The inviscid limit of incompressible fluid flow in an annulus

Sara Frietze, Robert Gerrity, and Tiago Picon

August 18, 2006

Abstract

Incompressible, circularly symmetric fluid flow in a two-dimensional annulus $A = \{x \in \mathbb{R}^2 | 1 < |x| < 2\}$ with fixed outer boundary and rotating inner boundary is analyzed in the low-viscosity limit. We conclude that in the inviscid limit, velocity solutions to the governing equations are solutions to the corresponding Euler zero-viscosity equations. However, the vorticity production proves to be non-zero in the inviscid limit given appropriate non-trivial initial conditions—differing from Euler flow, which produces zero-vorticity at the boundary. Results from semigroup theory together with the ability to calculate boundary conditions for the vorticity equations (due to the simple symmetry of the system) are used to prove the above conclusions.

Contents

1 Introduction

In a summary of their recent work, Lopes, Mazzucato, and Nussenzveig [1] analyzed the low-viscosity limit of incompressible fluid flow in the unit disk with rotating boundary. In this paper, we seek to adapt their conclusions to a region with non-trivial topology—namely the annulus, A, of outer radius two and inner radius one. As in [1], we seek to show that solutions to the Navier-Stokes (NS) flow equations for the annulus converge to solutions of the Euler equations and that there exist vorticity-producing inner and outer boundary layers in the low-viscosity limit. Our analysis proceeds using the key assumption of circularly symmetric flows—greatly simplifying the analysis. The presence of a second boundary component in the annulus creates some added difficulties, not present in the disk, which must be addressed in our adaptation of the Lopes *et al.* analysis [1].

Our analysis begins in section 2 by considering the flow velocity solutions to the NS equations using a conversion to polar coordinates to capitalize on the assumed symmetry of the system as well as using a change of variables to produce Dirichlet boundary conditions. Semigroup theory is then used to demonstrate the existence of a solution which converges to an Euler solution as viscosity approaches zero.

We next consider the vorticity production in the same annulus system in section 3. It has been observed through physical experiments that low-viscosity fluid flows lead to a zero-vorticity euler flow in the center of the flow region bordered by vorticity-producing boundary layers. The goal of our analysis of the vorticity is to prove the existence of such a layer in the annulus case via mathematical analysis.

We conclude with theorems regarding the results of the above analysis as well as attached appendices containing details about the polar coordinates transformation, vector identities, and some discussion of the harmonic decomposition of the velocity field.

2 Velocity Analysis

2.1 Governing Equations

We consider the flow of a viscous, incompressible fluid in the annulus, $A = \{x \in \mathbb{R}^2 | 1 < |x| < 2\}$. The governing equations are:

$$
\begin{cases}
v_t = \nu\Delta v - \nu\frac{v}{r^2} & \text{in } A \times (0,T) \\
v(r,0) = v_0(r) & \text{in } A \\
v(r,t) = \frac{\beta(t)}{2\pi} & \text{for } r=1,\ t \in (0,T) \\
v(r,t) = 0 & \text{for } r=2,\ t \in (0,T),
\end{cases}
\tag{2.1}
$$

which are derived from the Navier-Stokes equations (see Appendix A) via a polar coordinates symmetry reduction as described in the following subsections.

2.2 Navier-Stokes Equations in Polar Coordinates

The first step in our analysis is to consider the Navier-Stokes system (A.1) in polar coordinates (r, θ) where the radius $r = |\mathbf{x}|$ and $\theta = \tan^{-1}(\frac{y}{x})$ (see Appendix B).

This manipulation decomposes the velocity field $\mathbf{u}$ into the radial component of velocity u and the angular component v, and gives (A.1) in polar coordinates:

$$
\begin{cases}
u_t + uu_r + \frac{v}{r}u_\theta - \frac{v^2}{r} = -p_r + \nu(\Delta u - \frac{1}{r^2}u - \frac{2}{r^2}v_\theta), \\
v_t + uv_r + \frac{v}{r}v_\theta - \frac{uv}{r} = -\frac{1}{r}p_\theta + \nu(\Delta v - \frac{1}{r^2}v + \frac{2}{r^2}u_\theta), \\
u_r + \frac{u}{r} + \frac{v_\theta}{r} = 0.
\end{cases}
\tag{2.2}
$$

Note that the boundary conditions and the specific domain for each equation have been omitted; these will be addressed below.

