MATH 581 EXAM #1 Fall 2005

ON THE FRONT OF YOUR BLUEBOOK write: (1) your name, (2) your student ID number, (3) lecture section, and (5) a grading table. You must work all the problems on the exam. Show ALL your work in your bluebook and **BOX IN YOUR FINAL ANSWERS**. A correct answer with no relevant work may receive no credit, while an incorrect answer accompanied by some correct work may receive partial credit. Textbooks, classnotes, crib sheets, or calculators are NOT permitted.

1. (20 points) Let M be a partitioned matrix defined as follows

$$M = \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$$

where $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{m\times n}$, and $C \in \mathbf{R}^{m\times m}$.

   a. For which choices of A, B, and C is M invertible?

   b. If M is invertible, what is M^{-1}?

   c. Application: Find the inverse of M given by

$$M = \left[\begin{array}{cc|ccc} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ \hline 1 & 2 & 1 & 0 & 0 \\ 3 & 4 & 0 & 1 & 0 \\ 5 & 6 & 0 & 0 & 1 \end{array}\right]$$

2. (15 points) Let $x^T = (x_1, x_2, ..., x_n) \in \mathbf{R}^n$. Determine whether these expressions define norms on $\mathbf{R}^n$.

   a. $||x|| = \max\{|x_2|, |x_3|, ..., |x_n|\}$

   b. $||x|| = \sum_{i=1}^{n} |x_i|^3$

   c. $||x|| = \left\{\sum_{i=1}^{n} |x_i|^{\frac{1}{2}}\right\}^2$

3. (20 points)

   Let A and B be two $n \times n$ real matrices and $||\cdot||$ a matrix norm subordinate to a vector norm. Show that if $||AB - I|| = r < 1$, then

$$||A^{-1} - B|| \le \frac{r}{1-r}||B||$$

HEY, THERE'S MORE—TURN THE PAGE OVER!

4. (25 points) Let $\mathbf{R}^{n\times n}$ be the set of $n \times n$ real matrices equipped with the following norm

$$\forall A \in \mathbf{R}^{n\times n}: \qquad ||A|| = \max_{1\le i,j\le n} |A_{ij}|$$

where $A = (A_{ij})$.

a. Show that the function f defined by

$$\begin{cases} f \;:\; \mathbf{R}^{n\times n} \longrightarrow \mathbf{R}^{n\times n} \\ \quad X \longrightarrow f(X) \;=\; X^T X \end{cases}$$

is differentiable in $\mathbf{R}^{n\times n}$ and find its derivative.

b. Is f a $\mathcal{C}^1$ function?

5. (20 points) Let x and y be in $\mathbf{R}^n$ and define

$$\begin{cases} \Psi \;:\; \mathbf{R} \longrightarrow \mathbf{R} \\ \quad \alpha \longrightarrow \Psi(\alpha) \;=\; ||x - \alpha y||_2 \end{cases}$$

Show that Ψ is minimized when $\alpha \;=\; \dfrac{x^T y}{y^T y}$.

6. **(BONUS QUESTION)** (20 points)

a. Prove that for any vector norm and its subordinate matrix norm, and for any $n \times n$ matrix A, there corresponds a vector $x \neq 0$ such that $||Ax|| \;=\; ||A||\,||x||$.

b. Prove that the set of invertible $n \times n$ matrices is an open set in the set of all $n \times n$ matrices.