Composite Membranes

Aldo Lopes
UFMG, aldoelopes@hotmail.com

Russell E. Howes, Brigham Young University
rhowes@byu.edu

Cynthia Shepherd, California State University - Northridge
Cynthia.Shepherd.87@csun.edu

8th August 2006

Contents

1 Introduction **3**

2 The Dirichlet Problem **4**

2.1 Spectral Theorem 6

3 Variations **7**

4 Variations With Fixed Boundary **10**

5 Critical Eigenvalues **12**

1 Introduction

The study of vibrating membranes is an aspect of mathematical physics that is primarily concerned with the geometries of the membrane corresponding to its frequencies. Membranes consisting of two materials of different densities are called composite membranes.

We are interested in determining the shape of membranes with minimal eigenvalues; in particular the first eigenvalue, or fundamental frequency, of such membranes.

In this paper[1] we will be studying a membrane Ω divided into two subsets, a lower density D and a higher density D^c. The boundary between D and D^c is a hypersurface and will be called Γ. We will first look at the associated Dirichlet boundary PDE problem.

We will review the Spectral Theorem, which is an important result related to our work and then consider variations that fix the boundary of Ω and preserves the volume of D and show that such variations do exist under our given specifications.

The main purpose of this paper is to determine conditions on D and its corresponding first eigenfunction, u, for the critical points of the smallest eigenvalue, λ. We will also prove that for any critical point of λ, Γ is a level set of u.

[1]Partially supported by NSF grant OISE: 0526008.

2 The Dirichlet Problem

The composite membrane eigenvalue problem for a bounded domain $\Omega \subset \mathbb{R}^n$ is the Dirichlet boundary PDE problem:

$$(*)_{\Omega,\alpha,D} \begin{cases} -\Delta u + \alpha\chi_D u &= \lambda u \quad \text{in } \Omega \\ u &= 0 \text{ in } \quad \partial\Omega \end{cases}$$

u is a normalised eigenfunction of the above problem, D is the region of Ω previously defined, and we are interested in minimising λ_0, the lowest eigenvalue or lowest possible value of λ for a given D. We will only worry about the eigenfunction u associated with λ_0, and we want to find what configurations of u and D will make $\lambda_0' = 0$.

χ_D is the characteristic function:

$$\chi_D(x) = \begin{cases} 0 & \text{if } x \in D^c \, (= \Omega - D) \\ 1 & \text{if } x \in D \end{cases}$$

Let $\Omega \subset \mathbb{R}^n$ be an open, connected and bounded set. We define N as the boundary of Ω, i.e., $\partial\Omega = N$. Here N is smooth and a C^∞ hypersurface. In fact, $N = f^{-1}(a)$, where a is a regular value of the function $f : U \to \mathbb{R}$ with $N \subset U$.

$F : (-\epsilon, \epsilon) \times \bar{\Omega} \to \mathbb{R}^n$ a C^∞ function is the variation of Ω. If is necessary, we can consider the extension of F: $\bar{F} : (-\epsilon, \epsilon) \times U \to \mathbb{R}^n$ a C^∞ function and $\bar{\Omega} \subset U \subset \mathbb{R}^n$ is an open set.

We define now the follwing function that for each time:

$$F_t = F(t, \cdot) : \bar{\Omega} \to \mathbb{R}^n$$

$$F_t(\bar{\Omega}) = \bar{\Omega}_t$$

Restricting the time if necessary, one always has that $F_t : \bar{\Omega} \to \mathbb{R}^n$ is a diffeomorphism onto its image:

$$F_t : \bar{\Omega} \to \bar{\Omega}_t$$

For each time we consider $N = \partial\Omega$ and $N_t = \partial\bar{\Omega}_t$ also a C^∞ hypersurface. And $\Gamma = \partial D \cap \Omega$.

$F_t : N \to N_t$ is also a diffeomorphism.

Let's start by defining some functions. We previously defined $F : (-\epsilon, \epsilon) \times \bar{\Omega} \longrightarrow \bar{\Omega}$.