2.3 Symmetry Reduction and Change of Variables

The benefit of converting the cartesian system (A.1) to the polar system (2.2) lies in the simple symmetry of our system. Because we have assumed the flow to be circularly symmetric within the annulus A, we have that none of the variables (angular velocity, radial velocity, or pressure) vary with respect to θ. Additionally, we know through a geometric argument that the velocity must be entirely tangent to the radius, therefore the radial velocity, u must be zero. We can also show that $u = 0$ analytically as $(ru)_r = 0$ and $u(2,t) = 0$.

This symmetry reduction simplifies the polar NS equations (2.2) to:

$$
\begin{cases}
\frac{v^2}{r} & = p_r \\
v_t & = \nu(v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v).
\end{cases}
\tag{2.3}
$$

The first equation simply gives a formula for pressure:

$$p(r,t) = \int_1^r \frac{v(s,t)^2}{s}\, ds,$$

which can be decoupled from the system. Substituting the polar Laplacian, removing this portion, and once again including the previously omitted boundary and initial conditions gives the previously stated governing equations (2.1).

The simplified system (2.1) is no longer nonlinear in polar coordinates and now has the form of a non-homogeneous perturbed heat equation with two boundary conditions. One last manipulation remains to replace this non-homogeneity with homogeneous Dirichlet boundary conditions. To do so, we begin by defining:

$$W(r,t) = v(r,t) - \frac{4}{3}\beta(t) + \frac{\beta(t)}{3}r^2$$

Therefore, W solves the system below:

$$\begin{cases} W_t = \nu(\Delta - \frac{1}{r^2})W + \nu\beta(t) - \frac{4\nu\beta(t)}{3r^2} - \frac{4}{3}\beta'(t) + \frac{\beta'(t)}{3}r^2 & \text{in } A \times (0,T) \\ W(r,0) = W_0(r) = v_0 - \frac{4}{3}\beta(0) + \frac{\beta(0)}{3}r^2 & \text{in } A \\ W(1,t) = W(2,t) = 0 & \text{for } t \in (0,T). \end{cases} \tag{2.4}$$

Setting $Q = \nu(\Delta - \frac{1}{r^2})$; $f(r) = \nu - \frac{4\nu}{3r^2}$ and $g(r) = -\frac{4}{3} + \frac{1}{3}r^2$;
$a_0 = -\frac{4}{3}\beta(0)$; and $b_0(r) = \frac{\beta(0)}{3}r^2$ gives the modified version of (2.4) below:

$$\begin{cases} W_t = QW + \beta(t)f(r) + \beta'(t)g(r) & \text{in } A \times (0,T) \\ W_0 = V_0 + a_0 + b_0(r) & \text{in } A \\ W(r,t) = 0 & \text{for } r = 1,\ r = 2,\ t \in (0,T). \end{cases} \tag{2.5}$$

We now focus our analysis on the operator Q to formulate the following proposition.

Proposition 1. *Given a strongly elliptical operator $-Q$ of order 2 with domain equal to $H^2(A) \bigcap H_0^1(A)$, then Q is the infinitesimal generator of an analytic semigroup of operators on $L^2(A)$ and the corresponding velocity solutions are $C^2(\overline{A})$.*

The proof of this proposition follows from results in chapter 7 of Pazy's semigroup text [2] together with a bootstrap-style continuity argument and Sobolev embedding.

