UPPER MAXWELLIAN BOUNDS FOR THE SPATIALLY HOMOGENEOUS BOLTZMANN EQUATION

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Abstract. We consider solutions of the spatially homogeneous Boltzmann equation for hard potentials with an angular cutoff. We establish that if the initial datum is bounded from above by a Maxwellian distribution, the solution remains bounded for all times by another time-independent Maxwellian distribution. Our main technique is based on a comparison principle which uses a certain monotonicity property of the linear Boltzmann equation.

1. Introduction and main result

The nonlinear Boltzmann equation is a classical model for a gas at low or moderate densities. It takes the form

$$\partial_t f + v \cdot \nabla_x f = Q(f), \tag{1}$$

where $f = f(x, v, t)$, $(x, v) \in \Omega \times \mathbb{R}^n$, $t \geq 0$, is the time-dependent density of particles in the (x, v)-space, $n \geq 2$, $\Omega \subseteq \mathbb{R}^n$, and $Q(f)$ is a quadratic integral operator expressing the effect of instantaneous binary collisions of particles, to be introduced below.

Although some of our results deal with more general situations, we will be mostly concerned with a special class of solutions that are independent of the spatial variable (spatially homogeneous solutions). In this case $f = f(v, t)$ and one can study the initial-value problem

$$\partial_t f = Q(f), \quad f|_{t=0} = f_0, \tag{2}$$

where $0 \leq f_0 \in L^1(\mathbb{R}^n)$. The spatially homogeneous theory is very well developed although not complete. In the present paper we shall solve one of the most noticeable open problems remaining in the field, by establishing the following result.

Theorem 1. *Assume that $0 \leq f_0(v) \leq M_0(v)$, for a. a. $v \in \mathbb{R}^n$, where $M_0(v) = e^{-a_0|v|^2+c_0}$ is the density of a Maxwellian distribution, $a_0 > 0$, $c_0 \in \mathbb{R}$. Let $f(v,t)$, $v \in \mathbb{R}^n$, $t \geq 0$ be the unique solution of equation* (2) *for hard potentials with the angular cutoff assumptions* (5), (7), *that preserves the initial mass and energy* (12). *Then there are constants $a > 0$ and $c \in \mathbb{R}$ such that $f(v,t) \leq M(v)$, for a. a. $v \in \mathbb{R}^n$ and for all $t \geq 0$, where $M(v) = e^{-a|v|^2+c}$.*

Before going on, let us make a few comments about the interest of these bounds. The problem of qualitative behavior of solutions of the spatially homogeneous Boltzmann equation (2) has been extensively studied, starting from the pioneering work by Carleman [8,9]; for later developments we refer to the works [1,9,12,13,15,28,29] and the reviews in [10,32]. The upper bounds stated in Theorem 1 are very natural, since the Maxwellian functions

$$M(v) = e^{-a|v|^2+b\cdot v+c} \quad \text{with} \quad a > 0,\ b \in \mathbb{R}^n,\ c \in \mathbb{R},$$

are the only steady density solutions for the problem (2). Solutions in classes of functions bounded above by Maxwellians also appear in the contexts of short-time existence for the spatially inhomogeneous problem [22,24], rigorous validity of the Boltzmann equation [10,26], global existence for small data [4,21,23], perturbations of the spatially-homogeneous states [3]. The latter reference in particular points out a gap in the theory of the spatially-homogeneous problem, namely that the approach to equilibrium is not easily characterized in classes of functions with Maxwellian tails. The present work aims to at least partially remedy this situation, and to develop a technique that would allow one to obtain further results in this direction.

We will return to the discussion of solutions of (2), but first we need to introduce the particular form of the collision term $Q(f)$ and to give a more precise meaning to the concepts appearing in the formulation of Theorem 1. We set

$$Q(f)\,(v,t) = \int_{\mathbb{R}^n}\int_{S^{n-1}} (f'_*\,f' - f_*\,f)\,B(v-v_*,\sigma)\,d\sigma\,dv_*, \tag{3}$$

where, adopting common shorthand notations, $f = f(v,t)$, $f' = f(v',t)$, $f_* = f(v_*,t)$, $f'_* = f(v'_*,t)$, and

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\,\sigma\,, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\,\sigma\,, \tag{4}$$

where $\sigma \in S^{n-1}$ is the angular parameter determining the collision configuration. In the case of space-dependent densities, as in (1), the variable x appears (similarly to t above) as a parameter in each of f, f_*, f', f'_*; we shall often omit the t and x variables from the notation for brevity.

Many properties of the solutions of the Boltzmann equation depend crucially on certain features of the kernel B in (3). Its physical meaning is the product of the magnitude of the relative velocity and the effective scattering cross-section [25, §18]; this quantity characterizes the relative frequency of collisions between particles. Our assumptions on B fall in the category of "hard potentials with angular cutoff" (see [32] for the taxonomy and a discussion of the various types of kernels). More precisely, we assume that

$$B(v-v_*,\sigma) = |v-v_*|^\beta\,h(\cos\vartheta), \quad \cos\vartheta = \tfrac{(v-v_*)\cdot\sigma}{|v-v_*|}, \tag{5}$$

where $0 < \beta \le 1$ is a constant and h is a nonnegative function on $(-1,1)$ such that

$$h(z) + h(-z) \quad \text{is nondecreasing on } (0,1) \tag{6}$$

and

$$0 \le h(\cos\vartheta)\sin^\alpha\vartheta \le C, \quad \vartheta \in (0,\pi), \tag{7}$$

where $\alpha < n-1$ and C is a constant. Assumption (7) implies in particular that the integral $\int_{S^{n-1}} h(\cos\vartheta)\,d\sigma$ is finite; for convenience we normalize it by setting

$$\int_{S^{n-1}} h(\cos\vartheta)\,d\sigma = \omega_{n-2}\int_{-1}^{1} h(z)\,(1-z^2)^{\frac{n-3}{2}}\,dz = 1, \tag{8}$$

where ω_{n-2} is the measure of the $(n-2)$-dimensional sphere. The classical hard-sphere model in $\mathbb{R}^n$ satisfies (5) with $\beta = 1$ and (7) with $\alpha = n-3$.

Notice that we can write $Q(f) = Q^+(f) - Q^-(f)$, where $Q^+(f)$ is the "gain" term, and $Q^-(f)$ is the "loss" term,

$$Q^+(f) = \int_{\mathbb{R}^n}\int_{S^{n-1}} f' f'_* \, B(v - v_*, \sigma)\, d\sigma\, dv_*, \quad Q^-(f) = (f * |v|^\beta)\, f,$$

where $*$ denotes the convolution in v. Because of the symmetry $\sigma \mapsto -\sigma$ in the integral defining $Q^+(f)$ we can restrict the σ-integration above to the half-sphere $\{\cos\vartheta > 0\}$ if we simultaneously replace $B(v - v_*, \sigma)$ by

$$\overline{B}(v - v_*, \sigma) := (B(v - v_*, \sigma) + B(v - v_*, -\sigma))\, 1_{\{\cos\vartheta > 0\}}.$$

It will be convenient to introduce the following (nonsymmetric) bilinear forms of the collision terms,

$$Q^+(f, g) = \int_{\mathbb{R}^n}\int_{S^{n-1}} f'_* \, g' \, \overline{B}(v - v_*, \sigma)\, d\sigma\, dv_*, \quad Q^-(f, g) = (f * |v|^\beta)\, g, \tag{9}$$

for which obviously $Q^\pm(f) = Q^\pm(f, f)$.

We say that a nonnegative function $f \in C([0,\infty); L^1(\mathbb{R}^n))$, such that $(1+|v|^2)f \in L^\infty((0,\infty); L^1(\mathbb{R}^n))$, is a (mild) solution of (2) if for almost all $v \in \mathbb{R}^n$

$$f(v, 0) = f_0(v); \qquad f(v, t) - f(v, s) = \int_s^t Q(f)(v, \tau)\, d\tau, \tag{10}$$

for all $0 \le s < t$. Notice that the conditions on f imply (in the spatially-homogeneous case!) that

$$Q^+(f),\, Q^-(f) \in L^\infty((0,\infty); L^1(\mathbb{R}^n)), \tag{11}$$

so the integral form in (10) is well-defined. This also implies that f is weakly differentiable with respect to t and that the differential equation (2) holds in the sense of distributions on $\mathbb{R}^n \times (0,\infty)$.

The existence of a unique solution satisfying the conservations of mass and energy,

$$\int_{\mathbb{R}^n} f(v, t)\, dv = \int_{\mathbb{R}^n} f_0(v)\, dv, \qquad \int_{\mathbb{R}^n} f(v, t)\, |v|^2\, dv = \int_{\mathbb{R}^n} f_0(v)\, |v|^2\, dv \tag{12}$$

is guaranteed by a theorem by Mischler and Wennberg [28], for all $f_0 \ge 0$ for which the above integrals are finite. The second condition in (12) is also necessary for the uniqueness [33]. Of course, our initial data are significantly more regular than the general class discussed in [28], and we could refer to the older results by Arkeryd and Carleman [1,2,9]. In particular, the condition for the uniqueness (and the energy conservation) in [1] is that a moment of a sufficiently high order, say $\int_{\mathbb{R}^n} f(v,t)\, |v|^4\, dv$, is uniformly bounded in time.

The following theorem summarizes the main results about qualitative properties of solutions in the case of "hard potentials with cutoff" known before this work.

Theorem 2. *Let $f(v,t)$, $v \in \mathbb{R}^n$, $t \ge 0$, $(n \ge 2)$ be a solution of* (2) *that satisfies* (12)*, and let the kernel B in the Boltzmann operator* (3) *satisfy* (5), (7)*. Then*

(i) *if* $f_0 \in L^\infty(\mathbb{R}^n)$ *then* $f(t,\cdot) \in L^\infty(\mathbb{R}^n)$, $t \geq 0$. *Moreover, if* $(1+|v|)^s f_0 \in L^\infty(\mathbb{R}^n_v)$ *for some* $s > s_0$, *then* $(1+|v|)^s f(v,t) \in L^\infty(\mathbb{R}^n_v)$, $t \geq 0$. *Here* s_0 *is a constant dependent on the dimension* n.