Definition 1. *For some fixed $t \in (-\epsilon, \epsilon)$, we define $F_t : \bar{\Omega} \longrightarrow \bar{\Omega}$ as $F_t = F(t, x)$.*

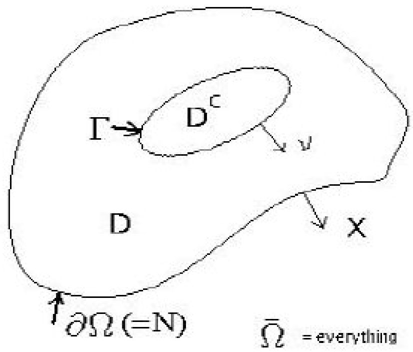

Figure 1: The membrane

Let's place some constraints on our F_t: For any $t_1, t_2, (t_1 + t_2) \in (-\epsilon, \epsilon)$:

$$F_{t_1} \circ F_{t_2} = F_{t_1+t_2}$$

This structure makes things nice for us–it makes the F_ts associative, and gives us for each F_t an inverse, so that $F_t \circ F_{-t} = F_0 = F$. We will refer to F_0 as F from here on out. Defining F_t leads to a natural definition of D_t and thus χ_{D_t}, and prompts us to redefine the first eigenfunction:

Definition 2. *For some fixed $t \in (-\epsilon, \epsilon)$, we define $u_t : \bar{\Omega} \longrightarrow \mathbb{R}$ as $u_t = u(t, x)$. We will newly define $u = u_0$.*

A few observations: We want to rewrite χ_{D_t} as a deformation of χ_D. We notice that if $x = F_t(y)$, $y = F_{-t}(x) = F_t^{-1}(x)$ and

$$\chi_D(y) = \chi_{D_t} \circ F_t(y)$$

$$\chi_D \circ (F_t)^{-1}(x) = \chi_{D_t}(x) = \chi_D \circ F_{-t}$$

2.1 Spectral Theorem

Usually one also has a boundary condition coming from the original problem. We will consider the case where the membrane is fixed along $\partial\Omega$,i.e., $F_t(x) = 0$ for all $x \in \partial\Omega$ and all $t \in \mathbb{R}$. This gives rise to the same condition for u, the Dirichlet boundary condition. Thus, one is bound to study the possible solutions of the PDE problem

$$\begin{cases} -\Delta u &= \lambda u \quad \text{in}\,\Omega \\ u &= 0 \quad \text{in}\,\partial\Omega \end{cases}$$

Analogous to the linear algbra concepts, a nontrivial u satisfyng such conditions will be called an eigenfunction of the Laplacian and the associated real number λ an eigenvalue.

Recall that given two functions ϕ and ψ in Ω wich go to zero at its boundary, their L^2 - inner product is given by

$$\langle \phi, \psi \rangle = \int_\Omega \varphi\psi dx\, dy$$

One then has the classical

Theorem 2.1. ***(Spectral Theorem for the Laplacian with the Dirichlet boundary condition)***. *There is an unbounded discrete set of (positive) eigenvalues*

$$(0 <)\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$$

and associated set of eigenfunctions u_i, which may be taken mutually orthogonal and with norm equal to 1, with respect to the L^2 inner product in Ω, which is complete in the sense that any smooth function $v : \Omega \to \mathbb{R}$ such that $v = 0$ at $\partial\Omega$ may be written as the Fourier series

$$v = \sum_{i=1}^{\infty} \langle v, u_i \rangle u_i$$

Remark. In the case of one space variable, i.e., of a vibrating string, the $u_i's$ are given by sine functions, the original Fourier series situation.

Observe that the (orthonormal) system of eigenfunctions plays the usual role of an orthonormal basis in finite dimensional linear algebra, when we use orthogonal projection to write a vector in that basis (in that case one has a finite sum of projected vectors). It is also called a *Hilbert basis* for the space os functions in Ω (with the Dirichlet boundary condition).

The numbers λ_i give the frequency of the basic solutions (called harmonics) for the vibrating membrane. For this reason, λ_1, the first eigenvalue, is also called the fundamental frequency or pitch of the membrane

3 Variations

We say that F preserves the volume if

$$vol(D_t) = vol(D) \qquad \forall\, t \in (-\epsilon, \epsilon)$$

Since Γ bounds D, there exists $\nu : \Gamma \to \mathbb{R}^n$, such that:

1. $\|\nu(p)\| = 1$
2. ν points to exterior of D
3. $\nu(p) \perp T_p\Gamma$

Where $T_p\Gamma$ is the tangent plane at p to Γ. ν is the exterior unit normal vector field along Γ.