Thus we can conclude that (2.5) has a unique solution. According to Duhamel's principle, this unique solution can be expressed as:

$$W(r,t) = e^{Qt}W_0(r) + \int_0^t e^{Q(t-s)} \left[\beta(s)f(r) + \beta'(s)g(r)\right] ds. \tag{2.6}$$

From this result, the convergence to the solution of the Euler equations as $\nu \to 0$ follows as in the work by Lopes *et al.*in [1] giving the following theorem:

Theorem 1. *Let* $\boldsymbol{u}$ *be the solution of the 2D Navier-Stokes equations in the annulus such that* $1 \leq |x| \leq 2$*. The inner boundary of the annulus,* $|x| = 1$*, is rotating with angular velocity* $\beta(t) \in AC([0,T])$*. On the outer boundary,* $|x| = 2$*, the angular velocity is zero. Assume that the initial vorticity* $\omega_0 \in L^1(A) \cap H^{-1}(A)$ *is radial so that the initial velocity* $u_0 \in L^2(A)$ *has circular symmetry. Then,* u_0 *is a steady solution of the 2D Euler equation and* $\boldsymbol{u}$ *converges strongly in* $L^\infty([0,T], L^2(A))$ *to* u_0*.*

3 Vorticity Analysis

3.1 Introduction to Vorticity

Previously we analyzed the fluid flow in an annulus as $\nu \to 0$. Our next goal is to examine the production of vorticity in the low-viscosity limit.

Our first task in this analysis is to define voriticty:

Definition The measure of the local rotation of 2D fluid flow is called **vorticity,** ω. Where

$$\overrightarrow{\omega} = \text{curl } \mathbf{u} = \nabla \times \mathbf{u}. \tag{3.1}$$

Note that $\overrightarrow{\omega}$ is a vector and that if $\mathbf{u}$ is in a plane, then $\mathbf{u} \perp \overrightarrow{\omega}$ and $\overrightarrow{\omega} = (0,0,\omega)$. We abuse notation and let $\omega = \tilde{v}_x - \tilde{u}_y$.

From our previous calculation that transformed the Navier-Stokes equations into polar coordinates, we have

$$\omega = \tilde{v}_x - \tilde{u}_y = v_r - \frac{v}{r} = \frac{1}{r}(rv)_r. \tag{3.2}$$

We begin this investigation with the equation for the evolution of vorticity in a viscous incompressible two-dimensional flow:

Proposition 2. *Given an incompressible, viscous, circularly symmmetric 2D fluid flow, the vorticity satisfies:*

$$w_t = \nu\Delta\omega \tag{3.3}$$

Proof. Using vector identities (C.1), the NS equations (A.1) may be rewritten as

$$\mathbf{u}_t + (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla(\frac{1}{2}\mathbf{u}^2) = -\nabla p + \nu\Delta\mathbf{u},$$

and by taking the curl we obtain

$$\omega_t + \nabla\times\omega\times\mathbf{u} + \nabla\times\left[\nabla\left(\frac{1}{2}\mathbf{u}^2\right)\right] = \nu\Delta\omega.$$

Using the vetor identities (C.2) and (C.3) in the third and fourth term respectively we have

$$\omega_t + (\mathbf{u}\cdot\nabla)\omega = \nu\Delta\omega.$$

Additionally, via a geometric argument, we know that $\nabla\omega$ is normal to level curves, which implies $\nabla\omega \perp$ circles, and we know that $\mathbf{u}$ is tangent to circles, both because of our symmetry assumption. Thus $\mathbf{u}\cdot\nabla\omega = 0$ and we reduce our vorticity equation to:

$$\begin{cases} \omega_t = \nu\Delta\omega, & \text{in } A\times(0,T); \\ \omega(0) = \omega_0, & \text{in } A \\ \omega_r(r,t) = \frac{\beta'(t)}{2\pi\nu}, & r=1 \\ \omega_r(r,t) = 0, & r=2. \end{cases} \tag{3.4}$$

where $\omega_0 = \text{curl } \mathbf{u}_0$.

□

The boundary conditions of (3.4) will be calculated in the next subsection (see (3.7) and (3.6)). The vorticity boundary conditions are often difficult to calculate, but in the case of this system, we use the circular symmetry and are able to overcome this difficulty as shown below.

We will use (3.4) to find an estimate of the flux of vorticity over the boundary. Eventually we will demonstrate the existence of a boundary layer where vorticity is produced surrounding an inner layer of zero-vorticity Euler flow.

3.2 Vorticity in polar coordinates

When estimating the production of vorticity in the annulus, our calculations are greatly simplified when we work in polar coordinates.

First we must find an expression to define ω_r.

