A New Consistent Discrete-Velocity Model for the Boltzmann Equation

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This paper discusses the convergence of a new discrete-velocity model to the Boltzmann equation. First the consistency of the collision integral approximation is proved. Based on this we prove the convergence of solutions for a modified model to renormalized solutions of the Boltzmann equation. In a numerical example, the solutions to the discrete problems are compared with the exact solution of the Boltzmann equation in the space-homogeneous case.

1 Introduction

The nonlinear Boltzmann equation is one of the fundamental equations of kinetic theory. It is used to describe evolutions of many-particle systems, such as rarefied gases, in which in which the main interaction mechanism is binary collisions. The equation reads as follows:

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla f = Q(f, f), \quad (x, t) \in \mathcal{D} \subseteq \mathbb{R}^3 \times \mathbb{R}, \quad \xi \in \mathbb{R}^3. \tag{1.1}$$

Here $f = f(\xi, x, t)$ is the one-paricle distribution function, which has the physical meaning of mass density of particles in the (ξ, x)-space. Variables ξ, x and t denote molecular velocity, position and time, respectively. $Q(f, f)$ is a non-linear (quadratic) integral operator, called collision operator. The basic properties of the collision operator are discussed below. A detailed presentation of the physical background, mathematical theory and applications of the Boltzmann equation can be found in the books on kinetic theory [13], [36].

One of the major problems associated with the Boltzmann equation is constructing an efficient numerical method for obtaining the solutions. Most widely used currently are the particle methods that are based on stochastic simulation. The best known examples are methods of Bird [4] (Direct Simulation Monte Carlo or DSMC) and Nanbu [25] with modifications by Babovsky [2]. The main drawbacks of such methods are their slow convergence and random fluctuations that affect the accuracy of the results. On the other hand, using other, deterministic techniques for solving the Boltzmann equation entails significant difficulties. Firstly, calculating multiple integrals that appear in the collision operator is a formidable task. Secondly, a large amount of information has to be stored, since the distribution function depends, in the physically meaningful cases, on seven scalar variables. On the current level of development, deterministic methods lose in efficeiency to the particle methods. However, a significant progress has been achieved in developing such methods in the last years [9, 26, 31, 35]. In fact, the deterministic approaches are the ones to be preferred when a high numerical accuracy is of paramount importance [27, 34, 33]. Finally, a number of approaches were developed to increase the efficiency of deterministic calculations without sacrificing the accuracy [9, 26, 29].

In this paper we focus on one deterministic approach for solving the Boltzmann equation: the one based on discrete-velocity models (DVMs). Such models have been studied for a long time in kinetic theory, [11, 8, 16, 10, 24]. They were introduced with the purpose of reproducing the formal properties of the Boltzmann equation using a finite number of discrete velocities. The idea of using DVMs as approximations of the Boltzmann equation was developed initially as a purely computational approach [1, 35, 6, 17, 20]. From a theoretical point of view, there was no justification of their relation to the Boltzmann equation, at least before the appearance of papers of Buet [9] and Bobylev, Palczewski and Schneider [7]. Buet studied the DVM introduced by Goldstein *et al.* [17] and gave a heuristic argument on why the convergence to the Boltzmann collision integral should be expected. The rigorous proof of convergence given in [7] was based on quite recent, detailed results from number theory. Rogier and Schneider [31] introduced another DVM with a two-dimensional velocity space, for which they also proved convergence. Their approach was later generalized to three dimensions by Michel and Schneider [22]. The convergence of solutions of the DVM [17] in the space-homogeneous case was established by Palczewski and Schneider [28]. For space-inhomogeneous solutions a convergence result was obtained by Mischler [23].

In this paper we introduce a DVM with properties analogous to the models studied in [7] and [31, 22]. It is based on a transformation of the velocity variables in the collision term, known as the Carleman tranfsorm [11]. Such a transformation allows us to formulate a model that is much easier to analyse theoretically than the previously known models, and is also easy to implement numerically. In Carleman's variables the integration over spheres in the collision term is replaced by integration over planes in $\mathbb{R}^3$. This allows us to reduce the problem of convergence to studying distribution of integer points on planes. The solution of the latter problem can be formulated in terms of linear Diophantine equations. Thus, the solution of the problem is obtained by a much lesser effort than in the case of spheres. The usefulness of the Carleman transformation in application to discrete-velocity models was also noticed previously by Wennberg and Golse [37], who studied a two-dimensional model analogous to ours. We notice that convergence of the present model (as well as the model [17]) in two dimensions is still an open question: our method works only in dimensions three or higher.

The main results of the paper are as follows: we prove the consistency (that is, the convergence of the collision operator for smooth distribution functions) in Section 3. We also prove the error estimates with bounds from $O(h^{1/4})$ to $O_\varepsilon(h^{1-\varepsilon})$ for distribution functions from the classes C^m with $m \geq 1$ and with compact support. We then pass to the analysis of convergence of solutions in Section 4. We use a finite volume reformulation of the model analogous to the one by Mischler [23] and prove the weak L^1 convergence of approximations to a renormalized solution of the Boltzmann equation. In Section 5 we present the results of numerical computations for a space homogeneous problem.

We consider the Boltzmann collision integral for a monoatomic gas in the following form:

$$Q(f,f)(\xi) = \int_{\mathbb{R}^3} \int_{S^{(2)}} \big(f(\xi')\, f(\eta') - f(\xi)\, f(\eta) \big)\, B(\xi - \eta, \omega)\, d\omega\, d\eta. \tag{1.2}$$

Here $\xi' = \xi - \omega(\omega \cdot (\xi - \eta))$ and $\eta' = \eta + \omega(\omega \cdot (\xi - \eta))$ are the velocities after the collision of particles with velocities ξ and η, and ω is the angular parameter of the collision.

The function $B(u, \omega)$, called the collision kernel, is assumed to be of the form

$$B(u, \omega) = b(|u|, \cos\vartheta), \quad \cos\vartheta = |u \cdot \omega|/|u|.$$

The exact form of this function is determined by the nature of the particle interactions, and the Boltzmann equation is usually studied under rather general assumptions about the function B. Typical examples of collision kernels are kernels of inverse power interactions with angular cut-off:

$$b(r,x) = r^{\gamma} a(x), \tag{1.3}$$

where $-3 < \gamma \le 1$ and a is a function in $L^1[0,1]$. In particular, the classical case of "hard sphere" particles corresponds to $b(r,x) = crx$, c being a constant.

In discrete-velocity models it is assumed that the velocities of the particles belong to a finite set $V = \{\xi_i\}_{i=1}^N \subseteq \mathbb{R}^3$. Thus, the distribution function $f(\xi,x,t)$ is replaced by an approximation $f_i(x,t),\ \xi_i \in V$. The values of f_i are determined from the following system of equations:

$$\frac{\partial f_i}{\partial t} + \xi_i \cdot \nabla f_i = Q_i(f,f), \quad (x,t) \in \mathcal{D} \subseteq \mathbb{R}^3 \times \mathbb{R},\ \xi_i \in V, \tag{1.4}$$

$$Q_i(f,f) = \sum_{jkl} A_{ij}^{kl} \left(f_k f_l - f_i f_j \right), \tag{1.5}$$

where the summation is taken over all indices corresponding to the discrete velocities in V. The coefficients A_{ij}^{kl} are assumed to be constants, and have the meaning of the rates of collisions that transform the pair of velocities (ξ_i, ξ_j) into (ξ_k, ξ_l).

It is known from the theory of DVMs [16] that if the coefficients A_{ij}^{kl} satisfy the symmetry relations

$$A_{ij}^{kl} = A_{ji}^{lk}, \quad A_{ij}^{kl} = A_{kl}^{ij}, \tag{1.6}$$

and also if the momentum ξ_i and the energy $\frac{1}{2}|\xi_i|^2$ are collision invariants:

$$\xi_i + \xi_j = \xi_k + \xi_l \ \text{ and } \ |\xi_i|^2 + |\xi_j|^2 = |\xi_k|^2 + |\xi_l|^2, \ \text{ if } \ A_{ij}^{kl} \neq 0, \tag{1.7}$$

then the following discrete analogues of the conservation laws and the entropy condition hold:

$$\sum_i Q_i(f,f) \begin{pmatrix} 1 \\ \xi_i \\ |\xi_i|^2 \end{pmatrix} = 0; \qquad \sum_i Q_i(f,f) \log f_i \le 0. \tag{1.8}$$

Another important formal property concerns with the structure of the collision invariants. We require that the model has no other collision invariants than the classical ones of mass, momentum vector and energy. In other words, the only vectors ψ_i for which one has

$$\psi_i + \psi_j = \psi_k + \psi_l \ \text{ if } \ A_{ij}^{kl} \neq 0. \tag{1.9}$$

are 1, ξ_i, $|\xi_i|^2$ and their linear combinations. For models with the above property the term *normal* was introduced by C. Cercignani [12]. The only possible equilibrium states of normal models are discrete Maxwellians [16], which is an important property from both theoretical and numerical points of view. Notice that the normality of a model is determined only by the distribution of the non-zero coefficients in the matrix A_{ij}^{kl}, and otherwise does not depend on their values.

2 The discrete-velocity model

In this section we construct a DVM that satisfies (1.6)–(1.9) and which will be the base for the convergence analysis in Sections 3 and 4. We formulate our model in general dimension $d \geq 2$. We will then consider the case $d = 3$ when proving consistency of the model.

We choose the discrete-velocity space V as a part of the regular cubic grid:

$$h\mathbb{Z}^d = \{hi \,|\, i = (i_1, \ldots, i_d) \in \mathbb{Z}^d\}.$$

It is then convenient to index the velocities by integer vectors $i = (i_1, \ldots, i_d)$. To obtain (1.7) we can simply set $A_{ij}^{kl} = 0$ if either of the conditions

$$\begin{aligned} i + j &= k + l, \\ |i|^2 + |j|^2 &= |k|^2 + |l|^2 \end{aligned} \tag{2.1}$$

is not satisfied.

