

Finding Unknown Angles

Geometry becomes more interesting when students start using geometric facts to find unknown lengths and angles. During this stage, roughly grades 5-8, students work on “unknown angle problems”. These problems are learning bonanzas. They initiate students in the art of deductive reasoning, solidify their understanding of geometry and measurement, and help introduce algebra.

You have already solved some unknown angle problems and seen how they are integrated into the Primary Math curriculum in grades 5 and 6. This chapter examines how unknown angle problems are used to develop geometry in grades 6 and 7.

From a teaching perspective, unknown angle problems are not just part of the geometry curriculum, they *are* the curriculum in grades 5-8; everything else is secondary. In these grades, teachers and textbooks introduce facts about angles within triangles and polygons, about parallel lines, about congruent and similar figures, and about circles. These are not simply facts to memorize: understanding emerges as students use them to solve problems. Thus teaching centers on *solving problems*.

Unknown angle problems are superbly suited for this purpose. Solutions require several steps, each applying a known fact to the given figure. As students do these problems the geometric facts spring to life; these facts become friends that can be called upon to help solve problems. Unknown angle problems are also enormous fun!

3.1 Unknown Angle Problems

An unknown angle problem is a puzzle consisting of a figure with the measures of some sides and angles given and with one angle — the unknown angle — marked with a letter. The student’s task is to find the measure of the unknown angle by applying basic geometric facts. Beginning exercises require only rudimentary facts, such as the fact that angles around a point add to 360° . As new geometric facts are introduced, they are added to the list of facts that are available as tools to solve unknown angle problems. As more knowledge is integrated, the problems become more challenging and more interesting.

This section examines the role of unknown angle problems in the Primary Math and New Elementary Math textbooks for grades 5-7. It includes a list of the geometric facts learned during this stage and a format for presenting “Teacher’s Solutions” to unknown angle problems. You will be asked to use this format for many homework problems.

Many elementary textbooks, including the Primary Math books, introduce new concepts using the following specific process.

Teaching sequence for introducing geometric facts

1. Review background knowledge and introduce any new terms needed.
2. Introduce the fact by an activity (measuring, folding, or cutting-and-rearranging) that serves to *clarify what the fact says* and *convince students that it is true*.
3. Summarize by stating the geometric fact in simple clear language.
4. Have students solve dozens of unknown angle problems:
 - a) simple problems using the fact alone,
 - b) multi-step problems using the fact alone,
 - c) multi-step problems combining the fact with previously-learned facts.

Step 3 takes only a few minutes, but it is the teacher’s most important input. In geometry, words have precise meanings; students’ success depends on knowing definitions and knowing how to apply them. One can even argue that geometry is included in the K-12 curriculum to teach students that *giving words precise meaning fosters clear thinking*. This lesson is applicable to all subjects.

After these preliminaries, the fun begins as students solve increasingly challenging problems (Step 4). As always in mathematics, the real learning occurs as students solve problems.

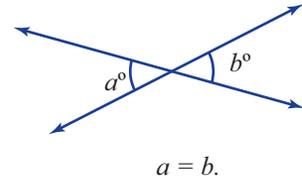
Geometry Facts — First List

As you have seen in homework problems, the basic facts about angles, triangles and quadrilaterals are presented in Primary Mathematics 5A and 5B. Below is a list of the facts learned at that stage. Each has a simple abbreviation. You will be expected to be consistent in using these abbreviations in your homework solutions.

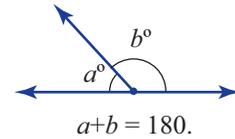
The list of facts is built around three exercises. These questions ask you to observe how these facts are justified at the grade 5 level (using folding, cutting, and measuring exercises) and to observe the type of problems students are asked to solve.