Proposition 3. *For all $(\boldsymbol{x}, t) \in A \times [0, T)$ we have,*

$$\omega_r = \frac{v_t}{\nu}.$$

Proof. Since $v(t) \in C^2(A)$ we have $\omega(t) \in C^1(A)$ and differentiating ω with respect to the radius gives:

$$\omega_r = v_{rr} + \frac{v_r}{r} - \frac{v}{r^2} \tag{3.5}$$

Moreover, $v_t \in C^1$, so we have $\frac{v_t(t)}{\nu} \in C^1(A)$. Therefore, by (2.3):

$$\omega_r = \frac{v_t}{\nu}.$$

□

We are able to proceed in calculating the vorticity boundary conditions due to this useful expression for ω_r in terms of v_t resulting from the polar coordinates transformation (2.3).

Thus, by (2.1):

$$\lim_{r \to 2} v(r, t) = 0 \;\Rightarrow\; \lim_{r \to 2} \frac{v_t(r, t)}{\nu} = 0.$$

And at $r = 1$,

$$\lim_{r \to 1} v(r, t) = \frac{\beta(t)}{2\pi} \;\Rightarrow\; \lim_{r \to 2} \frac{v_t(r, t)}{\nu} = \frac{\beta'(t)}{2\pi\nu}.$$

We conclude that

$$\omega_r(2, t) = 0 \tag{3.6}$$

and

$$\omega_r(1, t) = \frac{\beta'(t)}{2\pi\nu}. \tag{3.7}$$

The result for $\omega_r(1, t)$ can also be calculated using vector identities and (3.6) (see Appendix D).

3.3 Estimate of Vorticity Production

This section examines the production of vorticity at the boundary in the limit of vanishing viscosity. Given the above boundary condition results for ω_r, we can now calculate an estimate for the vorticity production in the annulus.

Theorem 2. *For any* $0 < t \leq T$ *we have*

$$\|\omega(\cdot,t)\|_{L^1(A)} \leq \|\beta(t)\|_{AC(0,t)} + \|\omega_0\|_{L^1(A)}$$

Proof. To calculate the L^1 estimate of vorticity, consider a convex mollifier function $\phi_\epsilon = \phi_\epsilon(s)$, $(\phi_\epsilon^{''}(s) \geq 0)$ such that $\phi_\epsilon(s) \to |s|$ as $\epsilon \to 0$ (see Appendix E).

Thus applying ϕ'_ϵ to (3.4) and integrating gives:

$$\begin{aligned}\frac{d}{dt}\left(\int_A \phi_\epsilon(\omega)dx\right) &= \int_A d_t(\phi_\epsilon(\omega))dx = \int_A \phi_\epsilon^{'}(\omega)(\omega_t)dx \\ &= \nu \int_A \phi_\epsilon^{'}(\omega)\Delta\omega dx.\end{aligned}$$

Using the following vector identity

$$\begin{aligned}\phi_\epsilon^{'}(\omega)\Delta\omega &= \phi_\epsilon^{'}(\omega)\mathrm{div}(\nabla\omega) \\ &= \mathrm{div}[\phi_\epsilon^{'}(\omega)\nabla\omega] - \nabla(\phi_\epsilon^{'})(\omega)\cdot\nabla\omega \\ &= \mathrm{div}[\phi_\epsilon^{'}(\omega)\nabla\omega] - \phi_\epsilon^{''}(\omega)|\nabla\omega|^2\end{aligned}$$

and the divergence theorem then produces,

$$\frac{d}{dt}\left(\int_A \phi_\epsilon(\omega)dx\right) = \nu\int_{\partial A}\left[\phi_\epsilon^{'}(\omega)\nabla\omega\cdot\hat{n}\right]ds - \nu\int_A \phi_\epsilon^{''}(\omega)|\nabla\omega|^2 ds,$$

where $\partial A =$ the boundary of $A = \{|x| = 1 \cup |x| = 2\}$. Thus,

$$\int_A \phi_\epsilon^{''}(\omega)|\nabla\omega|^2 ds \geq 0$$

therefore,

$$\frac{d}{dt}\left(\int_A \phi_\epsilon(\omega)dx\right) \leq \nu \int_{|x|=2} \left[\phi_\epsilon^{'}(\omega)\nabla\omega \cdot \frac{x}{2}\right] ds - \int_{|x|=1} \left[\phi_\epsilon^{'}(\omega)\nabla\omega \cdot x\right] ds$$