A choice of V that is natural for constructing a numerical method is taking all velocities inside a bounded set, say a cube or a sphere of a given (large) size. However, to analyse the convergence it is simpler to consider the infinite DVM with

$$V = V_h = h\mathbb{Z}^d.$$

Moreover, as noticed by Buet [9], if the bounded set used for truncating V satisfies certain symmetry conditions, the truncated DVM inherits the properties (1.6)–(1.9) from the infinite one. Therefore, we will study the consistency problem for the infinite DVM, and return to the models with finite number of velocities in Section 4.

The general approach of [7, 31, 22] for proving convergence of discrete collision terms is to view them as multidimensional quadrature formulas for the integral (1.2). To outline the difficulties that then arise we notice that for fixed i and j (pre-collisional velocities) equations (2.1) imply that k and l (post-collisional velocities) lie on a sphere of diameter $|i-j|$. Thus, a straightforward approach requires that the integral over the sphere in the collision term is approximated by using points with integer coordinates, which leads to difficult number-theoretical problems [7]. An alternative approach developed by Schneider et al. [31, 22] avoids these difficulties, but it still requires approximating integrals over spheres.

The motivation for introducing the present model is to to show that the problem of convergence can be solved without invoking integration over spheres. The key idea is to use the transformation of velocities that was introduced by Carleman [11]. It can be described as follows: If we fix in equations (2.1) one pre-collisional velocity, say i, and one post-collisional, say k, then the remaining two velocities lie on the planes (hyperplanes of codimension one) perpendicular to $i - k$. This can be shown easily by rewriting equations (2.1) in the following equivalent form:

$$\begin{aligned} j &= k + l - i, \\ (k - i) \cdot (l - i) &= 0. \end{aligned} \tag{2.2}$$

To obtain an analogous transformation of the collision term we perform the change of variables

$$(\eta,\ \omega)\ \mapsto\ (\xi',\eta') = \big(\xi - \omega(\omega\cdot(\xi-\eta)),\ \eta + \omega(\omega\cdot(\xi-\eta))\big).$$

We see that for fixed values of ξ and ξ', η' runs twice over the plane $E_{\xi\xi'}$ that passes through ξ and is perpendicular to $\xi-\xi'$, as ω runs over the unit sphere. It is convenient to modify the new variables as follows:

$$u = \xi' - \xi, \quad w = \eta' - \xi.$$

We also use the notation E_u for the plane through the origin perpendicular to the vector u. Thus, we obtain the following form of the bilinear collision integral (cf. Carleman [11]):

$$Q(f,g)(\xi) =$$
$$\int_{\mathbb{R}^d}\int_{E_u} (f(\xi+u)g(\xi+w) - f(\xi)g(\xi+u+w))B^c(u,w)\,dw\,du, \tag{2.3}$$

where

$$B^c(u,w) = 2\,|u|^{-2}\,b\left(\sqrt{u^2+w^2}, \frac{|u|}{\sqrt{u^2+w^2}}\right). \tag{2.4}$$

Fixing two functions f and g we denote the integrand in (2.3) by $F(\xi,u,w)$ and also denote

$$G(\xi,u) = \int_{E_u} F(\xi,u,w)\,dw, \tag{2.5}$$

so that

$$Q(f,g)(\xi) = \int_{\mathbb{R}^d} G(\xi,u)\,du. \tag{2.6}$$

For the integral (2.6) we apply the multidimensional rectangle rule, obtaining formally:

$$Q(f,g)(\xi) \approx h^d \sum_{k\in\mathbb{Z}^d} G(\xi,u_k). \tag{2.7}$$

Now for each ξ and $u \in V_h$ we approximate the integral (2.5) by using the values of w at those discrete points that lie on the planes E_k. For each k fixed, the intersection of V_h with the plane E_k is a two-dimensional lattice:

$$V_h \cap E_k = \left\{hl \in h\mathbb{Z}^d \;\middle|\; k\cdot l = 0\right\} = hL_k, \tag{2.8}$$

where

$$L_k = \left\{l \in \mathbb{Z}^d \;\middle|\; k\cdot l = 0\right\}. \tag{2.9}$$

We see that the lattice (2.9) is the set of solutions of a linear Diophantine equation $k\cdot l = 0$. Every such lattice has a basis $e_1,\dots,e_{d-1}$, such that

$$L_k = \{e_1m_1 + \dots + e_{d-1}m_{d-1} \,\big|\, e_1,\dots,e_{d-1} \in \mathbb{Z}^d,\ m_1,\dots,m_{d-1} \in \mathbb{Z}\},$$

where the vectors $e_1,\dots e_{d-1}$ are linearly independent as vectors of $\mathbb{R}^d$ [19]. This leads to the following approximation of the integral (2.5) by a lattice quadrature formula (which can be viewed as a generalization of the rectangle rule):

$$\int_{E_k} F(\xi, u_k, w)\, dw \;\approx\; h^{d-1}\Delta_k \sum_{l\in L_k} F(\xi, u_k, w_l). \tag{2.10}$$

Here $\Delta_k = \det(L_k)$ is the volume of the parallelepiped spanned by the vectors $e_1, \dots, e_{d-1}$ (such a parallelepiped is called a fundamental cell of the lattice). Although the basis of a lattice can be chosen in different ways, the value of Δ_k does not depend on this choice [19]. We also have the following explicit expression for Δ_k [32]:

$$\Delta_k = |k|/\mathfrak{g}(k). \tag{2.11}$$

Here $\mathfrak{g}(k)$ denotes the greatest common divisor of the components of the vector k.

Combining (2.7) and (2.10) we obtain the formal approximation:

$$\begin{aligned} Q(f,g)(\xi_i) \approx Q_{h,i}(f,g) = h^{2d-1} \sum_{k\in\mathbb{Z}^d} \Delta_k \sum_{l\in L_k} B^c_{jk} \\ \times\left(f(\xi_i+u_k)\, g(\xi_i+w_l) - f(\xi_i)\, g(\xi_i+u_k+w_l)\right). \end{aligned} \tag{2.12}$$

Here we set $B^c_{jk} = B^c(u_k, w_l)$. The sum in (2.12) can be rewritten in the form (1.5) by performing simple manipulations with indices. The coeffiecients A^{kl}_{ij} are then given by the following formula:

$$A^{kl}_{ij} = h^{2d-1}\Delta_{k-i} B^c_{k-i,l-i} \chi^{kl}_{ij}, \tag{2.13}$$

where χ^{kl}_{ij} is the indicator function that equals one if equations (2.2) are satisfied, and zero otherwise. (The right-hand side of (2.13) is not defined when $k = i$, so we set $A^{kl}_{ij} = 0$ in that case.)

The properties of the model can be formulated in the following Theorem:

Theorem 2.1 *The discrete-velocity model* (1.4), (1.5) *with the coefficients* A^{kl}_{ij} *defined by* (2.13) *satisfies the discrete mass, momentum, and energy conservation relations and the entropy property* (1.8). *The model has no other collision invariants than linear combinations of* 1, ξ_i *and* $|\xi_i|^2$.

Proof: Using (2.13) the symmetries (1.6) are easily verified. Because of the equivalence of equations (2.1) and (2.2), we can also get (1.7) and (1.8). The last statement of the Theorem can be proved along the lines of Buet [9] or Rogier and Schneider [31]. □

3 Convergence of the discrete collision operator

In this section we formulate and prove the main results about the formal consistency of the DVM with the Boltzmann equation. To simplify the presentation we consider only the case of a three-dimensional velocity space ($d = 3$). It is easy to see from the proofs that similar results could be obtained for each $d \geq 3$ (but not for $d = 2$). The first theorem deals with the case when the arguments of the the coliision operator are smooth and compactly supported functions. In this case we are able to prove estimates for the rate of convergence of the approximation.

Theorem 3.1 *Assume that the kernel B^c, defined by (2.4) satisfies the following condition:*

$$B^*(u, w) = |u|^\beta B^c(u, w) \in C^m(\mathbf{P}),$$

for some $0 \leq \beta < 3$ and $m \geq 1$. Here

$$\mathbf{P} = \{(u, w) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid u \cdot w = 0\},$$

equipped with the metric $(|u|^2+|w|^2)^{1/2}$. Take f and $g \in C^m(\mathbb{R}^3)$, with compact support. Then, for sufficiently small h,

$$|Q(f, g)(\xi_i) - Q_{h,i}(f, g)| \leq Ch^r,$$

where

$$r = \min\left(\frac{m}{m+3}, \beta, m/\left(m + 3 + \tfrac{\beta - m}{m+3-\beta}\right)\right),$$

and the constant C does not depend on h and ξ_i.

Remarks: *i) The assumption on B^c appears to be a natural one as soon as the functions in (2.6) and (2.5) are required to be integrable and smooth except at $u = 0$. It is satisfied for "hard sphere" particles with $\beta \geq 1$ and any $m \geq 1$ (in this case $B^c(u, w) = C|u|^{-1}$). ii) The third term in the expression for the exponent r can be omitted when $m \geq 3$. iii) In this Theorem, as well as everywhere below, the assumption of C^m-regularity can be easily relaxed to Lipschitz differentiability (Lipschitz continuity, if $m = 1$). iv) In the case of "hard spheres" and C^∞-integrands the Theorem states that the accuracy of the DVM approximation is $O_\varepsilon(h^{1-\varepsilon})$ for any $\varepsilon > 0$.*

Based on the result of Theorem 3.1 we can extend the class of functions for which the discrete collision sums can be proved to converge to the Boltzmann operator. We consider the spaces $C_r(\mathbb{R}^3)$ of bounded continuous functions with polynomial decay at infinity, equipped with the norms

$$\|f\|_r = \sup_{\xi \in \mathbb{R}^3} |f(\xi)\langle\xi\rangle^r|,$$

where $\langle\xi\rangle = (1 + \xi^2)^{1/2}$. We also consider the discrete norms where the supremum is taken over $\xi_i \in V_h$, that we also denote by $\|f\|_r$ abusing the notation. We then have the following:

Theorem 3.2 *Let the collision kernel B^c, as defined by (2.4), be such that $|u|^2 B^c(u, w) \in C(\mathbf{P})$, and for some $\gamma > 0$ the following condition is satisfied:*

$$\sup_{\rho,\vartheta} b(\rho, \cos\vartheta)\, \langle\rho\rangle^{-\gamma} = \lambda < +\infty.$$

Take f and $g \in C_r(\mathbb{R}^3)$. Then for all $\varepsilon > 0$

$$\|Q(f,g)(\xi_i) - Q_{h,i}(f,g)\|_{s-\varepsilon} \to 0 \text{ as } h \to 0,$$

where $s = r - \gamma - 1$.