(ii) *if the integral of* f *is nonzero, then for every* $t_0 > 0$ *there is a Maxwellian* $M(v) = Ke^{-\kappa|v|^2}$, $K > 0$, $\kappa > 0$ *such that*

$$f(v,t) \geq M(v), \quad t \geq t_0, \quad \text{for a. a. } v \in \mathbb{R}^n.$$

(iii) *for all* $t_0 > 0$ *and for all* $k > 1$, *the quantity* $m_k(t) = \int_{\mathbb{R}^n} f(v,t)\,|v|^{2k}\,dv$ *is bounded uniformly for* $t \geq t_0$; *moreover, this bound is uniform in* $t \geq 0$ *if* $m_k(0) < +\infty$.

(iv) *In the case* $n = 3$ *and* $B(v-v_*,\sigma) = c\,|v-v_*|$ *(hard spheres) or* $B(v-v_*,\sigma) = h(\frac{(v-v_*)\cdot\sigma}{|v-v_*|})$, $h \in L^1(-1,1)$ *(pseudo-Maxwell particles) if* f_0 *satisfies*

$$\frac{f_0}{M_0} \in L^1(\mathbb{R}^n)$$

for some Maxwellian $M_0(v) = e^{-a_0|v|^2}$, $a_0 > 0$, *then there exists constants* $a > 0$, C *such that*

$$\int_{\mathbb{R}^n} \frac{f(v,t)}{M(v)}\,dv \leq C,$$

where $M(v) = e^{-a|v|^2}$.

Part (i) of this theorem is due to Carleman [9] in the case of the hard spheres; the general case was studied by Arkeryd in [2]. Part (ii) is due to A. Pulvirenti and Wennberg [30]. Part (iii) is due to Desvillettes [12] under the additional assumption that a moment $m_{k_0}(t)$ of order $k_0 > 1$ is finite initially; this assumption was removed by Mischler and Wennberg [28]. Earlier result by Arkeryd [1] and Elmroth [15] state that all moments remain bounded uniformly in time, once they are finite initially. Finally, part (iv) is due to Bobylev [6]; we will give an extension of this result to the class of Boltzmann kernels satisfying (5)–(7) in Section 2.

The proofs of all statements included in Theorem 2 are based on constructive methods; all the constants appearing in the estimates can be made explicit. Results (ii) and (iii) express properties which hold true essentially independently of the initial datum, while (i) and (iv) express properties of propagation of decay, in some sense. As a consequence of (i) and (iv), if the initial data decays fast enough, then the solution is bounded and has finite square-exponential moments.

Our main contribution in the present work is to show that the estimates for the spatially homogeneous Boltzmann equation (precisely, parts (i) and (iv) of Theorem 2, together with the conservation of mass) imply Theorem 1. Since we do not use other properties of the spatially-homogeneous problem we can state our result in a more general, spatially inhomogeneous setting.

We consider solutions of (1) with the spatial domain $\Omega = \mathbb{T}^n$ (n-dimensional torus, or the unit hypercube with periodic boundary conditions), on an arbitrary finite time interval $[0,T]$. Spatially homogeneous solutions are then a special subclass characterized by the constant dependence on the x variable. To simplify the presentation, let us assume sufficient regularity (smoothness) of the solutions $f(x,v,t)$ with respect to the x and t variables; this is not a restriction in the setting of Theorem 1, and the requirements of smoothness will be relaxed significantly later on to include a sufficiently wide class of weak solutions of the spatially inhomogeneous problem.

Theorem 3. *Let $T>0$ and let $f\in C([0,T];L^1(\mathbb{T}^n\times\mathbb{R}^n))$, $f\geq 0$, be a (sufficiently regular) solution of the Boltzmann equation (1), with the initial condition*

$$f(x,v,0)=f_0(x,v)\leq M_0(v),\quad \text{for a. a. } (x,v)\in\mathbb{T}^n\times\mathbb{R}^n,$$

where $M_0(v)=e^{-a_0|v|^2+c_0}$, $a_0>0$, $c_0\in\mathbb{R}$. Assume that the solution $f(x,v,t)$ satisfies the estimates

$$\int_{\mathbb{R}^n} f(x,v,t)\,dv\;\geq\;\rho_0,\quad (x,t)\in\mathbb{T}^n\times[0,T], \tag{13}$$

and

$$\sup_{(x,t)\,\in\,\mathbb{T}^n\times[0,T]}\|f(x,v,t)\|_{L^\infty_v}\;\leq\;C_0,\qquad \sup_{(x,t)\in\mathbb{T}^n\times[0,T]}\int_{\mathbb{R}^n}\frac{f(x,v,t)}{M_1(v)}\,dv\leq C_1, \tag{14}$$

where $M_1(v)=e^{-a_1|v|^2+c_1}$ and $0<a_1<a_0$, c_1, ρ_0, C_0, C_1 are constants. Then for any $0<a<a_1$, for any $t\in[0,T]$

$$f(x,v,t)\leq M(v),\quad \text{for a. a. } (x,v)\;\in\;\mathbb{T}^n\times\mathbb{R}^n,$$

where $M(v)=e^{-a|v|^2+c}$, and the constant c depends on a, a_0, c_0, a_1, c_1, ρ_0, C_0 and C_1 only.

Remark. The regularity assumptions in Theorem 3 are not particularly restrictive. The precise conditions in the spatially inhomogeneous case are that f is a mild (renormalized) solution of (1) that is dissipative in the sense of P.-L. Lions (see Definition 10 in Section 3). A sufficient condition that is naturally satisfied in the spatially-homogeneous case is that (11) holds in addition to (10).

The plan of the paper is as follows. In Section 2 we extend property (iv) from Theorem 2 to the class of Boltzmann kernels satisfying (5)–(7). This involves a rather technical analysis, specific to the spatially-homogeneous problem, which is based mainly on a development of the ideas from [5–7]. The result of Section 2, however, illustrates an important point that the type of behavior described by Theorem 1 is not a particular feature of the hard-sphere model, but rather a generic phenomenon that holds for a wide class of collision kernels of "hard" type. The key step occurs in Section 3: there we introduce the technique based on a comparison principle which plays a crucial role in the derivation of pointwise estimates. In Section 4 we prove a weighted bound for the collision term, based on the Carleman representation of the gain operator, which is used in the comparison argument. Finally, some classical results used throughout the text are recalled in three Appendices.

Convention: Throughout the text, the function $\operatorname{sign} z$ is defined as 1 for $z>0$, -1 for $z<0$ and an arbitrary fixed value in $[-1,1]$ for $z=0$.

2. Weighted L^1 estimates of solutions

The aim of this section is the following result, originally due to Bobylev in the case of the "hard spheres" and Maxwell molecules [5, 6].

Theorem 4. *Let $f(v,t)$, $v \in \mathbb{R}^n$, $t \ge 0$ ($n \ge 2$) be a solution of the spatially homogeneous Boltzmann equation* (2) *with the collision kernel B satisfying* (5)–(7) *and with the initial datum $f_0 \ge 0$ such that*

$$\frac{f_0}{M_0} \in L^1(\mathbb{R}^n) \tag{15}$$

for a certain Maxwellian $M_0(v) = e^{-a_0|v|^2}$, where a_0 is a positive constant. Then there exist constants C, $a > 0$, such that

$$\int_{\mathbb{R}^n} \frac{f(v,t)}{M(v)}\, dv \le C, \quad t \ge 0, \tag{16}$$

where $M(v) = e^{-a|v|^2}$.

Our approach to the problem is based on the analysis of the sequence of moments,

$$m_k(t) = \int_{\mathbb{R}^n} f(v,t)\, |v|^{2k}\, dv, \quad k = 0, 1 \ldots, \tag{17}$$

and particularly, of the growth of $m_k(t)$ as $k \to \infty$. The relation between the moments (17) and the weighted averages (16) is given by the formal expansion

$$\int_{\mathbb{R}^n} \frac{f(v,t)}{M(v)}\, dv = \sum_{k=0}^{\infty} \frac{m_k(t)}{k!}\, a^k. \tag{18}$$

In view of (18), to prove Theorem 4 it suffices to show

$$\overline{\lim_{k\to\infty}} \sup_{t \ge 0} \frac{m_k(t)}{k!\, A^k} = 0, \quad \text{for some } A \text{ large enough.} \tag{19}$$

Our proof of (19) is to a large extent a refinement of the original approach in [6]. One particular technical aspect which allows us to simplify some of the arguments is the systematic use of the interpolation inequalities

$$\left(\tfrac{m_{k_1}(t)}{m_0}\right)^{\frac{1}{k_1}} \le \left(\tfrac{m_k(t)}{m_0}\right)^{\frac{1}{k}} \le \left(\tfrac{m_{k_2}(t)}{m_0}\right)^{\frac{1}{k_2}} \quad k_1 \le k \le k_2, \tag{20}$$

which follow directly from (17) by application of either Hölder or Jensen's inequalities.

It is well-known that if the kernel $B(|v - v_*|, \cos\vartheta)$ in (3) is constant in the first argument (the case of the Maxwell, or pseudo-Maxwell, particles) then the equations for the moments $m_k(t)$ with integer k form a closed infinite system of ODE. This property no longer holds if the kernel B depends on $|v - v_*|$, and one has to work with inequalities instead of equations. If the kernel B has the homogeneity $|v - v_*|^\beta$, one also generally has to consider the moments

$$m_k(t) \quad \text{with} \quad k = j + \tfrac{\beta}{2}\, l, \quad j, l = 0, 1 \ldots \tag{21}$$

Since the total mass is conserved, $m_0(t) = m_0 = \text{const}$; we shall enumerate the rest of the moments (21) by a single index k_n, $n = 1, 2\ldots$, in the increasing order, and introduce the notation

$$J = \{k_n : n = 1, 2 \ldots\}$$

for the index set. Also, introduce the normalized moments

$$z_k(t) = \frac{m_k(t)}{\Gamma(k+b)}, \quad k \geq 0, \tag{22}$$

where the constant $b > 0$ will be chosen below depending on α in (7). For $b = 1$ and k nonnegative integer we have $z_k(t) = m_k(t)/k!$ which is the normalization appearing in (19). Also, as is easy to verify by Stirling's formula,

$$\Gamma(k+b) \sim k^{b-1}\, \Gamma(k+1), \quad k \to \infty, \tag{23}$$

so the particular choice of b is irrelevant for (19).

Given $k = k_n \in J$ we set $\bar{z}^{(k)}(t) = (z_{k_1}(t), \ldots, z_{k_{n-1}}(t))$, a vector with $n-1$ components.

By the assumptions on $m_k(0)$, we have

$$z_k(0) \leq C_0\, q_0^k, \quad k \in J, \tag{24}$$

for certain constants C_0, q_0. We shall show that the geometric growth of the normalized moments is preserved uniformly in time, due to the structure of the system of differential inequalities satisfied by $z_k(t)$; this will imply (19). The key step is the following.

