1. (20 points)
   a. Derive the general solution $u$ of the given equation by using an appropriate change of variables
      
      \[ a \frac{\partial u}{\partial t} + b \frac{\partial u}{\partial x} = u \; ; \; a, b \neq 0 \]

   b. Find the solution which is equal to \( \frac{1}{1 + a^2 x^2} \) along the x-axis.

2. (20 points) Use the method of characteristic curves to solve the following equation

   \[ e^{-x^2} \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = 0 \]

3. (30 points) Using the method of separation of variables, solve the following non homogeneous heat boundary value problem

   \[
   \begin{align*}
   \frac{\partial u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 ; & \quad 0 < x < L , \; t > 0 \\
   u(0,t) = T_1 ; & \quad t \geq 0 \\
   u(L,t) = T_2 ; & \quad t \geq 0 \\
   u(x,0) = f(x) ; & \quad 0 \leq x \leq L
   \end{align*}
   \]

   (BVP) \hspace{1cm} (1)

   where $c, L, T_1,$ and $T_2$ are positive numbers.

HEY, THERE'S MORE—TURN THE PAGE OVER!
4. (30 points) Let \( f \) be an integrable function in \( \mathbb{R} \). We define \( \mathcal{F}f \), the Fourier transform of \( f \), as follows

\[
\mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{ix\xi} \, dx ; \quad \xi \in \mathbb{R}
\]

a. Find the Fourier transform of \( f(x) = e^{-|x|} ; \quad x \in \mathbb{R} \)

b. Let \( f \) be an even function. Prove that \( \mathcal{F}\mathcal{F}f = f \) and deduce the Fourier transform of \( f(x) = \frac{1}{1+x^2} \).

c. Use the Fourier transform to solve the following initial value problem

\[
\begin{align*}
\frac{\partial^2 u}{\partial t \partial x} - \frac{\partial^2 u}{\partial x^2} &= 0 ; \quad -\infty < x < +\infty , \ t > 0 \\
u(x,0) &= \sqrt{\frac{\pi}{2}} e^{-|x|} ; \quad -\infty < x < +\infty
\end{align*}
\]

Simplify the expression of the solution as much as you can.

5. **Bonus Questions** (10 points) For \( a \neq 0 \), consider \( f(x) = e^{-\frac{ax^2}{2}} \). Show that

\[
\mathcal{F}f(\xi) = \frac{1}{\sqrt{a}} e^{-\frac{\xi^2}{2a}} ; \quad \xi \in \mathbb{R}
\]