Now, let $F : (-\epsilon, \epsilon) \times \bar{\Omega}$ and $p \in \bar{\Omega}$. We will denote by

$$X(p) = \frac{\partial F}{\partial t}(0, p) \in \mathbb{R}^n$$

the *variation vector* of F. $X(p)$ is the initial velocity of variation at points of $\bar{\Omega}$. $X : N \to \mathbb{R}^n$ is a C^∞ vector field on N.

We have a formula for the volume of D_t:

$$v(t) = \int_{D_t} dx$$

Now we have the following important lemma.

Lemma 3.1.

$$v'(0) = \int_D divX\, dx = \int_\Gamma \langle X, \nu \rangle\, dx$$

Proof. We will show two different proofs.

(1)

As in the formula above, we have:

$$v'(t) = \frac{d}{dt}\left(\int_{D_t} dx\right) = \int_\Omega (\chi_{D_t})'\, dx = \int_\Omega (\chi_{D_t} \circ F_{-t})'\, dx$$

The function χ_D is not smooth neither continuous. Then we consider a smooth function $g_\delta : \Gamma \to \mathbb{R}$. $dg_\delta : \mathbb{R}^n \to \mathbb{R}$ is a linear application. Also we have:

$$\lim_{\delta \to 0} g_\delta = \chi_D$$

Now we can compute this integral

$$\int_\Omega (g_\delta \circ F_{-t})' \, dx$$

Remember that $X = \frac{\partial F}{\partial t}$ is the variation vector. Then we have:

$$\int_\Omega \frac{\partial}{\partial t}\bigg|_{t=0} g_\delta(F(-t,x))\, dx = \int_\Omega dg_\delta(x)\frac{\partial F}{\partial t}(0,x).(-1)\, dx$$

$$= -\int_\Omega \langle \nabla g_\delta(x), X(x)\rangle \, dx = -\int_\Omega X(g_\delta)(x)\, dx$$

But we have

$$div(g_\delta X) = \langle \nabla g_\delta, X\rangle + g_\delta div X$$

And

$$\int_\Omega div(g_\delta X) = \int_{\partial\Omega} g_\delta \langle X, \nu\rangle = 0$$

because X is zero along the boundary $\partial\Omega$. As we have $\lim_{\delta\to 0} g_\delta = \chi_D$, then the integral converges too. It means

$$\int_\Omega (\chi_D \circ F_{-t})' \, dx = \int_\Omega (\chi_D) div X dx = \int_\Gamma \langle X, \nu\rangle \, dS$$

(2)

Using change of variables we have:

$$v(t) = \int_D JF_t dy$$

Where $JF_t = |det(dF_t)|$. Then

$$v'(t) = \frac{d}{dt}\int_{D_t} dy = \frac{d}{dt}\int_D JF_t \, dx = \int_D \frac{d}{dt} JF_t \, dx$$

Now, we will work only with $\frac{d}{dt}JF_t$. dF_t is the Jacobian matrix for the function F_t.

$$F_t : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$$

$$F_t(x_1, x_2, \ldots, x_n) = (F_t^1 x, F_t^2 x, \ldots, F_t^n x)$$

Where $x = (x_1, x_2, \ldots, x_n)$ and $t \in (-\varepsilon, \varepsilon)$.

$$dF_t = \begin{pmatrix} (F_t^1)_1 & (F_t^1)_2 & \cdots & (F_t^1)_n \\ (F_t^2)_1 & (F_t^2)_2 & \cdots & (F_t^2)_n \\ \vdots & \vdots & \vdots & \\ (F_t^n)_1 & (F_t^n)_2 & \cdots & (F_t^n)_n \end{pmatrix}$$

Where $(F_t^i)_j = \frac{\partial F_t^i}{\partial x_j}$ and $i, j = 1, \ldots, n$.
If we look at the determinant of dF_t by its columns, we get:

$$det(dF_t) = \; det[(F_t)_1 \; (F_t)_2 \cdots (F_t)_n]$$

Now, we can differentiate with respect to t:

$$\frac{d}{dt}det(dF_t) = det[(F_t)_1' \; (F_t)_2 \cdots (F_t)_n] + \cdots + det[(F_t)_1 \; (F_t)_2 \cdots (F_t)_n']$$