Lemma 5. *Let the sequence of nonnegative functions $z_k \in C^1([0,\infty))$, $k \in J$, satisfy*

$$z_k'(t) \leq -\, A_k\, z_k^{1+\frac{\beta}{2k}}(t) + B_k\, f_k(\bar{z}^{(k)}(t)), \quad k \in J, \quad k \geq k_* \tag{25}$$

and

$$z_k(t) \leq C_1\, q_1^k, \quad k \in J, \quad k < k_*, \tag{26}$$

where $k_* > \frac{\beta}{2}$, C_1 and q_1 are positive constants, A_k, B_k are positive sequences satisfying

$$\frac{A_k}{B_k} \geq C_1^{1-\frac{\beta}{2k}}, \quad k \in J, \quad k \geq k_*, \tag{27}$$

and f_k are continuous functions of their arguments such that

$$f_k(\bar{z}^{(k)}) \;\leq\; C^2\, q^{k+\frac{\beta}{2}}, \quad \text{whenever} \quad z_k \leq C q^k, \quad k \in J, \quad k \geq k_*. \tag{28}$$

Assume that the sequence $z_k(0)$ satisfies (24). Then $z_k(t) \leq C q^k$, $k \in J$, $t \geq 0$, where $C = \max\{C_0, C_1\}$ and $q = \max\{q_0, q_1\}$.

Proof. Without loss of generality we can assume that $C_1 = C_0$ and $q_1 = q_0$. The proof will be achieved by induction on $k \in J$, $k \geq k_*$. For $k = k_*$ conditions (26) and (28) imply

$$z_k'(t) \leq -\, A_k\, z_k^{1+\frac{\beta}{2k}}(t) + B_k\, C_1^2\, q_1^{k+\frac{\beta}{2}}.$$

By a comparison argument for Bernoulli-type ordinary differential equations (cf. [6]),

$$z_k(t) \leq \max\{z_k(0), z_k^*\}, \tag{29}$$

where z_k^* is determined from the equation

$$A_k\,(z_k^*)^{1+\frac{\beta}{2k}} = B_k\,C_1^2\,q_1^{k+\frac{\beta}{2}}$$

Using condition (27) it is easy to verify that $z_k^* \le C_1\,q_1^k$, which in view of (29) and (24) implies $z_k(t) \le C_1 q_1^k$, $k = k_*$. This provides the basis for the induction. The induction step follows by repeating the same reasoning for any $k > k_*$. The proof is complete. □

Next, we shall verify the conditions of Lemma 5 for the sequence of the moments corresponding to a solution of the Boltzmann equation. The proof of the time-regularity of the moments is standard; we refer the reader to Appendix B for the details. We can also use the known property that the moments of every order are uniformly bounded in time (part (iii) of Theorem 2) to deduce (26). The main difficulty is then to obtain the system (25) and to make sure that the necessary estimates hold for the constants.

Let us first make some general comments about the time-evolution of the quantities $\int_{\mathbb{R}^n} f(v,t)\,\Psi(|v|^2)\,dv$, where $\Psi : \mathbb{R}_+ \to \mathbb{R}$ is a convex function. Multiplying equation (2) by $\Psi(|v|^2)$ and integrating with respect to v we obtain, after standard changes of variables,

$$\frac{d}{dt}\int_{\mathbb{R}^n} f(v,t)\,\Psi(|v|^2)\,dv = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v,t)\,f(v_*,t)\,R_\Psi(v,v_*)\,dv\,dv_*, \tag{30}$$

where

$$R_\Psi(v,v_*) = |v-v_*|^\beta\,\big(G_\Psi(v,v_*) - L_\Psi(v,v_*)\big),$$

$$G_\Psi(v,v_*) = \frac{1}{2}\int_{S^{n-1}} \big(\Psi(|v_*'|^2) + \Psi(|v'|^2)\big)\,b(\cos\theta)\,d\sigma$$

and

$$L_\Psi(v,v_*) = \frac{1}{2}\big(\Psi(|v|^2) + \Psi(|v_*|^2)\big)$$

Since the expression for $G_\Psi(v,v_*)$ is clearly the most complicated part of (30) we look for a simpler upper bound. This is achieved by means of the following estimate.

Lemma 6. *Let $\Psi : \mathbb{R}_+ \to \mathbb{R}$ be convex and assume that the function $\bar b(z)$ is nondecreasing on $(0,1)$. Then*

$$G_\Psi(v,v_*) \le \int_{-1}^{1} \Psi\Big(\big(|v|^2 + |v_*|^2\big)\,\frac{1+z}{2}\Big)\,\bar b(z)\,(1-z^2)^{\frac{n-3}{2}}\,dz.$$

Proof. See [7, Lemma 1] for the case $n = 3$; the extension to general n is straightforward. □

Recall that the mass m_0 and the energy m_1 are constant for the solution $f(v,t)$. We will also use a lower bound for the moments of order $\alpha \le 1$.

Lemma 7 (Cf. [6] for the case $\alpha = 1$)**.** *The solution of* (2) *satisfies*

$$\int_{\mathbb{R}^n} f(v_*,t)\,|v-v_*|^\alpha\,dv_* \ge c_\alpha \int_{\mathbb{R}^n} f_0(v_*)\,|v-v_*|^\alpha\,dv_*, \quad v \in \mathbb{R}^n,$$

for any $\alpha \in (0,1]$.

Proof. By translating the solution $f(v_*,t)$ in the velocity space, we can reduce the proof to the case $v=0$. We will establish the estimates

$$m_\alpha(t) \geq c_\alpha\, m_\alpha(0), \tag{31}$$

for $0<\alpha\leq 1$. Notice that $\Psi(z)=-z^\alpha$ is a convex function. Then, by the previous computation, and using Lemma 6,

$$m'_\alpha(t) \geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v,t)\, f(v_*,t)\; |v-v_*|^\beta \left(\frac{a_\alpha}{2}\left(|v|^2+|v_*|^2\right)^\alpha - \frac{1}{2}\left(|v|^{2\alpha}+|v_*|^{2\alpha}\right)\right) dv\, dv_*$$

where $a_\alpha = 2\int_{-1}^{1}(\frac{1+z}{2})^\alpha\, \bar{b}(z)\,(1-z^2)^{\frac{n-3}{2}}\,dz > 1$. We shall estimate the integrand above in order to obtain an expression involving $m_\alpha(t)$ and similar quantities. For this we notice that since $(x+y)^\beta \leq x^\beta + y^\beta$, for $\beta\in[0,1]$, then

$$|v-v_*|^\beta \leq (|v|+|v_*|)^\beta \leq |v|^\beta + |v_*|^\beta.$$

Also,

$$|v-v_*|^\beta \geq \big|\,|v|^\beta - |v_*|^\beta\big| \quad \text{and} \quad (|v|^2+|v_*|^2)^\alpha \geq \big|\,|v|^{2\alpha}-|v_*|^{2\alpha}\big|.$$

Therefore

$$\begin{aligned}
&|v-v_*|^\beta \left(\frac{a_\alpha}{2}\left(|v|^2+|v_*|^2\right)^\alpha - \frac{1}{2}\left(|v|^{2\alpha}+|v_*|^{2\alpha}\right)\right)\\
&\quad\geq \frac{a_\alpha}{2}\left(|v|^\beta-|v_*|^\beta\right)\left(|v|^{2\alpha}-|v_*|^{2\alpha}\right) - \frac{1}{2}\left(|v|^\beta+|v_*|^\beta\right)\left(|v|^{2\alpha}+|v_*|^{2\alpha}\right)\\
&\quad= \frac{a_\alpha-1}{2}\left(|v|^{\beta+2\alpha}+|v_*|^{\beta+2\alpha}\right) - \frac{a_\alpha+1}{2}\left(|v|^\beta|v_*|^{2\alpha}+|v|^{2\alpha}|v_*|^\beta\right)
\end{aligned}$$

and we obtain

$$m'_\alpha(t) \;\geq\; (a_\alpha-1)\, m_0\, m_{\alpha+\frac{\beta}{2}}(t) - (a_\alpha+1)\, m_{\frac{\beta}{2}}(t)\, m_\alpha(t)\;.$$

In the particular case $\beta=1$ we have

$$m'_{\frac12}(t) \geq (a_{\frac12}-1)\, m_0 m_1 - (a_{\frac12}+1)\, m^2_{\frac12}(t),$$

(m_0 and m_1 are constants, by the conservation of mass and energy). Therefore,

$$m_{\frac12}(t) \geq \min\left\{ m_{\frac12}(0), \left(\frac{a_{\frac12}-1}{a_{\frac12}+1}\, m_0\, m_1\right)^{\frac12}\right\} \geq \min\left\{1, \left(\frac{a_{\frac12}-1}{a_{\frac12}+1}\right)^{\frac12}\right\} m_{\frac12}(0)\;,$$

since $m_0 m_1 \geq m_{\frac12}(0)^2$. (This is the argument of Bobylev.) To achieve the proof for $\beta<1$ we iterate this argument, applying it with $\alpha=\frac{j\beta}{2}$, $j=1\ldots$, until $\frac{(j+1)\beta}{2}\geq 1$. Consider first the case of the terminal j, when

$$\alpha_0 = \frac{j\beta}{2} < 1 \leq \frac{(j+1)\beta}{2}\;.$$

In that case

$$\begin{aligned}
m'_{\alpha_0}(t) &\geq (a_{\alpha_0}-1)\, m_0\, m_{\alpha_0+\frac{\beta}{2}}(t) - (a_{\alpha_0}+1)\, m_{\beta/2}(t)\, m_{\alpha_0}(t)\\
&\geq (a_{\alpha_0}-1)\, m_0^{2-(\alpha_0+\frac{\beta}{2})}\, m_1^{\alpha_0+\frac{\beta}{2}} - (a_{\alpha_0}+1)\, m_0^{1-\frac{\beta}{2\alpha_0}}\, m_{\alpha_0}^{1+\frac{\beta}{2\alpha_0}}(t)
\end{aligned}$$