Remark: *The appearance of the exponent $s = r-\gamma-1$ is due to the fact that we are only able to prove that $Q_{h,i}$ decays as $\langle \xi_i \rangle^{-s}$ at infinity (Lemma 3.5). It is possible that a stronger estimate with $s = r - \gamma$ is in fact true; however the present result is sufficient for the purposes of proving convergence in Section 4.*

Proof of Theorem 3.1: It is convenient to introduce the function

$$F^*(\xi, u, w) = (f(\xi+u)g(\xi+w) - f(\xi)g(\xi+u+w))B^*(u,w). \tag{3.1}$$

By assumptions of the Theorem, F^* is a function in $C^m(\mathbb{R}^3 \times \mathbf{P})$. Also if $\operatorname{supp} f \in B(0,R)$, we have by simple geometrical arguments:

$$\operatorname{supp} F^*(\cdot, u, w) \in B(0, R^*), \quad \text{and} \quad \operatorname{supp} F^*(\xi, \cdot, \cdot) \in B(0, R^*),$$

where $R^* = R\sqrt{2}$. This means that the discrete collision sums include only a finite number of terms. The error of the DVM approximation can be then represented as follows:

$$\begin{aligned}
&\big|Q(f,g)(\xi_i) - Q_{h,i}(f,g)\big| = \Big| \int_{\mathbb{R}^3} |u|^{-\beta} \int_{E_u} F^*(\xi_i, u, w)\, du\, dw \\
&- h^5 \sum_{k \in \mathbb{Z}^3,\, k \neq 0} |u_k|^{-\beta}\, \Delta_k \sum_{l \in L_k} F^*(\xi_i, u_k, w_l) \Big| \\
&\leq \Big| \sum_{|k| \leq R^*/h} \mathcal{S}_k(\xi_i, h \,|\, G) \Big| + \Big| \sum_{0<|k| \leq R^*/h} h^3\, \mathcal{R}_k(\xi_i, h \,|\, F^*) \Big| = \mathcal{S} + \mathcal{R},
\end{aligned} \tag{3.2}$$

where

$$\mathcal{R}_k(\xi, h \,|\, F^*) = |u_k|^{-\beta} \Big(\int_{E_k} F^*(\xi, u_k, w)\, dw - h^2 \Delta_k \sum_{l \in L_k} F^*(\xi, u_k, w_l) \Big),$$

$$\mathcal{S}_k(\xi, h \,|\, G) = \int_{\mathcal{C}_k} G(\xi, u)\, du - h^3\, G(\xi, u_k)$$

Here $\mathcal{C}_k$ is the cube with the side length h centered at u_k, and E_k, L_k, and G are as defined in the previous Section (we can set $G(\xi, 0)$ equal zero to avoid complications in the notations). Now, the term $\mathcal{R}$ represents the total error due to the discretization of the plane integrals, and $\mathcal{S}$ can be viewed as the error of the three-dimensional rectangle formula applied to the function G.

The further proof is based on Lemmas 3.1–3.4 given below. To estimate each of the $\mathcal{R}_k$-terms we use a standard approach, subdividing the plane E_k into cells obtained by translating the fundamental cell into each point of the lattice.

We then obtain an estimate for the $\mathcal{R}$-term by collecting the estimates for each cell and using Lemma 3.2. An analogous estimate for the $\mathcal{S}$-term is obtained in Lemma 3.4. The following lemma is a standard tool for estimating errors of numerical integration. It can be obtained by a straightforward application of the classical result due to Bramble and Hilbert (see [14, Theorem 4.1.3]; cf. also [30, Theorem 3.1]).

Lemma 3.1 *Let $A_{k,h}$ be the linear mapping that transforms the fundamental cell of the lattice hL_k into the unit square in $\mathbb{R}^2$, and let $\|A_{k,h}\|$ denote the norm of this mapping. Then*

$$|\mathcal{R}_k(\xi, h \,|\, F^*)| \leq C|u_k|^{-\beta}\|A_{k,h}\|,$$

where C only depends on F^*, and does not depend on k, h, or ξ.

A direct application of Lemma 3.1 to estimating the $\mathcal{R}$-term would require taking the sum over all k. The problem that we then have to deal with is that $\|A_{k,h}\|$ (which is essentially the linear size of the fundamental cell) does not tend to zero uniformly as $h \to 0$. This can be seen easily by taking $k = (k_1, k_2, 0)$, where k_1 and k_2 are relatively prime, and so that $|k|$ is "large" (on the order of R^*/h). For such k, the linear size of the fundamental cell is on the order of R^*. Therefore, we need an argument that could be used to show that this kind of situation is untypical, and that $\|A_{k,h}\|$ converges to zero in a certain average sense.

Lemma 3.2 *Let $a \in \mathbb{Z}^d$ and consider the linear Diophantine equation $a \cdot x = 0$. Then there exists a basis of solutions $\{e_1, \ldots, e_{d-1}\}$, called reduced basis, such that*

$$|e_1| \ldots |e_{d-1}| \leq c_d \, |a|,$$

where c_d is a constant that depends only on the dimension d.

Proof. Let L_a be a lattice of solutions to the equation $a \cdot x = 0$. Hermite's theorem [19, p.71] states that there is a basis $\{e_1, \ldots, e_{d-1}\}$, such that

$$|e_1| \ldots |e_{d-1}| \leq c_d \, \det(L_a), \tag{3.3}$$

where $\det(L_a)$ is the determinant of the lattice. Since the determinant of the lattice is given by formula (2.11) the Lemma is proved. □

Remark: *As an easy consequence of the Lemma we obtain the inequality*

$$|e_i| \leq k_d |a|, \quad i = 1 \ldots d-1.$$

Using it we can get a "worst case" estimate for the norm of $A_{k,h}$:

$$\|A_{k,h}\| \leq c'_d h|k| = c'_d |u_k|, \tag{3.4}$$

where c'_d is a constant that depends on the dimension.

In the proof of next lemma we show using the bound of Lemma 3.2 that in fact "for most k" one gains a factor of h to a certain power in the above estimate.

Lemma 3.3 *Let F^* be a C^1 function with compact support. Then, for sufficiently small h,*

$$\Big| \sum_{k\in\mathbb{Z}^3,\ k\neq 0} h^3\, \mathcal{R}_k(\xi, h \,|\, F^*)\Big| \le C h^r,$$

where

$$r = \min\Big(\frac{1}{4}, 1/\big(4 + \tfrac{2-\beta}{1+\beta}\big)\Big).$$

Here the constant C depends only on F^*, and does not depend on h or ξ.

Proof. Since the function F^* has compact support, the sum is in fact taken over a finite set $\mathcal{A}_h = \big\{k \in \mathbb{Z}^3 \bigm| |k| \le N\big\}$, where $N = R^*/h$. For each $k \in \mathcal{A}_h$ take $\{e_1^{(k)}, e_2^{(k)}\}$, a reduced basis of the lattice L_k. Let $e_{\max}^{(k)}$ and $e_{\min}^{(k)}$ be the basis vectors with maximal and minimal Euclidean norms, respectively. For each $0 < \alpha < 1$ define the set $\mathcal{B}_h = \big\{k \in \mathcal{A}_h \bigm| |e_{\min}^{(k)}| \ge |k|^\alpha\big\}$. By Lemma 3.2 for each $k \in \mathcal{B}_h$ we have the inequality:

$$|e_{\max}^{(k)}| \le c_3 |k|^{1-\alpha}.$$

This translates into the following bound for the norm of $A_{k,h}$:

$$\|A_{k,h}\| \le 2h|e_{\max}^{(k)}| \le c_3' |u_k|^{1-\alpha} h^\alpha,$$

which allows us to estimate the sum over all $k \in \mathcal{B}_h$:

$$\begin{aligned}\Big|\sum_{k\in\mathcal{B}_h} h^3\, \mathcal{R}_k(\xi, h \,|\, F^*)\Big| &\le C h^\alpha \sum_{k\in\mathcal{B}_h} h^3 |u_k|^{-\beta+1-\alpha} \\ &\le C h^\alpha \int_{|u|\le R} |u|^{-\beta+1-\alpha}\, du \le C h^\alpha.\end{aligned}$$

To estimate the sum over the complement of $\mathcal{B}_h$ we can use the "worst case" bound (3.4) for $\|A_{k,h}\|$ and estimate the number of terms in the sum. We have

$$\mathcal{A}_h \setminus \mathcal{B}_h \subseteq \big\{k \in \mathcal{A}_h \bigm| |e_{\min}^{(k)}| < N^\alpha\big\} \stackrel{\mathrm{def}}{=} \mathcal{D}_h.$$