Where $(F_t)_k'$ is the derivative of the k-th column with respect to t. We can take a look at one of the terms of the sum above:

$$det[(F_t)_1 \cdots (F_t^1)_k' \cdots (F_t)_n] = det \begin{pmatrix} (F_t^1)_1 & \cdots & (F_t^1)_k' & \cdots & (F_t^1)_n \\ (F_t^2)_1 & \cdots & (F_t^2)_k' & \cdots & (F_t^2)_n \\ \vdots & \vdots & \vdots & & \\ (F_t^n)_1 & & (F_t^n)_k' & \cdots & (F_t^n)_n \end{pmatrix}$$

We can compute the determinant of the matrix above as a sum as below, where $\mathcal{P}$ is the set of all permutations of the index i and σ is its permutation.

$$\sum_{\sigma \in \mathcal{P}} \varepsilon_\sigma (F_t^1)_{\sigma(1)} (F_t^2)_{\sigma(2)} \cdots (F_t^k)_{\sigma(k)}' \cdots (F_t^n)_{\sigma(n)}$$

Where $\varepsilon_\sigma = \pm 1$ is the signal of the permutation.We need this computation only when $t = 0$. In this time we have for $i, j = 1, ..., n$:

$$\begin{cases} (F_0^i)_j = 0 & i \neq j \\ (F_0^i)_i = 1 & \end{cases}$$

This means that $F_0 = Id|_{\bar{\Omega}}$. So, it is clear that:

$$\frac{d}{dt}det \; dF_t \; |_{t=o} = (F_0^1)' + (F_0^2)' + \cdots + (F_0^n)'$$

It is easy to see that

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \cdots, \frac{\partial}{\partial x_n} \right) \cdot \left(\frac{dF_0^1}{dt}, \frac{dF_0^2}{dt}, \cdots, \frac{dF_0^n}{dt} \right) = \nabla \cdot F_0$$

The end of this proof it is made by the Gauss-Green Theorem. If we called $F_0^i = X_i$ for $i = 1, ..., n$ and $div \; X = X_1' + ... + X_n'$:

$$v'(0) = \int_D div \; X \; dx = \int_\Gamma \langle X, \nu \rangle \; dS$$

$\square$

4 Variations With Fixed Boundary

Lemma 4.1. *Let $f : \Gamma \to \mathbb{R}$ be a piecewise smooth function such that*

$$\int_\Gamma f dS = 0$$

then there exists a volume-preserving normal variation whose variation vector is fN. If, in addition, $f \equiv 0$ on $\partial\Omega$, the variation can be chosen so that it fixes the boundary $\partial\Omega$.

Proof. Let $g = 0$ on $\partial\Omega$ and $\int_D g dM \neq 0$. We can let $g = 1$ on Γ. Now, in order to extend $tf + \bar{t}g$ over all of $\bar{\Omega}$, Take any point $q \in \bar{\Omega}$. We will define $d(\Gamma, q)$ to be the length of the normal vector to Γ passing through q (touching Γ at point p; thus we also define a function $p(q)$ mapping $\bar{\Omega}$ to Γ.) We also define $d(p, \partial\Omega)$ for any point $p \in \Gamma$ as the length of the vector normal to Γ at p whose endpoint is on $\partial\Omega$ (we will say that the length is negative if q is inside Γ.) We will also define the "bump" function,

$$B : [0, 1] \to \mathbb{R}$$

so that $B(x) = 1$ if $x \leq \frac{1}{3}$, $B(x) = 0$ if $x \geq \frac{2}{3}$, and $B(x)$ is C^∞ on $[0, 1]$.

We then define our extension function,

$$\phi(q) = B\left(\frac{d(\Gamma, q)}{d(p(q), \partial\Omega)}\right)$$

It is easily shown that $\phi(q)$ is uniformly 1 on Γ, 0 on $\partial\Omega$, and C^∞ on $\bar{\Omega}$.

Now, set $x(t, \bar{t}) = x_0 + \phi(q)(tf + \bar{t}g)N$, $g : \bar{\Omega} \to \mathbb{R}$ is a piecewise smooth function.

The volume function is

$$V_D(t, \bar{t}) = \int_\Gamma \langle x, N \rangle \, dS$$

Since we are considering the case where the volume is constant, we have

$$V_D(t, \bar{t}) = V_D(0, 0)$$

Then we have

$$\left.\frac{\partial V_D}{\partial \bar{t}}\right|_{(0,0)} = \int_D dM \neq 0$$

Now we want to apply the Implicit Function Theorem.