Therefore,

$$m_{\alpha_0}(t) \geq \min\left\{ m_{\alpha_0}(0), \left(\frac{a_{\alpha_0}-1}{a_{\alpha_0}+1} m_0^{(\frac{1}{\alpha_0}-1)(\alpha_0+\frac{\beta}{2})} m_1^{\alpha_0+\frac{\beta}{2}} \right)^{\frac{1}{1+\frac{\beta}{2\alpha_0}}} \right\}$$

$$\geq \min\left\{ 1, \left(\frac{a_{\alpha_0}-1}{a_{\alpha_0}+1} \right)^{\frac{1}{1+\frac{\beta}{2\alpha_0}}} \right\} m_{\alpha_0}(0) = \left(\frac{a_{\alpha_0}-1}{a_{\alpha_0}+1} \right)^{\frac{1}{1+\frac{\beta}{2\alpha_0}}} m_{\alpha_0}(0) \; .$$

Further, take $\alpha_1 = \alpha_0 - \frac{\beta}{2} > 0$. Then

$$m'_{\alpha_1}(t) \geq (a_{\alpha_1}-1)\, m_0\, m_{\alpha_0}(t) - (a_{\alpha_1}+1)\, m_0^{1-\frac{\beta}{2\alpha_1}}\, m_{\alpha_1}^{1+\frac{\beta}{2\alpha_1}}(t),$$

so

$$m_{\alpha_1}(t) \geq \min\left\{ m_{\alpha_1}(0), \left(\left(\frac{a_{\alpha_1}-1}{a_{\alpha_1}+1} \right) m_0^{\frac{\beta}{2\alpha_1}} m_{\alpha_0}(t) \right)^{\frac{1}{1+\frac{\beta}{2\alpha_1}}} \right\}$$

$$\geq \min\left\{ m_{\alpha_1}(0), \left(\left(\frac{a_{\alpha_1}-1}{a_{\alpha_1}+1} \right) \left(\frac{a_{\alpha_0}-1}{a_{\alpha_0}+1} \right)^{\frac{\alpha_0}{\alpha_0+\frac{\beta}{2}}} m_0^{\frac{\beta}{2\alpha_1}} m_{\alpha_0}(0) \right)^{\frac{1}{1+\frac{\beta}{2\alpha_1}}} \right\}$$

$$\geq \left(\frac{a_{\alpha_1}-1}{a_{\alpha_1}+1} \right)^{\frac{\alpha_1}{\alpha_1+\frac{\beta}{2}}} \left(\frac{a_{\alpha_0}-1}{a_{\alpha_0}+1} \right)^{\frac{\alpha_1}{\alpha_0+\frac{\beta}{2}}} m_{\alpha_1}(0) \; .$$

The rest of the proof can be obtained by induction. This establishes (31) for all $\alpha \in (0,1]$ and completes the proof. □

In the particular case $\Psi(z) = z^k$, $k \geq 1$, we obtain the following inequalities

$$m'_k(t) \leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(v,t) f(v_*,t)\, \bar{R}_k(v,v_*)\, dv\, dv_*, \tag{32}$$

where

$$\bar{R}_k(v,v_*) = \frac{1}{2}\, |v-v_*|^\beta \big(a_k (|v|^2 + |v_*|^2)^k - |v|^{2k} - |v_*|^{2k} \big), \tag{33}$$

and the constant a_k is defined by

$$a_k = \int_{-1}^{1} \Big(\frac{1+z}{2} \Big)^k \, \bar{b}(z)\, (1-z^2)^{\frac{n-3}{2}}\, dz, \tag{34}$$

satisfies $a_k \leq 1$ for $k \geq 1$ and is strictly decreasing as k increases. We notice the inequalities $|v-v_*|^\beta \leq |v|^\beta + |v_*|^\beta$,

$$(|v|^2 + |v_*|^2)^k - |v|^{2k} - |v_*|^{2k} \leq \sum_{i=1}^{[\frac{k+1}{2}]} \binom{k}{j} (|v|^{2j} |v_*|^{2(k-j)} + |v|^{2(k-j)} |v_*|^j),$$

where $[\cdot]$ denotes the integer part (cf. [7]). Also, by Lemma 7,

$$\int_{\mathbb{R}^n} f(v_*,t)\, |v-v_*|^\beta\, dv_* \geq c_\beta \int f_0(v_*)\, |v-v_*|^\beta\, dv_* \geq \nu_0\, (1+|v|^\beta),$$

where ν_0 is a constant depending on β and f_0. Using these inequalities in (32), (33) we obtain

$$m'_k(t) \leq -(1-a_k)\, \nu_0\, m_{k+\frac{\beta}{2}}(t) + a_k\, S_k(t) \tag{35}$$

where

$$S_k(t) = \sum_{j=1}^{[\frac{k+1}{2}]} \binom{k}{j} \big(m_{j+\frac{\beta}{2}}(t)\, m_{k-j}(t) + m_{k-j+\frac{\beta}{2}}(t)\, m_j(t) \big).$$

The crucial estimate for the sum $S_k(t)$ is provided by the following Lemma.

Lemma 8. *For $b > 0$ fixed set $z_k(t) = m_k(t)/\Gamma(k+b)$, $k \geq 1$. Then*

$$S_k(t) \leq C_b\, \Gamma(k + \frac{\beta}{2} + 2b)\, Z_k(t), \quad k \geq 1,$$

where

$$Z_k(t) = \max_{1 \leq j \leq [\frac{k+1}{2}]} \{ z_{j+\frac{\beta}{2}}(t)\, z_{k-j}(t), z_j(t)\, z_{k-j+\frac{\beta}{2}}(t) \} \tag{36}$$

and C_b is a constant depending on b.

Proof. See [7, Lemma 4]. □

Proof of Theorem 4. Using the interpolation inequality $m_{k+\frac{\beta}{2}}(t) \geq m_0^{-\frac{\beta}{2k}}\, m_k(t)^{1+\frac{\beta}{2k}}$ and Lemma 8 we derive from (35) the inequalities

$$z_k'(t) \leq -(1-a_k)\, \nu_0\, m_0^{-\frac{\beta}{2k}}\, \Gamma(k+b)^{\frac{\beta}{2k}}\, z_k^{1+\frac{\beta}{2}}(t) + a_k\, C_b\, \frac{\Gamma(k+\frac{\beta}{2}+2b)}{\Gamma(k+b)}\, Z_k(t). \tag{37}$$

Notice that for $k \in J$, $k > 1 + \frac{\beta}{2}$ the term $Z_k(t)$ is of the form $f_k(\bar{z}^{(k)}(t))$ as in Lemma 5, since the highest order of moment entering (36) is $k - 1 + \frac{\beta}{2}$. It is also clear the the function f_k defined in this way is a continuous function of its arguments. Thus, we can identify (37) with (25) by setting

$$A_k = (1-a_k)\, \nu_0\, m_0^{-\frac{\beta}{2k}}\, \Gamma(k+b)^{\frac{\beta}{2k}}, \qquad B_k = a_k\, C_b\, \frac{\Gamma(k+\frac{\beta}{2}+2b)}{\Gamma(k+b)}. \tag{38}$$

We would like to apply Lemma 5 to the sequence of functions $z_k(t)$. It remains to verify that the sequences of constants A_k and B_k appearing in (37) satisfy (27). To this end we show that

$$\frac{A_k}{B_k} \geq c_0, \quad k > k_*, \tag{39}$$

for any $k_* > 1 + \frac{\beta}{2}$ and a sufficiently small c_0; then (27) would follow by choosing $C_0 = C_1$ small enough and $q_0 = q_1$ large enough in (24), (26). Indeed, by using (23),

$$\Gamma(k+b)^{-\frac{\beta}{2k}} \sim k^{\beta/2} \quad \text{and} \quad \frac{\Gamma(k+\frac{\beta}{2}+2b)}{\Gamma(k+b)} \sim k^{\frac{\beta}{2}+b}, \quad k \to \infty. \tag{40}$$

To estimate the constant a_k in (37) we recall that $\bar{b}(z) \leq C\,(1-z^2)^{-\alpha}$, $\alpha < n-1$ and setting in (34), $s = \frac{z+1}{2}$, $\varepsilon = n - 1 - \alpha > 0$, we have

$$\begin{aligned} a_k &= C\, 2^{-1+\varepsilon} \int_0^1 s^{k-1+\frac{\varepsilon}{2}} (1-s)^{-1+\frac{\varepsilon}{2}}\, ds = C\, 2^{-1+\varepsilon} B(k+\tfrac{\varepsilon}{2}, \tfrac{\varepsilon}{2}) \\ &= C\, 2^{-1+\varepsilon}\, \frac{\Gamma(k+\frac{\varepsilon}{2})\, \Gamma(\frac{\varepsilon}{2})}{\Gamma(k+\varepsilon)} \asymp k^{-\frac{\varepsilon}{2}}, \quad k \to \infty. \end{aligned} \tag{41}$$

We fix $0 < b < \varepsilon/2$; the corresponding constants A_k, B_k satisfy the inequalities

$$A_k \geq \bar{A}\, k^{\frac{\beta}{2}}, \quad B_k \leq \bar{B}\, k^{\frac{\beta}{2}+b-\frac{\varepsilon}{2}}, \quad k \geq k_*, \tag{42}$$

where $k_* > 1+\frac{\beta}{2}$, and $\bar{A}$ and $\bar{B}$ are absolute constants which can be estimated based on (38) and the asymptotic relations (40) and (41). From (42) we obtain (39) if we choose $c_0 = \bar{A}\bar{B}^{-1}k_*^{\frac{\varepsilon}{2}-b}$.

We conclude the proof of Theorem 4 by applying Lemma 5. □

3. Comparison principle for the Boltzmann equation

In this section we introduce the important technique of comparison that will allow us to obtain pointwise estimates of the solutions. It is based on a certain monotonicity property of the semigroup associated with a *linear* Boltzmann equation. The argument is roughly as follows: if f is a solution of (1), $f|_{t=0} = f_0$, and g satisfies

$$(\partial_t + v\cdot\nabla_x)\, g = Q(f,g), \quad g|_{t=0} = g_0, \tag{43}$$

then $u = f - g$ is a solution of

$$(\partial_t + v\cdot\nabla_x)\, u = Q(f,u), \quad u|_{t=0} = u_0, \tag{44}$$

where $u_0 = f_0 - g_0$. We will show that as long as $f \geq 0$ solutions of (44) satisfy the order-preserving property,

$$\text{if} \quad u_0 \leq 0 \quad \text{then} \quad u \leq 0. \tag{45}$$

This translates into the following estimates for the solutions of (1):

$$\text{if} \quad f_0 \leq g_0 \quad \text{then} \quad f \leq g. \tag{46}$$

For the purposes of the estimate (46) the equality in (43) can be replaced by the inequality ($\geq$); a similar approach can be utilized to obtain lower estimates of the solutions.