To estimate the number of points in $\mathcal{D}_h$ we notice that

$$N(\big\{j \in \mathbb{Z}^3 \bigm| |j| \le n\big\}) \le A n^3, \tag{3.5}$$

and for every k,

$$N(E_{u_k} \cap \big\{j \in \mathbb{Z}^3 \bigm| |j| \le n\big\}) \le B n^2, \tag{3.6}$$

where N denotes the number of elements in the set, and the constants A and B do not depend on k. Using (3.5) and (3.6) we conclude that the number of different vectors $e_{\min}^{(k)}$ satisfying the inequality $|e_{\min}^{(k)}| < N^\alpha$ is estimated as $AN^{3\alpha}$, and the number of vectors $k \in \mathcal{A}_h$ on each plane $\big\{k \in \mathbb{Z}^3 \bigm| e_{\min}^{(k)} \cdot k = 0\big\}$ is estimated as BN^2. This yields

$$N(\mathcal{D}_h) \le C N^{2+3\alpha}.$$

If $\beta \leq 1$ we obtain

$$\Big|\sum_{k\in\mathcal{D}_h} h^3\,\mathcal{R}_k(\xi,h\,|\,F^*)\Big| \leq C\sum_{k\in\mathcal{D}_h} h^3\,|u_k|^{-\beta+1}$$
$$\leq Ch^3(R^*)^{-\beta+1}\left(\frac{R^*}{h}\right)^{2+3\alpha} = Ch^{1-3\alpha}.$$

By letting $\alpha = \frac{1}{4}$ we obtain the estimate $C(F)h^{1/4}$ for both parts of the sum over $\mathcal{A}_h$. If $\beta > 1$, then $|u_k|^{-\beta+1}$ is unbounded on $\mathcal{D}_h$, and the contributions of "large" and "small" k should be considered separately. For this purpose let us take a constant $0 < \sigma < 1$, and split the above sum in the following way:

$$\sum_{k\in\mathcal{D}_h} h^3|u_k|^{-\beta+1} \leq \sum_{k\leq N^{1-\sigma}} h^3|u_k|^{-\beta+1} + \sum_{\substack{N^{1-\sigma}\leq k\leq N\\ k\in\mathcal{D}_h}} h^3|u_k|^{-\beta+1}$$
$$\leq C\int_{|u|\leq(R^*)^{1-\sigma}h^\sigma} |u|^{-\beta+1}\,du + h^3\left(\frac{R^*}{h}\right)^{2+3\alpha}\max_{N^{1-\sigma}\leq k\leq N}|u_k|^{-\beta+1}$$
$$\leq C(R^*)\big(h^{\sigma(4-\beta)} + h^{1-3\alpha-\sigma(1-\beta)}\big).$$

Thus, if $\sigma = \alpha/(4-\beta)$, and $\alpha = 1/\big(4+\frac{\beta-1}{4-\beta}\big)$, we find

$$\Big|\sum_{k\in\mathcal{D}_h} h^3\,\mathcal{R}_k(\xi,h\,|\,F^*)\Big| \leq Ch^\alpha.$$

By combining the estimates for $\mathcal{B}_h$ and $\mathcal{D}_h$, we complete the proof of the lemma. □

Remark. *For functions F^* from the class C^m with $m \geq 2$ the estimate of Lemma 3.3 can be improved. Arguing along the lines of [30] we obtain that the bound of Lemma 3.1 can be changed to*

$$|\mathcal{R}_k(\xi,h\,|\,F^*)| \leq C|u_k|^{-\beta}\|A_{k,h}\|^m.$$

By applying this inequality in the proof of Lemma 3.3, we find that for $m = 2$ the exponent r can be improved to $r = \min\Big(\frac{2}{5}, 2/\big(5+\frac{\beta-2}{5-\beta}\big)\Big)$, and for $m \geq 3$ we obtain $r = \frac{m}{m+3}$.

Now to complete the proof of Theorem 3.1 we need an estimate for the $\mathcal{S}$-term in (3.2). Such an estimate is provided by next lemma.

Lemma 3.4 *Under the assumptions of Theorem 3.1 we have*

$$\Big|\sum_{k\in\mathbb{Z}^3} \mathcal{S}_k(\xi,h\,|\,G)\Big| \leq Ch^r,$$

where $r = \min(1, 3-\beta)$. Here the constant C depends on the function F^*, but not on ξ or h.

Proof: The function $G(\xi, u)$ is supported in $B(0, R^*) \times B(0, R^*)$, and its Lipschitz constant over each cube $\mathcal{C}_k$ can be estimated as

$$C \max_{\mathcal{C}_k} |u|^{-1-\beta}$$

by using the following inequality:

$$|G(\xi, u) - G(\xi, u')| \leq C|u|^{-1}(|u|^{-\beta} + |u'|^{-\beta})|u - u'|, \quad u, u' \neq 0,$$

that can be proved by a direct calculation. We then obtain by arguing along the lines of Raviart [30]:

$$\begin{aligned} \Big| \sum_{k \in \mathbb{Z}^3} \mathcal{S}_k(\xi, h \,|\, G) \Big| &\leq \sum_{|k| \leq R^*/h} \Big| \int_{\mathcal{C}_k} G(\xi, u)\, du - h^3\, G(\xi, u_k) \Big| \\ &\leq \int_{\mathcal{C}_0} G(\xi, u)\, du + Ch \sum_{0 < |k| \leq R^*/h} h^3 \max_{\mathcal{C}_k} |u|^{-1-\beta}. \end{aligned} \tag{3.7}$$

The integral over $\mathcal{C}_0$ is bounded as

$$C \int_{\mathcal{C}_0} |u|^{-3+\beta}\, du \leq \ C \int_{\mathcal{C}_0} |u|^{-\beta}\, du = C_1 h^{3-\beta}.$$

Standard arguments show that the last sum in (3.7) is bounded by $C_2 h^r$, so the estimate of the lemma follows. $\square$

The proof of Theorem 3.1 is now complete. To prove Theorem 3.2 we need the following stability result in the spaces C_r.

Lemma 3.5 *Under the assumptions of Theorem 3.2, for all f and $g \in C_r(\mathbb{R}^3)$ with $r > 3 + \gamma$, we have*

$$\|Q_h(f, g)\|_{r-1-\gamma} \leq C\, \lambda\, \|f\|_r \|g\|_r$$

where $\lambda = \sup_{\rho, x} b(\rho, x)\, \langle \rho \rangle^{-\gamma}$ *and the constant C depends on r, but not on f, g or h.*

Proof. The kernel B^c can be estimated as follows:

$$B^c(u, w) \leq 2\lambda |u|^{-2} \big\langle \sqrt{u^2 + w^2} \big\rangle^{\gamma} \leq 2\lambda |u|^{-2} \big(\langle u \rangle^{\gamma} + \langle \xi \rangle^{\gamma} + \langle \xi + w \rangle^{\gamma} \big).$$

Due to the homogeneity of Q_h, it suffices to estimate $\frac{1}{\lambda} Q_h(\langle \cdot \rangle^{-r}, \langle \cdot \rangle^{-r})$. The "gain" and "loss" terms Q_h^+ and Q_h^- can be dealt with separately. We give the proofs only for the Q_h^+ term; the estimates for Q_h^- can be obtained in the same way. It is sufficient to consider only the terms corresponding to the first term in the above estimate of B^c, the other two being similar. Thus, the problem comes down to estimating expressions like

$$h^3 \sum_{\xi' \in V_h} \langle \xi' \rangle^{-r} |\xi' - \xi|^{-2} \langle \xi' - \xi \rangle^{\gamma}\, h^2 \Delta_k \sum_{\eta' \in \xi + hL_k} \langle \eta' \rangle^{-r}, \tag{3.8}$$

where $k = (\xi' - \xi)/h$. For any monotonically decreasing function $\varphi(x)$ defined on $[0,\infty)$, we have:

$$h\sum_{n=0}^{\infty}\varphi(hn) \le h\varphi(0) + \int_0^{\infty}\varphi(x)\,dx.$$

Accordingly, the inner sum over the lattice points in (3.8) can be estimated as follows:

$$\begin{aligned} h^2\Delta_k \sum_{\eta'\in\xi+hL_k}\langle\eta'\rangle^{-r} &\le 2\,h^2\Delta_k \max_{\xi+E_{\xi'-\xi}}\langle\eta'\rangle^{-r} \\ +\,2\,\max(|e_1|,|e_2|) &\sup_{l\subseteq\xi+E_{\xi'-\xi}}\int_l\langle\eta'\rangle^{-r}\,d\eta' + \int_{\xi+E_{\xi'-\xi}}\langle\eta'\rangle^{-r}\,d\eta' \\ \le\, 2h|\xi'-\xi|\langle\eta'_{\max}\rangle^{-r} &+ c_1|\xi'-\xi|\langle\eta'_{\max}\rangle^{-r+1} + c_2\langle\eta'_{\max}\rangle^{-r+2}, \end{aligned} \tag{3.9}$$

where the supremum is taken over all lines l on the plane $\xi + E_{\xi'-\xi}$ passing through the point ξ, and $\eta'_{\max}$ is the point where $\langle\eta'\rangle^{-r}$ attains its maximum on the plane $\xi + E_{\xi'-\xi}$. From the collision geometry, $|\eta'_{\max}| = |\xi|\cos\theta$, where θ is the angle between ξ and $\xi' - \xi$. The calculations for the three terms in the estimate (3.9) are performed in a similar way. Therefore, we confine ourself to considering the second term in (3.9). We set $\mathcal{E}(\xi,\xi') = \langle\xi'\rangle^{-r}|\xi'-\xi|^{-1}\langle\xi'-\xi\rangle^{\gamma}\langle|\xi|\cos\theta\rangle^{-r+1}$, and use the estimate [30]

$$\left|h^3\sum_{\xi'\in V_h}\mathcal{E}(\xi,\xi') - \int_{\mathbb{R}^3}\mathcal{E}(\xi,\xi')\,d\xi'\right| \le Ch\int_{\mathbb{R}^3}|\nabla_{\xi'}\mathcal{E}(\xi,\xi')|\,d\xi.$$