We have $V_D(t,\bar{t}) = V_D(0,0)$ so, let

$$G : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

$$G := V_D(t,\bar{t}) - V_D(0,0)$$

continuously differentiable on an interval containing $(0,0)$ and $G(0,0) = 0$.

The Jacobian is

$$dV_D = \left[\frac{\partial V_D}{\partial t} \frac{\partial V_D}{\partial \bar{t}} \right]$$

So, our $M = \left[\frac{\partial V_D}{\partial \bar{t}} \right]$

$$det(M) \mid_{\bar{t}=0} = \left. \frac{\partial V_D}{\partial \bar{t}} \right|_{\bar{t}=0} = \int_D g dM \neq 0 \text{ by definition.}$$

Then, by the Implicit Function Theorem, there is an open set $(-\varepsilon, \varepsilon) \subset \mathbb{R}$ containing $\bar{t} = 0$ such that $\forall t \epsilon (-\varepsilon, \varepsilon)$ $\exists!$ $\varphi(t) \epsilon (-\varepsilon, \varepsilon)$ such that $\varphi(t) = \bar{t}$ and $G(t, \varphi(t)) = 0 \Rightarrow V(t, \varphi(t)) = V_D(0,0)$. □

5 Critical Eigenvalues

Theorem 5.1. *Let $D \subset \Omega \subset \mathbb{R}^n$ be bounded, measurable and with a smooth boundary Γ,u,f as defined above. Then*

$$\lambda'(0) = \alpha \int_{\Gamma} u^2 f \, dS$$

Proof. Observe that

$$-\Delta u_0 + \alpha \chi_D u_0 = \lambda u_0$$

$$-\Delta u_t + \alpha \chi_{D_t} u_t = \lambda u_t$$

We multiply the first equation by $-u_t$ and the second by u, then add them and integrate both side over $\bar{\Omega}$ to obtain

$$\int_{\Omega} u_t \Delta u_0 - u_0 \Delta u_t + \alpha \int_{\Omega} (\chi_{D_t} - \chi_D) u_0 u_t = (\lambda(t) - \lambda) \int_{\Omega} u_0 u_t$$

The first integrand on the left-hand side of the equation is equal to the divergence of $u \nabla u_t - u_t \nabla u$. By the divergence theorem, this integral has value 0 since both u and u_t are uniformly 0 on $\partial\Omega$. Our equation reduces to

$$\alpha \int_{\Omega} (\chi_{D_t} - \chi_D) u_0 u_t = (\lambda(t) - \lambda) \int_{\Omega} u_0 u_t$$

We take the derivative of both sides with respect to t, and then terms cancel out when we evaluate at t = 0:

$$\alpha \left(\int_{\Omega} \frac{d}{dt} \left(\chi'_{D_t}\right) u_0 u_t + \int_{\Omega} \chi_{D_t} u_0 u_t' - \int_{\Omega} \chi_D u_0 u_t' \right) =$$

$$= \lambda'(t) \int_{\Omega} u_0 u_t' + \lambda(t) \int_{\Omega} u_0 u_t' - \lambda_0 \int_{\Omega} u_0 u_t'$$

Evaluate at t = 0:

$$\alpha \int_{\Omega} \frac{d}{dt} \left(\chi'_{D_t}\right)_{t=0} u^2 = \lambda'(0) \int_{\Omega} u^2 = \lambda'(0)$$

since the L^2-norm of u is 1. The question remains, how can we take the derivative of a non-differentiable function? Well, we can approximate χ_D by a C^∞ function, g_ϵ, by defining g_ϵ to be equal to χ_D on all points $p \in \Omega$ where $d(p, \Gamma) \geq \epsilon$ and C^∞ on all points (on the ring around Γ, g slopes from 0 to

1). Clearly as $\epsilon \to 0$, $g_\epsilon \to \chi_D$, but g_ϵ is still a C^∞ function which we can differentiate.

