ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section (4) your instructor's name, and (5) a grading table. You must work all of the problems on the exam. Show ALL of your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are <u>not permitted</u>.

1. (20 points)

   a. Derive the general solution u of the given equation by using an appropriate change of variables

   $$a\frac{\partial u}{\partial t} + b\frac{\partial u}{\partial x} = u \; ; \qquad a, b \neq 0$$

   b. Find the solution which is equal to $\dfrac{1}{1+a^2x^2}$ along the x-axis.

2. (20 points) Use the method of *characteristic curves* to solve the following equation

   $$e^{x^2}\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

3. (30 points) Using he method of separation of variables, solve the following non homogeneous heat boundary value problem

   $$\text{(BVP)} \quad \begin{cases} \dfrac{\partial u}{\partial t} - c^2\dfrac{\partial^2 u}{\partial x^2} = 0 \; ; & 0 < x < L \; , \; t > 0 \\ u(0,t) = T_1 \; ; & t \geq 0 \\ u(L,t) = T_2 \; ; & t \geq 0 \\ u(x,0) = f(x) \; ; & 0 \leq x \leq L \end{cases} \qquad (1)$$

   where c, L, T_1, and T_2 are positive numbers.

HEY, THERE'S MORE—TURN THE PAGE OVER!

4. (30 points) Let f be an integrable function in $\mathbf{R}$. We define $\mathcal{F}f$, the Fourier transform of f, as follows

$$\mathcal{F}f(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{\mathrm{i}x\xi}\, dx \;; \quad \xi \in \mathbf{R}$$

a. Find the Fourier transform of $f(x) = e^{-|x|}$; $\quad x \in \mathbf{R}$

b. let f be an even function. Prove that $\mathcal{F}\mathcal{F}f = f$ and deduce the Fourier transform of $f(x) = \dfrac{1}{1+x^2}$.

c. Use the Fourier transform to solve the following initial value problem

$$\text{(IVP)} \quad \begin{cases} \dfrac{\partial^2 u}{\partial t \partial x} - \dfrac{\partial^2 u}{\partial x^2} = 0 \;; & -\infty < x < +\infty \;,\; t > 0 \\[2ex] u(x,0) = \sqrt{\dfrac{\pi}{2}} e^{-|x|} \;; & -\infty < x < +\infty \end{cases} \qquad (2)$$

Simplify the expression of the solution as much as you can.

5. **Bonus Questions** (10 points) For $a \neq 0$, consider $f(x) = e^{-\dfrac{ax^2}{2}}$. Show that

$$\mathcal{F}f(\xi) = \frac{1}{\sqrt{a}} e^{-\dfrac{\xi^2}{2a}} \;; \quad \xi \in \mathbf{R}$$