Thus, to estimate the sum over $\xi' \in V_h$ one needs only the bounds for $\int\mathcal{E}(\xi,\xi')\,d\xi'$ and $\int|\nabla_{\xi'}\mathcal{E}(\xi,\xi')|\,d\xi'$. We split the integration over ξ' into two parts:

$$\int_{\mathbb{R}^3}\mathcal{E}(\xi,\xi')\,d\xi' = \int_{\{|\cos\theta|>\frac{1}{2}\}} + \int_{\{|\cos\theta|\le\frac{1}{2}\}}.$$

For the first integral on the right hand side, we have

$$\begin{aligned} &\int_{\{|\cos\theta|>\frac{1}{2}\}}\langle\xi'\rangle^{-r}|\xi'-\xi|^{-1}\langle\xi'-\xi\rangle^{\gamma}\langle|\xi|\cos\theta\rangle^{-r+1}\,d\xi' \\ &\le 2^{(r-1)/2}\langle\xi\rangle^{-r+1}\int_{\mathbb{R}^3}\langle\xi'\rangle^{-r}|\xi'-\xi|^{-1}\langle\xi'-\xi\rangle^{\gamma}\,d\xi' \le C_r\langle\xi\rangle^{-r+\gamma}. \end{aligned}$$

For the second one, using the spherical coordinates $\xi' - \xi = \rho\omega$, we get

$$\begin{aligned} &\int_{\{|\cos\theta|\le\frac{1}{2}\}}\langle\xi'\rangle^{-r}|\xi'-\xi|^{-1}\langle\xi'-\xi\rangle^{\gamma}\langle|\xi|\cos\theta\rangle^{-r+1}\,d\xi' \\ &\le 4\pi 2^{(r-1)/2}\int_0^{\infty}\rho\langle\rho\rangle^{\gamma}(|\xi|+\rho)^{-r}d\rho\int_0^{\frac{1}{2}}\langle|\xi|x\rangle^{-r+1}dx \\ &\qquad\le \frac{C_r}{|\xi|}\int_0^{\infty}\rho\langle\rho\rangle^{\gamma}(|\xi|+\rho)^{-r}d\rho \le C_r\langle\xi\rangle^{-r+\gamma+1}. \end{aligned}$$

We get the estimate for the integral of the gradient norm with the same exponent by using the same arguments. The estimate of the lemma then follows. □

Proof of Theorem 3.2: Take two functions f and $g \in C_r(\mathbb{R}^3)$. There are sequences of smooth functions with compact support f_n and g_n such that $\|f_n - f\|_{r-\varepsilon}$ and $\|g_n - g\|_{r-\varepsilon} \to 0$. Also, take a sequence B_n^c such that $|u|^2 B_n^c(u,w)$ is smooth and $\left\| |u|^2 \left\langle \sqrt{|u|^2 + |w|^2} \right\rangle^{-\gamma} (B_n^c - B^c) \right\|_{L^\infty} \to 0$. Then

$$\|Q_h(f,g) - Q(f,g)\|_{s-\varepsilon} \leq \|Q_h(f,g) - Q_h(f_n,g_n)\|_{s-\varepsilon}$$
$$+ \|Q_h(f_n,g_n) - Q(f_n,g_n)\|_{s-\varepsilon} + \|Q(f_n,g_n) - Q(f,g)\|_{s-\varepsilon}.$$

The second term on the left-hand side converges to 0 for every fixed n, and the other two converge to 0 as $n \to \infty$ uniformly in h, which proves the statement of the Theorem.

4 Convergence to the solutions of the Boltzmann equation

To study the convergence of solutions to discrete-velocty models we first need to make more precise the way in which the distribution function f is approximated by solutions f_i of the DVMs. In doing this we follow the aprroach of S. Mischler [23] and use the following finite-volume approximation. The velocity space $\mathbb{R}^3$ is split into cubes $\mathcal{C}_i$ centered at the points of V_h. The distribution function is approximated by the function f_h, piecewise constant in ξ:

$$f_h(\xi, x, t) = \sum_i f_i(x,t) \chi_{\mathcal{C}_i}(\xi).$$

The functions $f_i(x,t)$ are determined from the system of equations (1.4). It is convenient to rewrite this system as a single Boltzmann-like equation for the function f_h:

$$\begin{aligned} &\frac{\partial f_h}{\partial t} + v_h(\xi) \cdot \nabla_x f_h = Q_h(f_h, f_h) \\ &= \iiint \big(f_h(\xi')f_h(\eta') - f_h(\xi)f_h(\eta)\big) B_h(\xi,\eta,\xi',\eta')\, d\eta\, d\xi'\, d\eta', \end{aligned} \tag{4.1}$$

where $v_h(\xi) = \sum\limits_i \xi_i \chi_{\mathcal{C}_i}(\xi)$, and the kernel B_h is given by

$$B_h(\xi,\eta,\xi',\eta') = h^{-9} \sum_{ijkl} A_{ij}^{kl} \chi_{\mathcal{C}_i \times \mathcal{C}_j \times \mathcal{C}_k \times \mathcal{C}_l}(\xi,\eta,\xi',\eta'). \tag{4.2}$$

We can truncate "large" velocities in the model by defining for each $R > 0$ a new collision kernel B_n^R as follows:

$$B_n^R(\xi,\eta,\xi',\eta') = \chi_R B_{\frac{1}{n}}(\xi,\eta,\xi',\eta') \tag{4.3}$$

where χ_R is the indicator function of the set $\{|\xi| < R,\ |\eta| < R,\ |\xi'| < R,\ |\eta'| < R\}$. This yields a model with finite number of velocities. The same result is obtained by truncating the discrete-velocity space V_h in equations (1.4).

Now the question can be studied whether solutions f_h converge to solutions of the Boltzmann equation as $h \to 0$ and $R \to \infty$. We study this question in the setup of the Cauchy problem for the inhomogeneous Boltzmann equation, when the existence of renormalized solutions is guaranteed by the result of DiPerna and Lions [15]. There are two major problems that we have to tackle. The first one is common for all DVMs: there is no proof of global existence for an equation of type (4.1). One of the reasons for this is that the averaging techniques used in [15] do not apply to the discrete transport operators (4.1). We can get round this difficulty in a fashion similar to [23] by considering the following modified DVM:

$$\frac{\partial f_h}{\partial t} + v_h(\xi) \cdot \nabla_x f_h = G_\delta(\rho)\, Q_h(f_h, f_h), \tag{4.4}$$

where

$$G_\delta(\rho) = (1 + \delta\rho)^{-1}, \quad \delta > 0, \quad \text{and} \quad \rho = \int_{\mathbb{R}^3} f_\delta \, d\xi.$$

We can then try to prove that solutions of (4.4) converge to f as $n \to 0$, $R = R_n \to \infty$ and $\delta = \delta_n \to 0$. The second difficulty that arises is specific to our model. Namely, it is known that the Boltzmann collision operator with a kernel B truncated so that $B \le K$ satisfies the following estimate in $L^1(\mathbb{R}^3_\xi)$:

$$\|Q(f, g)\|_{L^1} \le C_K \|f\|_{L^1} \|g\|_{L^1}.$$

Surprisingly, such an estimate for Q_h, uniformly in h, does not appear to be available. This can be illustrated by the following computation that we show for the "gain" part of the collision operator for "hard spheres", to simplify the formulas. Taking f and g nonnegative we have:

$$\begin{aligned} \|Q_h^+(f,g)\|_{L^1_\xi} &= \int d\xi' \, f(\xi') \int d\eta' \, g(\eta') \int d\xi \, B_h(\xi, \eta, \xi', \eta') \\ &= \int d\xi' \, f(\xi') \int d\eta' \, g(\eta') \sum_{kl} \chi_{\mathfrak{C}_k}(\xi') \chi_{\mathfrak{C}_l}(\eta') \sum_{i \in S_{kl}} h \frac{1}{\mathfrak{g}(k-i)} \end{aligned} \tag{4.5}$$

Here S_{kl} denotes the set of integer points falling on the sphere that has the vector $k-l$ as the diameter. Estimating the factor $1/\mathfrak{g}(k-i)$ by one ($\mathfrak{g}(k-i) = 1$ for all vectors with relatively prime components) leads to the expression $h|S_{kl}|$, where $|S_{kl}|$ is the number of points on the integer sphere, that is known to be unbounded [7]. Therefore, to be able to control the L^1-norm of Q_h we have to introduce the following truncation:

$$B_{R,n,\alpha}(\xi, \eta, \xi', \eta') = \chi_\alpha B_{R,n}(\xi, \eta, \xi', \eta') \tag{4.6}$$

where χ_α is the characteristic function of the set $\{|S_{kl}| < \alpha n\}$. Notice that such a truncation preserves the symmetries of the equations, so (1.6)–(1.9) still hold.

Now taking an initial datum $f^{(0)} \geq 0$ such that

$$\int f^{(0)}(\xi, x)\big(1 + |\xi|^2 + |x|^2 + \log f^{(0)}(\xi, x)\big)\, dx\, d\xi \leq C,$$

we can consider the Cauchy problem for the Boltzmann equation (1.1) and also the following approximated problem:

$$\frac{\partial f_\delta}{\partial t} + \xi \cdot \nabla_x f_\delta = G_\delta(\rho)\, Q_\delta(f_\delta, f_\delta),$$
$$f_\delta(\xi, x, 0) = f_\delta^{(0)}(\xi, x),$$

where Q_δ is the collision operator (1.2) with the regularized kernel:

$$B_\delta(\xi, \eta, \omega) = \min\left\{\frac{1}{\delta}, B(\xi - \eta, \omega)\, \chi_{\{\delta \leq |\xi - \xi'| \leq |\xi - \eta| \leq 1/\delta\}}\right\} * \mu_\delta(\xi, \eta, \omega),$$

and $f_\delta^{(0)}$ is as follows:

$$f_\delta^{(0)} = f^{(0)} \chi_{\{|\xi| < 1/\delta\}} * \mu_\delta(\xi, x) + \delta \exp\big(-|x|^2 - |\xi|^2\big).$$

Here μ_δ denotes a C^∞-mollifier that approximates the delta-function at zero as $\delta \to 0$.

