1. (20 points) Find the limit of each of the following sequences. If a limit does not exist, write “DNE” for your answer.

   a. \( x_n = \frac{1}{\sqrt{n} + (-1)^n} \)
   
   b. \( y_n = \sqrt{n+1} - \sqrt{n} \)
   
   c. \( z_n = \frac{1}{n} + \sin(2n) \)
   
   d. \( t_n = \frac{\sqrt{2n^2 - 1}}{n+1} \)
   
   e. \( u_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \frac{n}{n^2+3} + \cdots + \frac{n}{n^2+n} \)

2. (20 points) Consider the sequence \((x_n)\) such that \(\forall n \in \mathbb{N}, x_n = (-1)^n\).

   a. Using the definition, prove that \((x_n)\) is a divergent sequence.
   
   b. Use an other method to establish the divergence of the sequence \((x_n)\).

3. (20 points) State whether each of the following statements are TRUE or FALSE. You do need to show your work when your answer is FALSE only.

   a. A subsequence of a bounded sequence is not necessarily convergent.
   
   b. Let \((x_n)\) be a divergent sequence. Then, \(\lim_{n \to +\infty} x_n = +\infty\)
   
   c. A convergent sequence is bounded.
   
   d. A bounded sequence has at least one convergent subsequence.
   
   e. If \(\{I_n\}_{n \in \mathbb{N}}\) is a nested sequence of nonempty closed intervals, then \(E = \bigcap_{n \in \mathbb{N}} I_n\) contains at least one number.

HEY, THERE’S MORE—TURN THE PAGE OVER!
4. (20 points) Let \( a \) be a positive number and consider the sequence \((x_n)\) defined by

\[
\begin{cases}
  x_1 > 0 \\
  x_{n+1} = \frac{1}{2} \left( x_n + \frac{a^2}{x_n} \right)
\end{cases}
\]

a. Show that \( x_n \geq a \); \( \forall n \geq 2 \)
b. Show that \((x_n)\) is a decreasing sequence.
c. Deduce that \((x_n)\) is a convergent sequence and find its limit.

5. (20 points) Consider \( a \) and \( b \) two real numbers such that \( 0 < a < b \). Let \((x_n)\) and \((y_n)\) be two sequences defined by

\[
x_1 = a \quad \text{and} \quad y_1 = b
\]

and for \( n \geq 1 \)

\[
\begin{cases}
  x_{n+1} = \sqrt{x_n y_n} \\
  y_{n+1} = \frac{x_n + y_n}{2}
\end{cases}
\]

a. Prove that \( 0 < x_n < y_n \); \( \forall n \in \mathbb{N} \)
b. Show that \((x_n)\) is an increasing sequence.
c. Show that \((y_n)\) is an decreasing sequence.
d. Show that \((x_n)\) and \((y_n)\) converge to the same limit.

6. **Bonus Questions** (10 points) Let \((x_n)\) and \((y_n)\) be two sequences such that \( \lim_{n \to \infty} x_n = x \) and \( \lim_{n \to \infty} y_n = y \). Consider the sequence \((z_n)\) defined by

\[
z_n = \sup(x_n, y_n) ; \quad \forall n \in \mathbb{N}
\]

Prove that \((z_n)\) is a convergent sequence and find its limit.