Of course, this scheme has to be implemented with suitable modifications. For instance, since we generally only have limited information about f we will require that g satisfies

$$(\partial_t + v\cdot\nabla_x)\, g \geq Q(f,g)$$

for *a class* of functions f satisfying the apriori estimates, on a certain "large" subset of the (x,v,t)-space (in the argument below this will be the the region of large velocities $\{|v| \geq R\}$). A different argument will have to be applied to obtain the estimates in the rest of the space. This will be the general strategy that we will pursue in the proof of Theorem 3 in the end of this section.

This approach originated in the work by one of the authors [32, page 103] in the context of lower bounds for the spatially-homogeneous equation without angular cutoff. It was then used for obtaining lower bounds for a model describing inelastic collisions [18, Section 7]. It is interesting to compare the present technique with other methods based on monotonicity applied to the Boltzmann equation, in particular the one by Kaniel and Shinbrot [24] (see also [21, 23]) and the pointwise estimates by Vedenjapin [31] (the result in the latter paper follows from our approach using

$g = e^{C(1+t)}$). The monotonicity property expressed by (45) has also an important relation to the concept of dissipative solutions introduced by P.-L. Lions [27].

We will first explain (45). We define the bilinear form in (44) by setting

$$Q(f,u)(x,v,t) = \int_{\mathbb{R}^n}\int_{S^{n-1}} (f'_* u' - f_* u)\, B(v-v_*,\sigma)\, d\sigma\, dv_*, \tag{47}$$

where as usual, $f'_* = f(x, v'_*, t)$, $u' = u(x, v', t)$, $f_* = f(x, v_*, t)$, $u = u(x,v,t)$. For the results of this section we do not need to assume the kernel B to satisfy (5)–(7); we can take a more general class of kernels with the usual symmetries, as described in [14], for instance. We fix the function $f(x,v,t) \geq 0$, $(x,v,t) \in \mathbb{T}^n \times \mathbb{R}^n \times [0,T]$, which we may for simplicity assume to be smooth in (x,t), bounded and rapidly decaying for $|v|$ large. We also assume that for every $u_0 \in D \subseteq L^1(\mathbb{T}^n \times \mathbb{R}^n)$ the initial-value problem (44) has a unique solution $u \in C([0,T]; L^1(\mathbb{T}^n \times \mathbb{R}^n))$, such that

$$Q^+(f,|u|),\, Q^-(f,|u|) \in L^1(\mathbb{T}^n \times \mathbb{R}^n \times [0,T]) \tag{48}$$

and the equation (44) is satisfied in the sense of distributions. Thus, we have a well-defined flow map (or semigroup)

$$\Phi_t :\, D \ni u_0 \mapsto u(t,\cdot,\cdot) \in L^1(\mathbb{T}^n \times \mathbb{R}^n), \quad t \in [0,T].$$

Conditions on the regularity of f and u will be relaxed significantly later on; this will be particularly relevant from the point of view of applications to the general weak solutions of the nonlinear spatially inhomogeneous problem.

It is straightforward to see that the bilinear collision term (47) satisfies

$$\int_{\mathbb{R}^n} Q(f,u)\, \operatorname{sign} u\, dv \leq 0, \tag{49}$$

for every $f \geq 0$ and every u so that $Q^+(f,|u|)$, $Q^-(f,|u|) \in L^1$. Indeed, inequality (49) follows immediately from the weak form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\int_{S^{n-1}} f_* u\, (\operatorname{sign} u' - \operatorname{sign} u)\, B(v-v_*,\sigma)\, d\sigma\, dv_* dv$$

by noticing that $u\,(\operatorname{sign} u' - \operatorname{sign} u) \leq 0$.

A direct application of (49) is the following nonexpansive property: for any $u_0,\ \nu_0 \in D$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big|\Phi_t(u_0) - \Phi_t(\nu_0)\big|\, dv\, dx \,\leq\, \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u_0 - \nu_0|\, dv\, dx, \quad t \in [0,T]. \tag{50}$$

To verify (50) we set $w = \Phi_t(u_0) - \Phi_t(\nu_0)$; then w satisfies

$$\partial_t w + v\cdot\nabla_x w = Q(f,w) \quad \text{on} \quad \mathbb{T}^n \times \mathbb{R}^n \times (0,T)$$

in the sense of distributions, and $Q(f,w) \in L^1$ by our assumptions. By a standard argument, $\forall\, t \in [0,T]$, for a. a. (x,v) the function $w^\sharp : s \mapsto w(x-(t-s)v, v, s)$, $s \in [0,T]$, is absolutely continuous, and we can apply the chain rule (see Appendix A) to obtain

$$\frac{d}{ds}\,|w^\sharp| \,=\, Q(f,w)^\sharp\, \operatorname{sign} w^\sharp, \quad s \in (0,T), \tag{51}$$

where $Q(f,w)^\sharp$ is defined similarly to $w^\sharp$. Integrating with respect to $s \in (0,t)$ and $(x,v) \in \mathbb{T}^n \times \mathbb{R}^n$ we obtain, after standard changes of variables,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w(x,v,t)|\,dv\,dx \;-\; \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w_0|\,dv\,dx$$
$$= \int_0^t \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} Q(f,w)\,\operatorname{sign} w\,dv\,dx\,ds \;\le\; 0,$$

where $w_0 = u_0 - \nu_0$, and we used the dissipative property (49). This establishes (50).

The same approach can be followed to obtain (45). Indeed, we have by (49) and the mass conservation

$$\int_{\mathbb{R}^n} Q(f,u)\,\tfrac{1}{2}(\operatorname{sign} u + 1)\,dv \;\le 0, \tag{52}$$

where $\frac{1}{2}(\operatorname{sign} u + 1)$ is the a. e. derivative of the Lipschitz-continuous function $u_+ = \max\{u, 0\}$. We then have

$$\frac{d}{ds}\,u_+^\sharp \;=\; Q(f,u)^\sharp\;\tfrac{1}{2}(\operatorname{sign} u + 1)^\sharp, \quad s \in (0,T),$$

and the integration yields

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_+(x,v,t)\,dv\,dx \le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_{0+}\,dv\,dx, \quad t \in [0,T],$$

which implies (45) for a. a. (x,v).

Remark. Relation (45) can be restated as the order-preserving property of Φ_t:

$$\forall\, u_0,\, \nu_0 \in D,\;\; u_0 \le \nu_0 \;\text{ implies }\; \Phi_t(u_0) \le \Phi_t(\nu_0), \quad t \in [0,T]. \tag{53}$$

In fact, the equivalence of (53) and (50) follows from a general principle applied to (nonlinear) maps that preserve integral, as described by Crandall and Tartar [11]. Inequality (45) (or (53)) can then be seen as a consequence of the results in [11], the preservation of the mass $\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f\,dv\,dx$ along solutions of (44), and (50).

The following localized version of the order-preserving property will be useful for the comparison argument.

Proposition 9. *Let* $f, u \in C([0,T]; L^1(\mathbb{T}^n \times \mathbb{R}^n))$ *satisfy*

$$f \ge 0; \quad \partial_t u + v\cdot\nabla_x u,\; Q^+(f,u),\; Q^-(f,u) \in L^1; \quad u|_{t=0} = u_0 \le 0,$$

and assume that for a certain (measurable) set $U \subseteq \mathbb{T}^n \times \mathbb{R}^n \times (0,T)$,

$$\partial_t u + v\cdot\nabla_x u - Q(f,u) \;\le\; 0 \quad \text{on} \quad U,$$

and

$$u \le 0 \quad \text{on} \quad U^c := \big(\mathbb{T}^n \times \mathbb{R}^n \times (0,T)\big) \setminus U.$$

Then $u(t,\cdot,\cdot) \le 0$ *a. e. on* $\mathbb{T}^n \times \mathbb{R}^n$*, for every* $t \in [0,T]$.

Proof. Let $D(u) = \partial_t u + v \cdot \nabla_x u$. We obtain by arguing as above,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_+(x,v,t)\,dv\,dx - \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_+(x,v,0)\,dv\,dx = \int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} D(u)\,\tfrac{1}{2}(\operatorname{sign} u + 1)\,dx\,dv\,ds.$$

We have $u_+|_{t=0} = 0$; also $\frac{1}{2}(\operatorname{sign} u + 1) = 0$ whenever $u < 0$ and $D(u) = 0$ outside of a set of zero measure in $\{u = 0\}$. Therefore, setting $U_t = \{(x,v,s) \in U : s \le t\}$ we have

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_+(x,v,t)\,dv\,dx = \iiint_{U_t} D(u)\,dx\,dv\,ds$$
$$\le \iiint_{U_t} Q(f,u)\,dx\,dv\,ds = \int_0^t\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} Q(f,u)\,\tfrac{1}{2}(\operatorname{sign} u + 1)\,dx\,dv\,ds \le 0,$$

for every $t \in [0,T]$, where we used the dissipative property (52). This shows that $u(t,\cdot,\cdot) \le 0$ almost everywhere. □

Proposition 9 is sufficient to formulate the comparison principle in the generality required for Theorem 1. However, the regularity assumptions, particularly the integrability conditions for the collision terms, can be relaxed to give a more general statement that is applicable to a wide class of weak solutions of (1). We first give a definition of weak solutions.

Definition 10. We say that $f \in C([0,T]; L^1(\mathbb{T}^n \times \mathbb{R}^n))$, $f \ge 0$, $f|v| \in L^1$, is a dissipative solution of the nonlinear Boltzmann equation (1) if f is a mild solution, and for every sufficiently regular function $g : \mathbb{T}^n \times \mathbb{R}^n \times [0,T] \to \mathbb{R}$,

$$\partial_t \int_{\mathbb{R}^n} |f - g|\,dv + \operatorname{div}_x \int_{\mathbb{R}^n} |f - g|\,v\,dv \le \int_{\mathbb{R}^n} \big(Q(f,g) - D(g)\big)\operatorname{sign}(f-g)\,dv, \tag{54}$$

in the sense of distributions, where $D(g) = \partial_t g + v \cdot \nabla_x g$, and sign(0) is assigned an abitrary value in $[-1,1]$.

Remark. Mild solutions in the spatially inhomogeneous case are defined via the integral form

$$f^\sharp(t_2) - f^\sharp(t_1) = \int_{t_1}^{t_2} Q(f)^\sharp(s)\,ds, \quad 0 \le t_1 < t_2 \le T,$$

a. e. in (x,v), in full analogy with the definition (10) (cf. also [14]). "Sufficiently regular" in the above definition precisely means that $g \in C([0,T]; L^1_{xv})$, $g|v|^2 \in L^\infty_t(L^1_{xv})$, $\partial_t g + v \cdot \nabla_x g \in L^1_{xvt}$ and that for any $f \in C([0,T]; L^1_{xv})$ such that $f|v|^2 \in L^\infty_t(L^1_{xv})$, $Q^+(f,|g|)$, $Q^-(f,|g|) \in L^1_{xvt}$ (these conditions can be extended in a rather technical way, as shown in [27]).