We consider the DVM given by the equation (4.4) with $h = \frac{1}{n}$ and the discrete kernel $B_n = B_{R,n,\delta,\alpha}$ obtained from $B_\delta(\xi, \eta, \omega)$ according to (2.13), (4.2), (4.3), and (4.6). The following approximation of $f^{(0)}$ can be taken as the initial datum in the Cauchy problem for the DVM:

$$f_{\delta n}^{(0)} = \sum_i \chi_{\mathcal{C}_i}(\xi) \int_{\mathcal{C}_i} f_\delta^{(0)}(\eta, x)\, d\eta. \tag{4.7}$$

The convergence of solutions to the DVM is established in the following theorem.

Theorem 4.1 *There are sequences $\delta_n \to 0$, $\alpha_n \to \infty$ and $R_n \to \infty$, such that the sequence $f_n = f_{R_n, n, \delta_n \alpha_n}$ of solutions to the DVM* (4.4) *with the initial data* (4.7)*, converges weakly in L^1 along a subsequence of $n \to \infty$, to f which is a renormalized solution of the Cauchy problem for the Boltzmann equation* (1.1) *with the initial datum $f^{(0)}$.*

Proof: First, we notice that the existence of global in time solutions to the DVM (4.4) is guaranteed by a contraction argument, similar to the one used in [15]. We prove that for every fixed $\delta > 0$, $T > 0$, and for every $\varepsilon > 0$, we can take $\alpha > 0$, $R > 0$ such that

$$\sup_{t \in [0,T]} \|f_{R,n,\delta,\alpha} - f_\delta\|_{L^1(\mathbb{R}^3_x \times \mathbb{R}^3_\xi)} < \varepsilon, \tag{4.8}$$

for an infinite subsequence of n. From this and the result of DiPerna and Lions [15], which says that $f_\delta \to f$ weakly in L^1, the statement of the theorem follows by a diagonalization argument.

To shorten notation we denote by F the solutions $f_{R,n,\delta,\alpha}$ to the DVM. The notation $\widetilde{Q}_\delta$ will be used for the operator $G_\delta(\rho)Q_\delta$, and analogously, $\widetilde{Q}_{R,n,\delta}$ is used for the discrete collision operators with $G_\delta(\rho_n)$ factors. Integrating the equations along characteristics we obtain:

$$\begin{aligned} f_\delta(\xi,x,t) - F(\xi,x,t) &= f_\delta^{(0)}(\xi,x) - f_{\delta n}^{(0)}(\xi,x) \\ &\quad + \int_0^t \big(\widetilde{Q}(f_\delta,f_\delta)^\sharp - \widetilde{Q}_{R,n,\delta}(F,F)^{\sharp n}\big)\,ds \\ = I_0 + \int_0^t I_1\,ds + \int_0^t I_2\,ds &+ \int_0^t I_3\,ds + \int_0^t I_4\,ds + \int_0^t I_5\,ds. \end{aligned} \tag{4.9}$$

Here the symbol $^\sharp$ denotes the argument $(\xi, x-\xi s, s)$, and $^{\sharp n}$ denotes $(\xi, x - v_n(\xi)s, s)$. We have also

$$\begin{aligned} I_0 &= f_\delta^{(0)}(\xi,x) - f_{\delta n}^{(0)}(\xi,x), \\ I_1 &= \widetilde{Q}_\delta(f_\delta,f_\delta)^\sharp - \widetilde{Q}_\delta(f_\delta,f_\delta)^{\sharp n}, \\ I_2 &= \widetilde{Q}_\delta(f_\delta,f_\delta)^{\sharp n} - \widetilde{Q}_{n,\delta}(f_\delta,f_\delta)^{\sharp n}, \\ I_3 &= \widetilde{Q}_{n,\delta}(f_\delta,f_\delta)^{\sharp n} - \widetilde{Q}_{n,\delta,\alpha}(f_\delta,f_\delta)^{\sharp n}, \\ I_4 &= \widetilde{Q}_{n,\delta,\alpha}(f_\delta,f_\delta)^{\sharp n} - \widetilde{Q}_{R,n,\delta,\alpha}(f_\delta,f_\delta)^{\sharp n}, \\ I_5 &= \widetilde{Q}_{R,n,\delta,\alpha}(f_\delta,f_\delta)^{\sharp n} - \widetilde{Q}_{R,n,\delta,\alpha}(F,F)^{\sharp n}. \end{aligned}$$

Here $\widetilde{Q}_{n,\delta,\alpha}$ denotes the collision operator of the infinite DVM, without the R truncation, and $\widetilde{Q}_{n,\delta}$ is the analogous operator without the α truncation.

Now, I_0 converges to zero in the L^1 norm, as $n \to \infty$. We next claim that the term I_5 can be estimated in the $L^1_{x\xi}$ norm as $\frac{6\alpha}{\delta^3}\|f_\delta - F\|_{L^1_{x\xi}}$. Indeed, since α is kept fixed, it is easy to see by repeating the calculation (4.5) for both "gain" and "loss" parts of the collision operator:

$$\|Q_{R,n,\delta,\alpha}(f,g)\|_{L^1_\xi} \le \frac{2\alpha}{\delta^2}\|f\|_{L^1_\xi}\|g\|_{L^1_\xi}.$$

This implies the following Lipschitz condition, uniform in n (cf. [15]):

$$\|\widetilde{Q}_{R,n,\delta,\alpha}(f,f) - \widetilde{Q}_{R,n,\delta,\alpha}(g,g)\|_{L^1_{x\xi}} \le \frac{6\alpha}{\delta^3}\|f-g\|_{L^1_{x\xi}},$$

and thereby proves our claim.

The remaining four terms contain only the limit solution f_δ which can be shown to be a function from the Schwartz class, smooth and rapidly decaying [15]. Thus, I_1 converges to zero uniformly as $n \to \infty$, by continuity. To estimate the effect of the R truncation, expressed by the I_4 term, we can notice that

$$\begin{aligned} &\widetilde{Q}_{n,\delta,\alpha}(f_\delta,f_\delta) - \widetilde{Q}_{R,n,\delta,\alpha}(f_\delta,f_\delta) \\ &= G_\delta(\rho_\delta)\big(Q_{n,\delta,\alpha}(f_\delta\chi_{\{|\xi|>R\}}, f_\delta) + Q_{n,\delta,\alpha}(f_\delta, f_\delta\chi_{\{|\xi|>R\}})\big). \end{aligned}$$

Thus, using Lemma 3.5, we find that for each δ and T fixed, I_4 converges to zero as $R \to \infty$ in $L^\infty(\mathbb{R}^3_x \times [0,T], C_r(\mathbb{R}^3_\xi))$ for every $r > 3$, uniformly in n. Also, by Theorem 3.2, $I_2 \to 0$ as $n \to 0$, in the same function space as I_4.

The last remaining term I_3 is a little bit more tricky. It suffices for us to show that $I_3 \to 0$ as $\alpha \to \infty$, uniformly in n. We introduce the function

$$\Phi_n(\xi,\eta) = \sum_{ij} \chi_{C_i}(\xi)\chi_{C_j}(\eta)\,\frac{1}{n}\,|S_{ij}|,$$

where $|S_{ij}|$ is the number of integer points on the sphere that appears in the α truncation. This function is not bounded in L^∞ uniformly in n; however, we can show the following:

$$\int_{|\xi-\eta|\le K} (\Phi_n(\xi,\eta))^2\, d\eta = \frac{1}{n^3} \sum_{|i-j|\le Kn} \left(\frac{1}{n}|S_{ij}|\right)^2$$

$$= \frac{1}{n^3} \sum_{m=1}^{(Kn)^2} \sum_{|i-j|=\sqrt{m}} \left(\frac{1}{n}|S_{ij}|\right)^2 = \frac{1}{n^5} \sum_{m=1}^{(Kn)^2} (r(m))^3 \le C_K.$$

Here $r(m)$ denotes the number of integer points on the sphere of radius $\sqrt{m}$, and the constant C_K does not depend on n. To get the third equality we used that the number of points on the sphere $\{|i-j| = \sqrt{m}\}$ equals $S_{ij} = r(m)$. The last inequality follows from the Gauss formula for $r(m)$ [18, Theorem 2, Chapter 4] and the estimates in Barban [3, Theorem 2]. (These estimates imply in fact that $\Phi_n \in L^p(\{|\xi-\eta| \le K\})$ for every $p < \infty$, with bounds independent on n). Now the required estimate for the term I_3 can be obtained as follows:

$$\|I_3\|_{L^1_{x\xi}} = \|Q_{\delta\alpha n}(f_\delta, f_\delta) - Q_{\delta n}(f_\delta, f_\delta)\|_{L^1_{x\xi}}$$

$$\le 2 \int\!\!\int\!\!\int |f_\delta(\xi')||f_\delta(\eta')|\,\Phi_n(\xi',\eta')\chi_{\{|\Phi_n|>\alpha\}} B_{n,\delta}\, d\xi' d\eta' dx$$

$$\le \frac{2}{\delta^2}\|f_\delta\|_{L^1_{x\xi}}\,\|f_\delta\|_{L^\infty_{x\xi}}\,\sup_{\xi'} \int_{|\xi'-\eta'|\le\frac{1}{\delta}} \Phi_n(\xi',\eta')\,\chi_{\{|\Phi_n|>\alpha\}}\, d\eta'$$

$$\le \frac{2}{\delta^2}\|f_\delta\|_{L^1_{x\xi}}\,\|f_\delta\|_{L^\infty_{x\xi}}\,\frac{1}{\alpha}\,\sup_{\xi'} \int_{|\xi'-\eta'|\le\frac{1}{\delta}} (\Phi_n(\xi',\eta'))^2\, d\eta' \le \frac{C_\delta}{\alpha}.$$

