbb{N}$

and

If $x_1 = \beta$, then $x_n = \beta$ ; $\forall n \in \mathbb{N}$

b. Assume that $x_1 \neq \alpha$ and $x_1 \neq \beta$. Consider the sequence $(y_n)$ defined by

$$y_n = \frac{x_n - \alpha}{x_n - \beta} ; \forall n \in \mathbb{N}$$

i. Find a relation between $y_n$ and $y_{n+1}$ and then establish the convergence of the sequence $(y_n)$.

ii. Deduce the limit of the sequence $(x_n)$. 