Dissipative solutions are known to exist globally [27], however they may generally not satisfy the conditions $Q^+(f)$, $Q^-(f) \in L^1$. The formal motivation for the definition of dissipative solutions is clear: the right-hand side of the Boltzmann equation can be written as

$$Q(f) = Q(f, f-g) + Q(f,g),$$

so we have

$$(\partial_t + v\cdot\nabla_x)(f-g) = Q(f,f-g) + Q(f,g) - D(g).$$

Multiplying the above equation by $\operatorname{sign}(f-g)$ and using relation (49) (note that $f \ge 0$) we see that every sufficiently regular solution of (1) should satisfy (54).

Dissipative solutions also satisfy the local form of the mass conservation,

$$\partial_t \int_{\mathbb{R}^n} f\,dv + \operatorname{div}_x \int_{\mathbb{R}^n} f\,v\,dv = 0, \tag{55}$$

in the sense of distributions.

Using the order-preserving property of Proposition 9 we establish the following comparison principle for dissipative solutions of the nonlinear Boltzmann equation.

Theorem 11. *Let $f \in C([0,T];L^1(\mathbb{T}^n\times\mathbb{R}^n))$ be a disssipative solution of* (1) *and let g be a sufficiently regular function, such that $f|_{t=0} \le g|_{t=0}$,*

$$\partial_t g + v\cdot\nabla_x g - Q(f,g) \ge 0 \quad \text{on } U$$

and $f \le g$ on U^c, where U is a measurable subset of $\mathbb{T}^n\times\mathbb{R}^n\times[0,T]$. Then $f\le g$ almost everywhere on $\mathbb{T}^n\times\mathbb{R}^n$, for every $t\in[0,T]$.

Remark. It is natural to call g a (localized) upper barrier. By reversing all inequalities in the above formulation one can also obtain a similar comparison principle for the lower barrier.

Proof. We use the notation $D(g) = \partial_t g + v\cdot\nabla_x g$, so that

$$\partial_t \int_{\mathbb{R}^n} g\,dv + \operatorname{div}_x \int_{\mathbb{R}^n} g\,v\,dv = \int_{\mathbb{R}^n} D(g)\,dv,$$

in the sense of distributions. Using the mass conservation (55) and the identity

$$(f-g)_+ = \tfrac{1}{2}\big(|f-g| + (f-g)\big)$$

we obtain, by combining the above relations with (54),

$$\begin{aligned}\partial_t \int_{\mathbb{R}^n} (f-g)_+\,dv &+ \operatorname{div}_x \int_{\mathbb{R}^n} (f-g)_+\,v\,dv \\ &\le \frac{1}{2}\int_{\mathbb{R}^n} \big(Q(f,g)-D(g)\big)\operatorname{sign}(f-g)\,dv - \frac{1}{2}\int_{\mathbb{R}^n} D(g)\,dv.\end{aligned}$$

Since $Q^\pm(f,|g|)$ are integrable, we have $\int_{\mathbb{R}^n} Q(f,g)\,dv = 0$, a. e. (x,t), and therefore,

$$\begin{aligned}\partial_t \int_{\mathbb{R}^n} (f-g)_+\,dv &+ \operatorname{div}_x \int_{\mathbb{R}^n} (f-g)_+\,v\,dv \\ &\le \int_{\mathbb{R}^n} \big(Q(f,g)-D(g)\big)\,\tfrac{1}{2}(\operatorname{sign}(f-g)+1)\,dv.\end{aligned} \tag{56}$$

We can set $\operatorname{sign}(0) = -1$ in (56) to avoid estimating the integral over the set $\{f=g\}$. Since $(f-g)_+\,v \in L^1(\mathbb{T}^n\times\mathbb{R}^n\times[0,T])$ we can integrate over x and t to obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f-g)_+(x,v,t)\,dv\,dx &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (f-g)_+(x,v,0)\,dv\,dx \\ &\quad + \iiint_{U_t} \big(Q(f,g)-D(g)\big)\,dx\,dv\,ds \le 0,\end{aligned} \tag{57}$$

where $U_t = \{(x, v, s) \in U : s \le t\}$ and we used that $\frac{1}{2}(\text{sign}(f-g)+1)$ vanishes for $f \le g$ and that $Q(f, g) - D(g) \le 0$ on U_t. The inequality in (57) implies that $f \le g$, a. e. $(x, v) \in \mathbb{T}^n \times \mathbb{R}^n$, for every $t \in [0, T]$. □

Theorem 11 is a cruicial ingredient in the proof of Theorem 3, which we give below.

Proof of Theorem 3. To apply Theorem 11 we set $U = \{(x, v, t) : |v| > R\}$, where R will be chosen large enough, and $g(x, v, t) = M(v)$, where $M(v) = e^{-a|v|^2+c}$, $0 < a < a_1$ is fixed and $c > c_0$ will be chosen sufficiently large, depending on R. To prove that g can be used as a barrier for the solution on U we need to verify the inequality

$$Q^+(f, g)(x, v, t) \le Q^-(f, g)(x, v, t), \quad (x, t) \in \mathbb{T}^n \times [0, T], \quad |v| > R. \tag{58}$$

First notice that, by elementary inequalities,

$$\begin{aligned} Q^-(f, g)(x, v, t) &= M(v) \int_{\mathbb{R}^n} f(x, v_*, t)\, |v - v_*|^\beta \, dv_* \\ &\ge M(v) \Big(\rho_0 |v|^\beta - \int_{\mathbb{R}^n} f(x, v_*, t)\, |v_*|^\beta \, dv_* \Big), \end{aligned}$$

where ρ_0 is the constant in (13). The last term can be controlled using the estimate for the integral of f/M_1 from (14) as follows,

$$\int_{\mathbb{R}^n} f(x, v_*, t)\, |v_*|^\beta \, dv_* \le L \int_{\mathbb{R}^n} \frac{f(x, v_*, t)}{M_1(v_*)} \, dv_* \le L\, C_1,$$

where $L = \max_{y \ge 0} y^\beta \, e^{-a_1 y^2 + c_1}$. Thus, we have

$$Q^-(f, g)(x, v, t) \ge M(v) \left(\rho_0 |v|^\beta - L\, C_1 \right).$$

The control of the "gain" term is more technical; we establish below in Lemma 12 the estimate

$$Q^+(f, g)(x, v, t) \le C\, (1 + |v|^{\beta - \varepsilon})\, M(v), \tag{59}$$

where $\varepsilon = \min\{\beta, n - 1 - \alpha\} > 0$. This implies that (58) holds if we set R to be the largest root of the equation

$$C + LC_1 + Cy^{\beta - \varepsilon} - \rho_0 y^\beta = 0.$$

Finally, we take $c = aR^2 + \log C_0$, where C_0 is the constant in (14); then it is easy to verify that

$$f(x, v, t) \le C_0 \le M(v), \quad (x, t) \in \mathbb{T}^n \times [0, T], \quad |v| \le R. \tag{60}$$

The conditions $0 < a < a_1 < a_0$ and $c \ge c_0$ guarantee that we have $f(x, v, 0) \le M(v)$. Together with the inequalities (58) and (60) this allows us to use Theorem 11 to conclude. □

4. A weighted estimate for the "gain" operator

To complete the proof of Theorem 3 we prove the following weighted estimate of the linear "gain" operator. The main technique is based on Carleman's form of the "gain" term (see Appendix C).

Lemma 12. *Let $B : \mathbb{R}^n \times S^{n-1} \to \mathbb{R}^+$, $n \geq 2$, be a measurable function that satisfies*

$$B(u,\sigma) \;\leq\; C\,(1+|u|^\beta)\,\frac{1}{|\sin\vartheta|^\alpha}\,1_{\{\cos\vartheta\geq 0\}}, \qquad \cos\vartheta = \tfrac{u\cdot\sigma}{|u|},$$

where $\beta > 0$ and $\alpha < n-1$. Define

$$Q^+(f,g)(v) \;=\; \int_{\mathbb{R}^n}\int_{S^{n-1}} f'_*\,g'\,B(v-v_*,\sigma)\,d\sigma\,dv_*,$$

and set $M(v) = e^{-a|v|^2}$, $a > 0$; $w_\varepsilon(v) = 1+|v|^{\beta-\varepsilon}$, where $\varepsilon = \min\{\beta, n-1-\alpha\} > 0$. Then

$$\Big\|\,\frac{Q^+(f,M)}{w_\varepsilon\,M}\,\Big\|_{L^\infty(\mathbb{R}^n)} \;\leq\; C\,\Big\|\,\frac{f\,w_\varepsilon}{M}\,\Big\|_{L^1(\mathbb{R}^n)}, \tag{61}$$

where C is an explicitly computable constant depending on n, α, β and a.