$$\alpha \int_\Omega \frac{d}{dt} (\chi_{D_t})_{t=0} u^2 = \alpha \int_\Omega \frac{d}{dt} (\chi_D \circ F_{-t})_{t=0} u^2 \approx \alpha \int_\Omega \frac{d}{dt} \left(u^2 g_\epsilon \circ F_{-t}\right)_{t=0}$$

$$= -\alpha \int_\Omega u^2 X (g_\epsilon) = -\alpha \int_\Omega \langle u^2 X, \nabla g_\epsilon \rangle$$

where X is the vector field associated with F. Now, by the properties of divergence,

$$div(g_\epsilon \circ u^2 X) = \langle u^2 X, \nabla g_\epsilon \rangle + g_\epsilon div(u^2 X)$$

$$\int_\Omega g_\epsilon div(u^2 X) = \int_\Omega div(g_\epsilon \circ u^2 X) - \int_\Omega \langle u^2 X, \nabla g_\epsilon \rangle$$

$$= \int_{\partial\Omega} g_\epsilon u^2 \langle X, \mu \rangle - \int_\Omega \langle u^2 X, \nabla g_\epsilon \rangle = - \int_\Omega \langle u^2 X, \nabla g_\epsilon \rangle$$

since $u \equiv 0$ on $\partial\Omega$. Thus

$$\lambda'(0) = -\alpha \int_\Omega \langle u^2 X, \nabla g_\epsilon \rangle = \alpha \int_\Omega g_\epsilon div(u^2 X)$$

$$\approx \alpha \int_\Omega \chi_D u^2 div X = \alpha \int_D u^2 div X = \alpha \int_{\partial D} u^2 \langle X, \nu \rangle$$

by the Divergence Theorem. We previously stated that $f = \langle X, \nu \rangle$. ∂D is made up of two parts, Γ and $\partial\Omega$. On $\partial\Omega$, $u \equiv 0$, so the previous result reduces to

$$\lambda'(0) = \alpha \int_\Gamma u^2 f dS$$

which is a nice simple expression for $\lambda'(0)$.

□

Definition 3. *(Ω, D) is a **critical point** if for all variations f where $\int_\Gamma f dS = 0$ we have $\lambda'(0) = 0$.*

Proposition 5.2. *(Ω, D) is a critical point for our problem if and only if $\Gamma = u^{-1}(c)$ where c is a constant.*

Proof. (⇒) Assume, in order to prove the contra-positive, that u is not constant over Γ. $\Rightarrow u^2$ is not constant over Γ. Then define

$$\bar{u}^2 = \frac{\int_\Gamma u^2 dS}{\int_\Gamma dS}$$

Now define positive, smooth functions ϕ, ψ such that

$$\phi = 0 \text{ when } u^2 - \bar{u}^2 \leq 0 \text{ and,}$$

$$\psi = 0 \text{ when } u^2 - \bar{u}^2 \geq 0$$

Then let $f = (\phi + \psi)(u^2 - \bar{u}^2)$

$$\Rightarrow \int_\Gamma (\phi + \psi)(u^2 - \bar{u}^2) = 0$$

Consider $\int_\Gamma u^2 f dS$

$$\int_\Gamma u^2 f dS = \int_\Gamma u^2(\phi+\psi)(u^2-\bar{u}^2) = \int_\Gamma (\phi+\psi)(u^2-\bar{u}^2)^2 + \int_\Gamma \bar{u}^2(\phi+\psi)(u^2-\bar{u}^2)$$

Then since $\bar{u}^2$ is constant, we can factor it out and that term goes to zero by our definition of f, leaving

$$\int_\Gamma u^2 f dS = \int_\Gamma (\phi + \psi)(u^2 - \bar{u}^2)^2 \neq 0$$

(⇐) Trivial, since if $u = c$ constant, $\lambda'(0)$ is clearly 0.

□

References

[1] Pedrosa, H.L. Renato, *Some Results Regarding Symmetry and Symmetry-breaking Proprieties of Optimal Composite Membranes*, Progress in Nonlinear Differential Equations and Their Applications, Vol 66, 429-442 (2005)

[2] Barbosa, João Lucas and do Carmo, Manfredo *Stability of Hypersurfaces with Constant Mean Curvature*, Mathematische Zeitschrift 185, 339-353 (1984)

This research was conducted in the IRES program at UNICAMP, Brazil, funded by NSF and CNPq, and was advised by Renato Pedrosa of the UNICAMP Mathematics Department. The authors would like to thank the Department of Mathematics for their hospitality.