It follows now, by collecting the estimates for all terms in (4.9), that

$$\|f_\delta - F\|_{L^1_{x\xi}}(t) \le \frac{\alpha}{\delta^3}\int_0^t \|f_\delta - F\|_{L^1_{x\xi}} ds + \varepsilon_{0,1,2}(n) + \varepsilon_3(\alpha,n) + \varepsilon_4(R,n),$$

where $\varepsilon_{0,1,2}(n) \to 0$ as $n \to \infty$, $\varepsilon_3(\alpha,n) \to 0$ as $\alpha \to \infty$ and $\varepsilon_4(R,n) \to 0$ as $R \to \infty$, uniformly in n . Thus (4.8) follows by Gronwall's lemma. This completes the proof of the theorem. □

5 Numerical results. Space-homogeneous relaxation

The accuracy of the collision integral approximation, as well as its computational cost, are illustrated by a comparison with the exact solution of the space-homogeneous Boltzmann equation for pseudo-Maxwell molecules, which was obtained by A. Bobylev [5] and, independently, by M. Krook and B. Wu [21]. This solution has the form

$$f(\xi,t) = (2\pi\tau)^{-\frac{3}{2}}\Big(1+\frac{1-\tau}{\tau}\big(\frac{\xi^2}{2\tau}-\frac{3}{2}\big)\Big)\exp\Big(-\frac{\xi^2}{2\tau}\Big), \tag{5.1}$$

$$\tau(t) = 1-\theta\exp(-\lambda t)$$

with $\theta \in [0, \frac{2}{5}]$, and $\lambda = \frac{\pi}{2}\int_{-1}^{1} g(z)(1-z^2)\,dz$. Here $g(z)$ is a function such that $B(v,\cos\vartheta) = \cos\vartheta\, g(\cos\vartheta)$. We use $g(z)\equiv\frac{1}{2}$; then $\lambda = \frac{\pi}{3}$ and $B^c(u,w) = |u|^{-1}(u^2+w^2)^{-\frac{1}{2}}$. We also take $\theta = \frac{2}{5}$ so that $f(0,0) = 0$. For the numerical scheme we take the points of the regular grid V_h in a cube of suitable size (determined by the parameters of the distribution function), and include in the model only those collisions for which the postcollisional velocities remain in the fixed cube. For the parameters of the solution (5.1) chosen as above, we take the cube with the side length 7.0 centered at the origin as the domain of computation. The size of the cube is kept fixed in the time-dependent problem. We denote by n the number of velocities in each direction; the total number of velocities is therefore $N = n^3$. To increase the numerical efficiency, we follow J. Schneider [22] in using the symmetry properties of the DVM coefficients in the numerical algorithm. All computations were performed on *Sun Dual UltraSparc 1700 / 167 MHz*.

Table 1: Relative errors of the collision term approximation and CPU times, in dependence on the number n of velocities in each direction.

n	8	10	12	14	16	18	20	22
error(Q,Q_h)	0.044	0.035	0.033	0.028	0.024	0.021	0.019	0.017
CPU time, sec.	0.38	1.96	7.25	23.01	66.43	135.47	286.94	638.13

In the first test we compare the exact value of the collision operator computed on $f(\xi,0)$ and the values obtained by using discrete-velocity models. We measure the difference between the exact and approximate data by calculating the relative maximum error

$$\text{error}(Q,Q_h) = \frac{2\max\limits_i |Q(\xi_i)-Q_i|}{\max\limits_i |Q(\xi_i)| + \max\limits_i |Q_i|}.$$

The results of the computations are shown in Table 1. Figure **??** illustrates the relation between the CPU time on the number of velocities N. The dependence is close to a power law with the exponent ≈ 2.43. Figure **??** shows the dependence of the relative error on $h = \frac{1}{n}$. The approximate value of the exponent obtained by the interpolation using $n = 12,\ 14,\ 16,\ 18,\ 20,\ 22$ is found to be ≈ 1.09.

The second problem we address is the time relaxation in a space homogeneous gas. In the numerical scheme, the discrete-velocity approximation of the collision term is combined with the forward Euler time-stepping method. The time step is chosen so as to make the time integration error negligible in comparison with the error of the discrete-velocity approximation. We compare numerical solutions withthe exact one (5.1); this is done by computing the relative mean error:

$$\mathrm{error}(f, f_h) = \frac{2\sum_i |f(\xi_i) - f_i|}{\sum_i |f(\xi_i)| + \sum_i |f_i|}.$$

as a function of time. We also compute the fourth-order moment

$$m_h^{(2,2,0)}(t) = h^3 \sum_i f_i(t)\xi_{ix}^2\xi_{iy}^2,$$

which is compared with the exact value of the moment calculated using (5.1). The results for the calculation of the moment are shown on Figure **??**. The relative error in calculating the moment is below $2.5 \cdot 10^{-3}$ for $N = 10$ and decreases with increasing N. Figure **??** presents the mean error in the distribution function in dependence on time.

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Figure 1

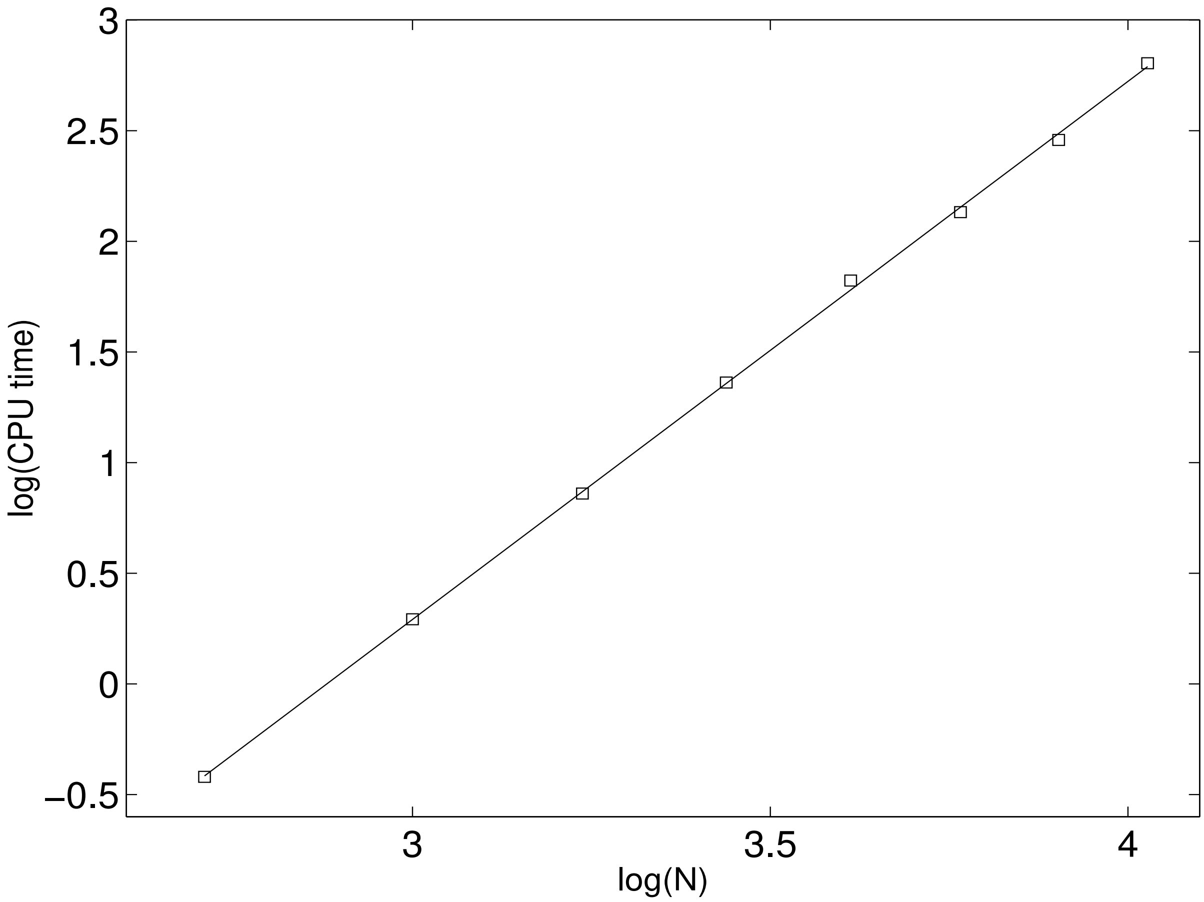

Figure 1: Squares: CPU time for calculation of the discrete collision operator vs. the number of discrete points. Solid line: interpolant with the slope 2.4319