Remark. For B satisfying the estimate with $\alpha = 0$ (for example, the kernel $\bar B$ for hard spheres in three dimensions) we have $\varepsilon = \beta$ for all $\beta \leq n-1$ and the weight $w_\varepsilon(v)$ is constant. The estimate of the Lemma then takes a particularly simple form,

$$\Big\|\,\frac{Q^+(f,M)}{M}\,\Big\|_{L^\infty_v} \;\leq\; C\,\Big\|\,\frac{f}{M}\,\Big\|_{L^1_v}.$$

For the quadratic "gain" term this implies the estimate

$$\Big\|\,\frac{Q^+(f)}{M}\,\Big\|_{L^\infty_v} \;\leq\; C\,\Big\|\,\frac{f}{M}\,\Big\|_{L^\infty_v}\,\Big\|\,\frac{f}{M}\,\Big\|_{L^1_v}.$$

Proof. By the Carleman representation formula,

$$Q^+(f,M)(v) = 2^{n-1}\int_{\mathbb{R}^n}\frac{f(v'_*)}{|v-v'_*|}\int_{E_{vv'_*}} M(v')\,\frac{B(v-v_*,\sigma)}{|v-v_*|^{n-2}}\,d\pi_{v'}\,,$$

where $E_{vv'_*}$ is the hyperplane

$$\{v'\in\mathbb{R}^n \,:\, (v-v')\cdot(v-v'_*) = 0\},$$

and $d\pi_{v'}$ denotes the usual Lebesgue measure on $E_{vv'_*}$. We then have

$$\frac{Q^+(f,M)(v)}{M(v)} \;=\; \int_{\mathbb{R}^n}\frac{f(v'_*)}{M(v'_*)}\,K(v,v'_*)\,dv'_*, \tag{62}$$

where

$$K(v,v'_*) = \frac{2^{n-1}}{|v-v'_*|}\int_{E_{vv'_*}} M(v_*)\,\frac{B(v-v_*,\sigma)}{|v-v_*|^{n-2}}\,d\pi_{v'}, \tag{63}$$

and we used that, by the energy conservation,

$$\frac{M(v')\,M(v'_*)}{M(v)} = M(v_*).$$

Note that in (63) the variables v_* and σ are expressed through v, v'_* and v' as follows,

$$v_* = v'_* + v' - v, \qquad \sigma = \frac{v' - v'_*}{|v' - v'_*|}.$$

Now to establish the Lemma it suffices to verify the inequality

$$K(v, v'_*) \le C\,(1 + |v - v'_*|^{\beta-\varepsilon}). \tag{64}$$

Indeed, since

$$1 + |v - v'_*|^{\beta-\varepsilon} \le (1 + |v|^{\beta-\varepsilon})\,(1 + |v'_*|^{\beta-\varepsilon}),$$

then (62) and (64) imply

$$Q^+(f, M)(v) \le C\,(1 + |v|^{\beta-\varepsilon})\,M(v) \int_{\mathbb{R}^n} \frac{f(v'_*)}{M(v'_*)}\,(1 + |v'_*|^{\beta-\varepsilon})\,dv'_*$$

which is equivalent to (61).

In the remainder of the proof we will therefore verify (64). Using the identity

$$(v - v_*) \cdot (v' - v_*) = |v - v'_*|^2 - |v - v'|^2$$

for $v' \in E_{vv'_*}$ and recalling that $B(v - v_*, \sigma)$ vanishes for $(v - v_*) \cdot \sigma < 0$ we see that the integration in (63) can be restricted to the disk

$$D_{vv'_*} = E_{vv'_*} \cap \{v' \in \mathbb{R}^n : |v - v'_*| \le |v - v'|\}.$$

We notice that for $v' \in D_{vv'_*}$,

$$\left|\tan\frac{\vartheta}{2}\right| = \frac{|v'_* - v_*|}{|v - v'_*|}, \qquad |\vartheta| \le \frac{\pi}{2},$$

where ϑ is the angle between the vectors $v - v_*$ and σ. This implies

$$\frac{1}{|\sin\vartheta|} \le \frac{1}{2}\,\frac{|v - v'_*|}{|v'_* - v_*|}$$

Thus, $K(v, v'_*) \le C\widetilde{K}(v, v'_*)$, where

$$\widetilde{K}(v, v'_*) = \frac{2^{n-1-\alpha}}{|v - v'_*|^{1-\alpha}} \int_{D_{vv'_*}} M(v_*)\,\frac{1 + |v - v_*|^\beta}{|v - v_*|^{n-2}}\,\frac{1}{|v'_* - v_*|^\alpha}\,d\pi_{v'}.$$

To estimate the above expression we consider two cases.

Case a) $|v - v'_*| \le 1$. Since for $v' \in D_{vv'_*}$

$$|v - v'_*| \le |v - v_*| \le \sqrt{2}\,|v - v'_*|$$

we have $1 + |v - v_*|^\beta \le 1 + 2^{\beta/2}$ and

$$|v - v_*|^{2-n} \le |v - v'_*|^{2-n}.$$

Therefore,

$$\widetilde{K}(v, v'_*) \le \frac{2^{n-1-\alpha}(1 + 2^{\beta/2})}{|v - v'_*|^{n-1-\alpha}} \int_{D_{vv'_*}} M(v_*)\,\frac{1}{|v'_* - v_*|^\alpha}\,d\pi_{v'}.$$

Since $M(v_*) \le 1$ the last integral is estimated above by

$$\int_{D_{vv'_*}} \frac{1}{|v'_* - v_*|^\alpha}\,d\pi_{v'} = \int_{\{w \in \mathbb{R}^{n-1} : |w| \le |v - v'_*|\}} \frac{1}{|w|^\alpha}\,dw = \tfrac{\omega_{n-2}}{n-1-\alpha}\,|v - v'_*|^{n-1-\alpha},$$

if $n-1-\alpha > 0$, i. e. $\alpha < n-1$. Here ω_{n-2} is the measure of the $(n-2)$-dimensional unit sphere. This implies the estimate

$$\widetilde{K}(v, v'_*) \le \frac{2^{n-1-\alpha}(1+2^{\beta/2})\,\omega_{n-2}}{n-1-\alpha}, \quad |v - v'_*| \le 1.$$

Case b) $|v - v'_*| > 1$. Then

$$1 + |v - v_*|^\beta \le 2\,|v - v_*|^\beta \le 2^{1+\frac{\beta}{2}}\,|v - v'_*|^\beta,$$

and we obtain, similarly to the previous case,

$$\widetilde{K}(v, v'_*) \le \frac{2^{n-\alpha+\frac{\beta}{2}}}{|v - v'_*|^{n-1-\alpha-\beta}} \int_{D_{vv'_*}} M(v_*)\,\frac{1}{|v'_* - v_*|^\alpha}\,d\pi_{v'}.$$

Since $M(v_*)$ is a radially decreasing function of $v_* \in \mathbb{R}^n$, and so is $|v_*|^{-\alpha}$,

$$\int_{D_{vv'_*}} M(v_*)\,|v'_* - v_*|^{-\alpha}\,d\pi_{v'} \le \int_{\mathbb{R}^{n-1}} \bar{M}(w)\,|w|^{-\alpha}\,dw$$
$$\le \int_{|w|\le 1} |w|^{-\alpha}\,dw + \int_{\mathbb{R}^{n-1}} \bar{M}(w)\,dw = \frac{\omega_{n-2}}{n-1-\alpha} + \left(\frac{\pi}{a}\right)^{\frac{n-1}{2}},$$

where $\bar{M}(w) = e^{-a|w|^2}$, $w \in \mathbb{R}^{n-1}$. Since $|v-v'_*|^{\beta+\alpha-n+1} \le |v-v_*|^{\beta-\varepsilon}$ this establishes the required estimate for Case b). $\square$

Appendix A: Some properties of weakly differentiable functions

Let $AC[a,b]$ denote the class of absolutely continuous real-valued functions defined on an interval $[a,b]$. Given $f \in AC[a,b]$ we set $[c,d] = f([a,b])$ and use the notation $\mathrm{Lip}[c,d]$ for the class of all Lipschitz continuous functions defined on $[c,d]$. By Rademacher's theorem, every function $\beta \in \mathrm{Lip}[c,d]$ is differentiable (in the classical sense) almost everywhere on (c,d); we agree to extend this derivative to a function β' defined everywhere on $[c,d]$ by assigning arbitrary *finite* values at the points where β is not differentiable. The function β' also coincides with the weak derivative of β almost everywhere on (c,d) The following chain rule was used in the arguments in Section 3.

Proposition 13. *Let $f \in AC[a,b]$ and $\beta \in \mathrm{Lip}[c,d]$. Then $\beta \circ f \in AC[a,b]$ and*

$$(\beta \circ f)' = (\beta' \circ f)\,f',$$

almost everywhere on (a,b).

Remark. 1) The seeming ambiguity in the above formulation occuring since $\beta' \circ f$ can assume arbitrarily assigned values on a set of positive measure is resolved by observing that whenever this happens then f' vanishes, except on a set of measure zero (see the proof below). 2) For the purposes of Section 3 we only need the chain rule for $\beta(y) = |y|$ and $\beta(y) = y_+$; these cases are covered in [17], and the proof for the case of piecewise-C^1 functions β can be found in [20]. We include a short proof that applies to the general case to make the presentation in Section 3 self-contained.

Proof. By the definition of absolutely continuous functions,

$$\forall\,\varepsilon>0\ \ \exists\,\delta>0\ \text{ such that }\ \forall\,n\in\mathbb{N},\ \ \forall\,\{(x_j,y_j)\subseteq[a,b]:j=1,\dots,n\},$$

a disjoint family,

$$\sum_{j=1}^{n}|y_j-x_j|<\delta\ \ \Rightarrow\ \ \sum_{j=1}^{n}|f(y_j)-f(x_j)|<\varepsilon.$$

Clearly then, since

$$|\,\beta(f(y_j))-\beta(f(x_j))\,|\le L\,|f(y_j)-f(x_j)|,$$

where L is the Lipschitz constant of β, the composition $\beta\circ f$ is absolutely continuous on $[a,b]$. By Lebesgue's differentiation theorem, f and $\beta\circ f$ are differentiable in the classical sense on a set with complement of measure zero in (a,b). Pick $x\in(a,b)$ from this set. We will consider two cases, depending on whether β is differentiable at $f(x)$ or not. In the first case we have

$$\begin{aligned}(\beta\circ f)'(x)&=\lim_{h\to0}\frac{\beta(f(x+h))-\beta(f(x))}{h}\\&=\lim_{h\to0}\frac{\beta(f(x+h))-\beta(f(x))}{f(x+h)-f(x)}\ \lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\beta'(f(x))f'(x).\end{aligned}$$

Let us further take A to be the set of y such that β is not differentiable at $f(y)$. We claim that $f'(x)$ vanishes for $x\in A$, except perhaps on a set of zero Lebesgue measure. Indeed, let $B=\{y\in A:|f'(y)|>0\}$; then

$$B=\bigcup_{n=1}^{\infty}B_n,\quad B_n=\{y\in B:\ |f(z)-f(y)|\ge\tfrac{|z-y|}{n}\ \text{for}\ |z-y|<\tfrac{1}{n}\,\}.$$

We prove the claim by showing that every set B_n has zero measure.

Fix an $n\in\mathbb{N}$. Since β is Lipschitz, we know that $f(A)$ is a set of measure zero. Given $\varepsilon>0$ we can then choose the intervals I_j, $j=1,\dots$, such that

$$f(A)\subseteq\bigcup_{j=1}^{\infty}I_j\quad\text{and}\quad\sum_{j=1}^{\infty}|I_j|<\varepsilon.$$

Let J be an interval of length $\frac{1}{n}$, and let $D=B_n\cap J$, $D_j=f^{-1}(I_j)\cap D$. Then, from the definition of B_n, $|D_j|\le n|I_j|$; therefore, $|D|\le n\varepsilon$ and $|B_n|\le n^2|b-a|\varepsilon$. Since ε is arbitrary this shows that $|B_n|=0$.