Taking $\omega(x,t) = \tilde{\omega}(r,t)$ again in polar coordinates and $\epsilon \to 0$,

$$\frac{d}{dt}\left(\int_A |\omega|dx\right) \leq \nu 4\pi \ \mathrm{sign}(\tilde{\omega}(2,t))\tilde{\omega}_r(2,t) - \nu 2\pi \ \mathrm{sign}(\tilde{\omega}(1,t))\tilde{\omega}_r(1,t)$$

but (3.6) implies that the first term is zero, and thus

$$\frac{d}{dt}\left(\int_A |\omega|dx\right) \leq -\nu 2\pi \ \mathrm{sign}(\tilde{\omega}(1,t))\tilde{\omega}_r(1,t).$$

Applying (3.7) to the right and integrating gives:

$$\int_A |\omega|dx - \int_A |\omega_0|dx \leq \int_0^t |\beta^{'}(s)|ds$$

therefore by definition of the L^1 norm,

$$\|\omega(\cdot,t)\|_{L^1(A)} \leq \|\beta\|_{AC(0,t)} + \|\omega_0\|_{L^1(A)}$$

□

4 Conclusion

In this paper, we analyzed the low-viscosity limit of incompressible, circularly symmetric fluid flow in the annulus, A with fixed outer and rotating inner boundaries. As a result of our analysis, we obtain two theorems that describe this particular fluid flow. In order to construct the theorems, we took advantage of the simple symmetry of the system. We also applied results from semigroup theory (Proposition 1), which results in well-behaved solutions of the governing equations for this system.

Theorem 1 concludes that for this annulus system, the velocity solutions of the Navier-Stokes equations converge strongly to the velocity solutions of the Euler equations as $\nu \to 0$.

Theorem 2 concludes that in the low-viscosity limit of incompressible, circularly symmetric fluid flow in the annulus, vorticity is produced beyond any initial vorticity. From previous analysis, we know that there is no additional vorticity production in the zero-viscosity case.

Therefore we can conclude that while the low-viscosity limit mimics the zero-viscosity case with respect to the velocity solutions, an analysis of the vorticity reveals a stark difference between infinitesimal viscosity and no viscosity in this annulus system.

Acknowledgements

This research was conducted during the IRES program at UNICAMP, Brazil, funded by the NSF, CNPq, and FAEPEX-UNICAMP grant 163/2006. The authors would like to thank the Department of Mathematics for their hospitality and advising professors Max Souza and Helena Lopes for their assistance. Sara Frietze was partially supported by NSF grants DGE:0440547 and DGE:0631397, Robert Gerrity was partially supported by NSF grant OISE:0526008, and Tiago Picon was partially supported by CNPq grant 451.154/2006-1.

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A Navier-Stokes Equations

In this and the following appendices, we outline some of the additional details behind the results discussed in the preceding paper.

We consider the annulus A in the plane centered at the origin such that, $A = \{\mathbf{x} \in \mathbb{R}^2 \mid 1 < |\mathbf{x}| < 2\}$. In order to describe the velocity of the fluid with viscosity ν we consider the 2D Navier-Stokes equations:

$$\begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{u}, & \text{in } A \times (0,T) \\ \text{div } \mathbf{u} = 0, & \text{in } A \times [0,T) \\ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(x), & \text{in } A \\ \mathbf{u}(\mathbf{x},t) = \frac{\beta(t)}{2\pi}, & \text{on } |\mathbf{x}| = 1 \\ \mathbf{u}(\mathbf{x},t) = 0, & \text{on } |\mathbf{x}| = 2. \end{cases} \tag{A.1}$$

$\mathbf{u}(\mathbf{x},t)$ is the fluid velocity vector field, ν is the kinematic viscosity of the fluid, p is the scalar fluid pressure, and ρ is the fluid density. We will assume $\rho = 1$ for this analysis (ρ could also be scaled out to similar effect). We make the assumption that the inner boundary is rotating with angular velocity $\frac{\beta(t)}{2\pi}$ (where $\beta(t)$ is a function of bounded variation) and the outer boundary is stationary.