Figure 2

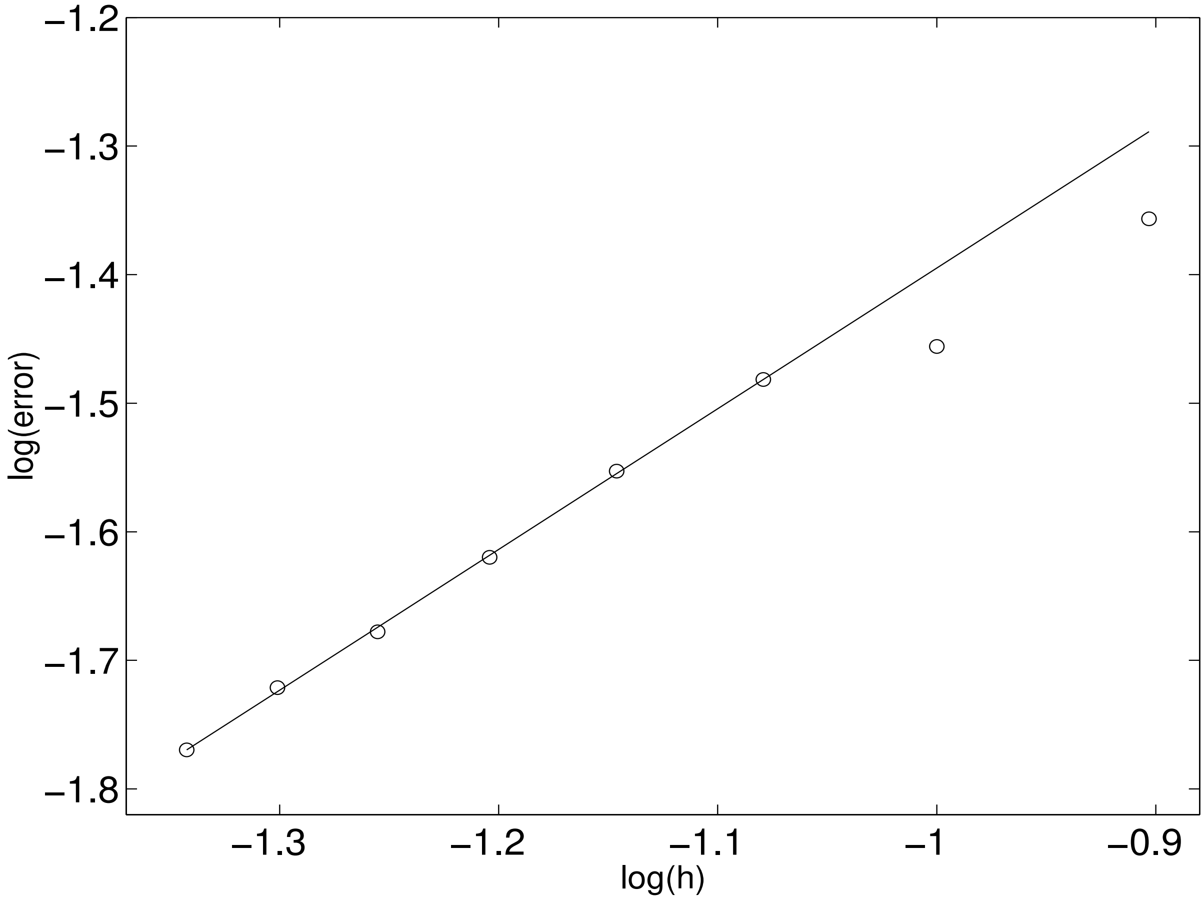

Figure 2: Circles: relative maximum error of Q_h vs. $h = \frac{1}{n}$. Solid line: interpolant with the slope 1.0943.

Figure 3

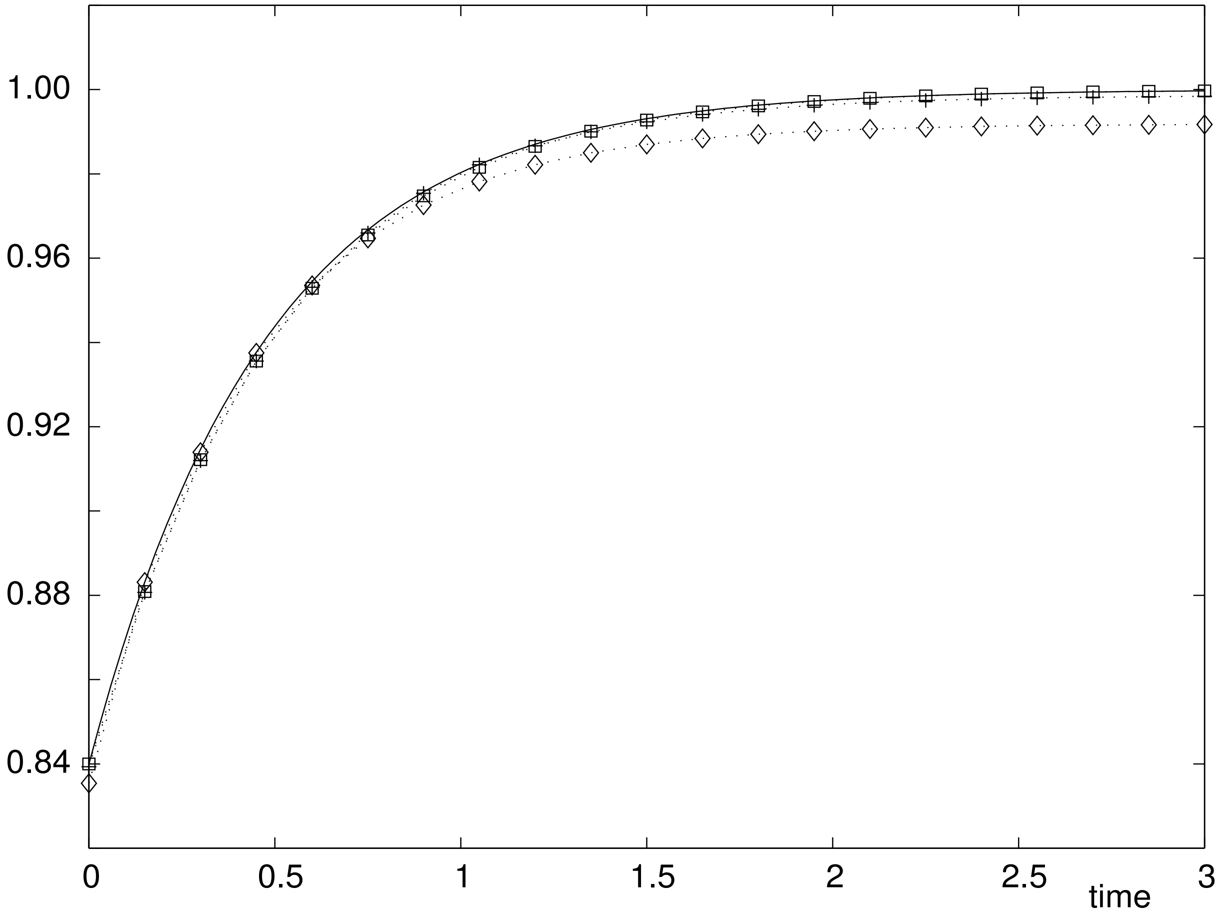

Figure 3: Fourth-order moment for the Bobylev-Krook-Wu solution, computational vs. exact. Solid line: exact; diamonds: $N = 6$; crosses: $N = 8$; squares: $N = 10$.

Figure 4

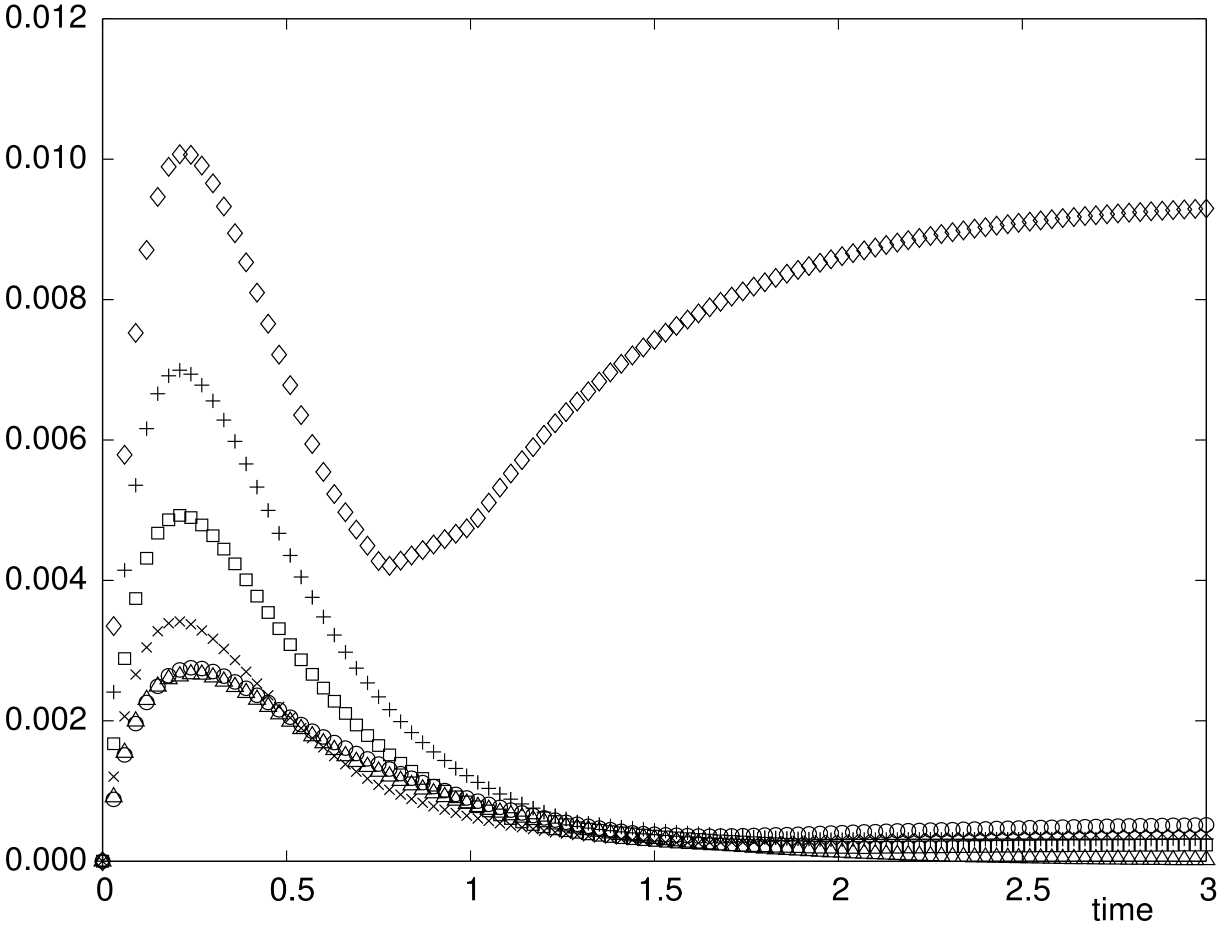

Figure 4: The relative mean error in the distribution function. Diamonds: $N = 6$; crosses: $N = 8$; squares: $N = 10$; $\times$-marks: $N = 12$; circles: $N = 14$; triangles: $N = 16$.