We now have that for a. a. $x\in A$

$$\Big|\frac{\beta(f(x+h))-\beta(f(x))}{h}\Big|\le L\,\Big|\frac{f(x+h)-f(x)}{h}\Big|$$

for $|h|$ small enough, so $(\beta\circ f)'(x)=0$ and $\beta'(f(x))f'(x)=0$. This proves the claim of the Lemma for a. a. $x\in(a,b)$. $\square$

Appendix B: Time regularity for the spatially homogeneous Boltzmann equation

We show that the solution of the Boltzmann equation (2) under the conditions of Theorem 1 is smooth with respect to time, together with its moments of any order.

For $k \geq 0$ we introduce the following weighted Lebesgue spaces

$$L^1_k(\mathbb{R}^n) = \Big\{ f \in L^1(\mathbb{R}^n) : \int_{\mathbb{R}^n} |f| \, (1+|v|^2)^k \, dv < +\infty \Big\} \tag{65}$$

with the norms defined by the integrals appearing in (65). The regularity result that we used in Section 2 is the following.

Proposition 14. *Let f be the unique solution of the Boltzmann equation* (2) *that preserves the total mass and energy. Assume that $f_0 \in L^1_k(\mathbb{R}^n)$, $k > 1 + \frac{\beta}{2}$. Then $f \in C^1\big([0,+\infty); L^1_p(\mathbb{R}^n)\big)$ for any $p < k - \frac{\beta}{2}$.*

The proof of Proposition 14 depends on the following continuity property of the nonlinear operator $Q(f)$.

Lemma 15. *Let the pair of positive numbers (k,p) satisfy $k > p + \frac{\beta}{2}$. Then $Q(f)$ is continuous on $L^1_k(\mathbb{R}^n)$ as a mapping $L^1_k(\mathbb{R}^n) \to L^1_p(\mathbb{R}^n)$. Moreover, we have the following Hölder estimate for any $f, g \in L^1_k(\mathbb{R}^n)$*

$$\|Q(f) - Q(g)\|_{L^1_p} \leq C_p \Big(\|f-g\|_{L^1}^{1-\frac{p+\frac{\beta}{2}}{k}} + \|f-g\|_{L^1} \Big),$$

where the constant C_p depends on p and on the upper bound of the L^1_k-norms of f and g.

Proof. Using the weak form of $Q(f)$ and $Q(g)$ we compute

$$\begin{aligned}
&\int_{\mathbb{R}^n} |Q(f) - Q(g)| \, (1+|v|^2)^p \, dv \\
&= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \int_{S^{n-1}} (ff_* - gg_*) \, B(v-v_*, \sigma) \Big(\mathrm{sign}\big(Q(f)' - Q(g)'\big)(1+|v'|^2)^p \\
&\qquad\qquad - \mathrm{sign}\big(Q(f) - Q(g)\big)(1+|v|^2)^p \Big) \, d\sigma \, dv \, dv_* \\
&\leq \; 2^{p+1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |ff_* - gg_*| \, |v-v_*|^\beta \, \big((1+|v|^2)^p + (1+|v_*|^2)^p \big) \, dv \, dv_*
\end{aligned}$$

Since

$$\begin{aligned}
|v-v_*|^\beta \, (1+|v|^2)^p &\leq (1+|v_*|^2)^{\frac{\beta}{2}} (1+|v|^2)^p + (1+|v|^2)^{p+\frac{\beta}{2}} \\
&\leq 2 \left((1+|v|^2)^{p+\frac{\beta}{2}} + (1+|v_*|^2)^{p+\frac{\beta}{2}} \right)
\end{aligned}$$

and $|ff_* - gg_*| \le \frac{1}{2}|f-g|\,|f_*+g_*| + \frac{1}{2}|f+g|\,|f_*-g_*|$, we obtain

$$\begin{aligned}
&\|Q(f)-Q(g)\|_{L^1_p} \\
&\le 2^{p+3}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f+g|\,|f_*-g_*|\,\big((1+|v|^2)^{p+\frac{\beta}{2}} + (1+|v_*|^2)^{p+\frac{\beta}{2}}\big)\,dv\,dv_* \\
&\le 2^{p+3}\,\|f+g\|_{L^1_k}\big(\|f-g\|_{L^1_{p+\frac{\beta}{2}}} + \|f-g\|_{L^1}\big).
\end{aligned}$$

We use the interpolation inequality (20) with $k_1 = p+\frac{\beta}{2}$ to get

$$\begin{aligned}
\|f-g\|_{L^1_{p+\frac{\beta}{2}}} &\le \|f-g\|_{L^1}^{1-\frac{p+\frac{\beta}{2}}{k}}\,\|f-g\|_{L^1_k}^{\frac{p+\frac{\beta}{2}}{k}} \\
&\le \big(\|f\|_{L^1_k} + \|g\|_{L^1_k}\big)^{\frac{p+\frac{\beta}{2}}{k}}\,\|f-g\|_{L^1}^{1-\frac{p+\frac{\beta}{2}}{k}}.
\end{aligned}$$

Substituting this bound into the previous estimate we obtain the Hölder estimate stated in the Lemma. This completes the proof. □

Proof of Proposition 14. We fix $T>0$. By the results of Arkeryd and Elmroth [1,15] (see part (iii) of Theorem 2), f belongs to $L^\infty([0,+\infty); L^1_k(\mathbb{R}^n))$. By Lemma 15,

$$(1+|v|^2)^p\,Q(f) \in L^1((0,T)\times\mathbb{R}^n), \quad \text{for} \quad p < k - \frac{\beta}{2} \tag{66}$$

The mild form of (2), together with the regularity condition (66) imply that f is weakly differentiable and $\partial_t f = Q(f)$ in the sense of distributions on $(0,T)\times\mathbb{R}^n$. Hence,

$$f \in W^{1,1}((0,T); L^1_p(\mathbb{R}^n))$$

and therefore (cf. [16, p. 286]), $f \in C\big([0,T]; L^1_p(\mathbb{R}^n)\big)$. By the continuity of $Q(f)$ established in Lemma 15 it follows that $\partial_t f \in C\big([0,T]; L^1_p(\mathbb{R}^n)\big)$, where $\partial_t f$ is the weak time-derivative of f. It is then easy to verify directly that f is strongly differentiable on $(0,T)$ with values in $L^1_p(\mathbb{R}^n)$, and its derivative is continuous on $[0,T]$. Since T is arbitrary, we obtain the conclusion of the Lemma. □

Remark. As a consequence of Proposition 14, if the moments of all orders are finite initially then they are continuously differentiable functions of time. By iterating the argument we used in the proof above one can show that in fact then $f \in C^\infty([0,\infty); L^1_k(\mathbb{R}^n))$, for any $k \ge 0$.

Appendix C: Carleman's representation

Lemma 16. *Let $Q^+(f,g)$ be defined by (9) and let $f=f(v)$ and $g=g(v)$, $v\in\mathbb{R}^n$ be smooth functions, decaying rapidly at infinity. Then*

$$Q^+(f,g)(v) = 2^{n-1}\int_{\mathbb{R}^n}\frac{f(v'_*)}{|v-v'_*|}\int_{E_{v,v'_*}}\frac{g(v')\,B(2v-v'-v'_*,\frac{v'-v'_*}{|v'-v'_*|})}{|v'-v'_*|^{n-2}}\,d\pi_{v'}\,dv'_*,$$

where E_{v,v'_*} is the hyperplane $\{v'\in\mathbb{R}^n \mid (v'-v)\cdot(v'_*-v)=0\}$ and $d\pi_{v'}$ denotes the Lebesgue measure on this hyperplane.

Proof. Using the change of variables $u = v - v_*$, and recalling the definition of the delta function of a quadratic form, see [19], we have

$$Q^+(f,g)(v) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(v'_*)\, g(v')\, B(u,k)\, \delta\big(\tfrac{|k|^2-1}{2}\big)\, dk\, du, \tag{67}$$

where $v' = v - u + \frac{1}{2}(u + |u|k)$ and $v'_* = v - \frac{1}{2}(u+|u|k)$. We further set $z = -\frac{1}{2}(u+|u|k)$; for every u fixed this defines a linear map $k \mapsto z$ with determinant $\big(\frac{|u|}{2}\big)^n$. We also have

$$k = -\frac{2z+u}{|u|} \quad \text{and} \quad \frac{|k|^2-1}{2} = \frac{|2z+u|^2 - |u|^2}{2|u|^2} = \frac{2z\cdot(z+u)}{|u|^2}.$$

With this change of variables the integral in (67) can be written as

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big(\tfrac{2}{|u|}\big)^n f(v+z)\, g(v-u-z)\, B(u, -\tfrac{2z+u}{|u|})\, \delta\big(\tfrac{2z\cdot(z+u)}{|u|^2}\big)\, dz\, du.$$

We set $y = -z-u$; then $|u| = |y+z|$ and $\delta\big(\frac{2z\cdot(z+u)}{|u|^2}\big) = \frac{|y+z|^2}{2}\,\delta(z\cdot y)$. Further, for any test function φ,

$$\int_{\mathbb{R}^n} \delta(z\cdot y)\,\varphi(y)\, dy = |z|^{-1}\int_{z\cdot y = 0} \varphi(y)\, d\pi_y,$$

where $d\pi_y$ is the Lebesgue measure on the hyperplane $\{y : z\cdot y = 0\}$. This yields

$$\begin{aligned} &Q^+(f,g)(v) \\ &= 2^{n-1}\int_{z\in\mathbb{R}^n}\int_{y\cdot z=0} f(v+z)g(v+y)\,|z|^{-1}|y+z|^{n-2}\, B(-y-z, \tfrac{y-z}{|y+z|})\, d\pi_y\, dz \end{aligned}$$

We now return to the original notations $v'_* = v+z$, $v' = v+y$ and perform the corresponding changes of variables to obtain the expression for $Q^+(f,g)$ stated in the Lemma. □

Remark. The above result takes a particularly simple form in the case of the hard-sphere model in $\mathbb{R}^3$; in that case $B(v-v_*,\sigma) = \frac{1}{4\pi}|v-v_*|$ and

$$Q^+(f,g)(v) = \int_{\mathbb{R}^3} \frac{f(v'_*)}{\pi|v-v'|}\int_{E_{v,v'_*}} g(v')\, d\pi_{v'}\, dv'_*.$$

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