B Polar Coordinates Transformation

In order to consider the Navier-Stokes equations in polar coordinates, we use the fact that:

$$\mathbf{u} = \tilde{u}(x,y)\overrightarrow{x} + \tilde{v}(x,y)\overrightarrow{y}$$

and define

$$\overrightarrow{x} = cos\theta\overrightarrow{e_r} - sin\theta\overrightarrow{e_\theta}$$

and

$$\overrightarrow{y} = sin\theta\overrightarrow{e_r} + cos\theta\overrightarrow{e_\theta}.$$

This transformation gives:

$$\mathbf{u} = [\tilde{u}(x,y)cos\theta + \tilde{v}(x,y)sin\theta]\overrightarrow{e_r} + [-\tilde{u}(x,y)sin\theta + \tilde{v}(x,y)cos\theta]\overrightarrow{e_\theta}$$

We now let

$$u = \tilde{u}(x,y)cos\theta + \tilde{v}(x,y)sin\theta \text{ and}$$
$$v = -\tilde{u}(x,y)sin\theta + \tilde{v}(x,y)cos\theta.$$

C Vector Identities

The following vector identities are used occasionally throughout our analysis.

$$(\mathbf{F}\cdot\nabla)\mathbf{F} = (\nabla\times\mathbf{F})\times\mathbf{F} + \nabla(\frac{1}{2}\mathbf{F}^2) \tag{C.1}$$

$$\nabla\times(\mathbf{F}\times\mathbf{G}) = (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G} + \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) \tag{C.2}$$

$$\nabla\times\nabla\phi = 0 \tag{C.3}$$

D Vector Identity Calculation of $\omega_r(1,t)$:

Lemma 1. *The vorticity satisfies*

$$\frac{\beta'(t)}{\nu} = 2\pi\tilde{\omega}_r(1,t) \tag{D.1}$$

Proof. Initially we make an estimate

$$\int_A \omega dx = \int_A (\nabla^\perp\cdot\mathbf{u})dx = -\int_A (\nabla\cdot\mathbf{u}^\perp)dx =$$

$$-\int_A div(\mathbf{u}^\perp)dx = -\left[\int_{|x|=2}(\mathbf{u}^\perp.\frac{x}{2})ds - \int_{|x|=1}(\mathbf{u}^\perp.x)ds\right] =$$

$$= -\left[\int_{|x|=2}(v.x^\perp)^\perp.\frac{x}{2}ds - \int_{|x|=1}(v.x^\perp)^\perp.xds\right] =$$

$$= v(2,t)\int_{|x|=2}(x\cdot\frac{x}{2})ds - v(1,t)\int_{|x|=2}(x\cdot x)ds = -v(1,t)2\pi,$$

that is,

$$-\beta(t) = \int_A \omega dx.$$

Now,

$$-\beta'(t) = \frac{d}{dt}\left(\int_A \omega dx\right) = \int_A \omega_t dx$$

$$= \nu \int_A \Delta\omega \; dx = \nu \left[\int_{|x|=2} (\nabla\omega \cdot \frac{x}{2}) ds - \int_{|x|=1} (\nabla\omega \cdot x) ds \right]$$

We take $\omega(x,t) = \tilde{\omega}(|x|,t)$. Thus $\nabla\omega = \tilde{\omega}_r(|x|,t)\frac{x}{|x|}$. Soon,

$$\frac{\beta'(t)}{\nu} = 2\pi\tilde{\omega}_r(1,t)$$

□

E An explicit $\phi_\epsilon(s)$ mollifier function

One function found to satisfy the requirements for the mollifier function in Theorem 2 is given below.

$$\phi_\epsilon(s) = \begin{cases} -s, & s < -\epsilon; \\ s, & s > \epsilon; \\ -\cos(\frac{\pi s}{2\epsilon})\frac{2\epsilon}{\pi}, & s \in [\epsilon, \epsilon]. \end{cases} \tag{E.1}$$

F Harmonic Decomposition

In addition to analyzing the velocity and vorticity of a fluid flow system, one often also considers the harmonic decomposition. While this was not included in our final analysis in the paper, a preliminary harmonic decomposition of the system is outlined below.

By the Hodge-Kodaira Decomposition Theorem, we know that there exists a unique stream function, Ψ such that

$$(u,v) = \nabla^{\perp}\Psi + H \tag{F.1}$$

where $\Delta\Psi = \omega$ and H is the harmonic part. Ψ must also satisfy $\nabla^{\perp}\Psi \perp H$. We also have

$$v(r,t) = \frac{1}{r}\int_1^r s\omega(s,t)ds + \frac{\beta(t)}{r} \tag{F.2}$$

and

$$v(r,t) = \Psi_r + \frac{B(t)}{r}. \tag{F.3}$$

Since, $\Delta\Psi = \omega$, we calculate Ψ_r using the definition of the laplacian of Ψ.

$$\Psi_r = \frac{1}{r}\int_1^r s\omega(s,t)\mathrm{d}s + \frac{A(t)}{r}. \tag{F.4}$$

Integrate both sides to find Ψ

$$\Psi = \int_1^r \frac{1}{z}\int_1^z s\omega(s,t)\mathrm{d}s + A(t)\ln r \tag{F.5}$$

where $\Psi(1,t) = 0$.

Moreover, we need to define the constant functions $A(t)$ and $B(t)$ so that Ψ will satisfy the following boundary conditions of the fluid velocity.

$$\begin{cases} v(1,t) = \beta(t), & r = 1; \\ v(2,t) = 0, & r = 2. \end{cases} \tag{F.6}$$

So at $r = 1, \beta(t) = A(t) + B(t)$. In order to solve for our constant functions, we utilize the fact that $\nabla^{\perp}\Psi \perp H$, which means $\int_1^2 r\Psi_r \frac{B(t)}{r}\mathrm{d}r = 0$. Substitute Ψ_r into this equation,

$$\int_1^2 \Psi_r B(t)\mathrm{d}r = \int_1^2 \left[\frac{1}{r}\int_1^r s\omega(s,t)\mathrm{d}s + \frac{A(t)}{r}\right] B(t)\mathrm{d}r = 0 \tag{F.7}$$

We integrate by parts to find:

$$0 = \int_1^2 \left[\int_1^r s\omega(s,t)ds + A(t)\right]\frac{B(t)}{r}\mathrm{d}r = B(t)\int_1^2 \left[\frac{1}{r}\int_1^r s\omega(s,t)ds + \frac{A(t)}{r}\right]\mathrm{d}r +$$
$$\left[\int_1^2 s\omega(s,t)ds + A(t)\right] B(t)ln2 \; - \left[\int_1^2 s\omega(s,t)ds + A(t)\right] B(t)\ln 2.$$

Now we can conclude that

$$B(t)\int_1^2 \left[\frac{1}{r}\int_1^r s\omega(s,t)ds + \frac{A(t)}{r}\right]\mathrm{d}r = 0.$$

Since B(t) is nonzero,

$$\int_1^2 \left[\frac{1}{r}\int_1^r s\omega(s,t)ds + \frac{A(t)}{r}\right]\mathrm{d}r = 0$$

and solve

$$A(t) = \frac{-1}{ln2}\int_1^2 \left[\frac{1}{r}\int_1^r s\omega(s,t)ds\right]\mathrm{d}r.$$

Thus,

$$B(t) = \beta(t) - A(t).$$

We need to ensure that the constant functions that we just defined satisfy $\Psi(2, t) = 0$.

$$\Psi(2, t) = \int_1^2 \frac{1}{z} \int_1^z s\omega(s, t)ds + A(t)\text{ln}2 \tag{F.8}$$

Substitute the previous calculation of $A(t)$:

$$\Psi(2, t) = \int_1^2 \frac{1}{z} \int_1^z s\omega(s, t)ds + ln2\frac{-1}{ln2} \int_1^2 \left[\frac{1}{r} \int_1^r s\omega(s, t)ds\right] \mathrm{d}r$$

Thus we have shown that $\Psi(2, t) = 0$.