EXERCISE 1.1 (Angle Facts). *The following three facts are introduced on pages 85–88 of Primary Mathematics 5A. How are these facts justified?*

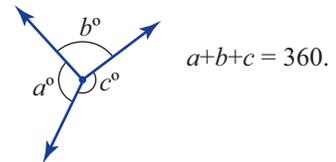
Vertical angles have equal measure.
(Abbreviation: **vert. \angle s.**)



The sum of adjacent angles on a straight line is 180° . (Abbreviation: **\angle s on a line.**)

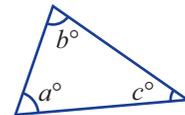


The sum of adjacent angles around a point is 360° .
(Abbreviation: **\angle s at a pt.**)



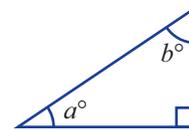
EXERCISE 1.2 (Triangle Facts). *The following five triangle facts are introduced on pages 57–64 in Primary Mathematics 5B. Locate the statement of each in your 5B book. What activity is used to justify the first fact? What wording is used for the fourth one?*

The angle sum of any triangle is 180° .
(Abbreviation: **\angle sum of Δ**)



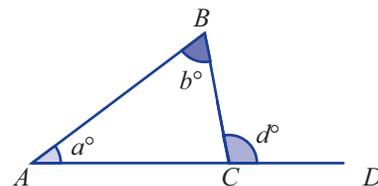
$$a + b + c = 180.$$

When one angle of a triangle is a right angle, the other two angles add up to 90° .
(Abbreviation: **\angle sum of rt. Δ .**)



$$a + b = 90.$$

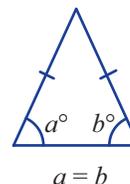
The exterior angle of a triangle is equal to the sum of the interior opposite angles.
(Abbreviation: **ext. \angle of Δ .**)



$$d = a + b.$$

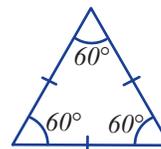
Base angles of an isosceles triangle are equal.

(Abbreviation: **base \angle s of isos. Δ .**)



Each interior angle of an equilateral triangle is 60° .

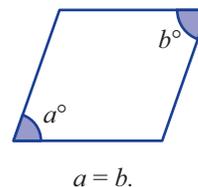
(Abbreviation: **equilat. Δ .**)



EXERCISE 1.3 (Quadrilateral Facts). *The next section of Primary Math 5B (pages 68–71) introduces two facts about 4-sided figures. Study the folding and cutting exercises given on page 70. How would you use these exercises in your class?*

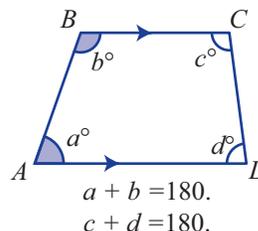
Opposite angles in a parallelogram are equal.

(Abbreviation: **opp. \angle s // -ogram.**)



Interior angles between two parallel sides in a trapezoid (or a parallelogram) are supplementary.

(Abbreviation: **int. \angle s, $\overline{BC} \parallel \overline{AD}$.)**



The “Teacher’s Solution” Format for Unknown Angle Problems

Teachers are obliged to present detailed solutions to problems for the benefit of their students. The teacher’s solutions must meet a different standard than the students’ solutions. Both teachers and students are expected to get the reasoning and the answer correct. But teacher-presented solutions must also communicate the thought process as clearly as possible.

In this book, solutions that meet this high standard are called **Teacher’s Solutions**. You will frequently be asked to write such Teacher’s Solutions in homework. If you are unsure how to do this, *look in the textbooks*: almost every solution presented in the Primary Math books, and all of the “Worked Examples” in the New Elementary Mathematics book, are Teacher’s Solutions.

You are already familiar with one type of Teacher’s Solution — bar diagrams. Bar diagrams are extraordinarily useful for communicating ideas about arithmetic. Teachers need similar devices for communicating geometric ideas. As a start, in this chapter you will be writing Teacher’s Solutions for unknown angle problems. Here is a simple example.

EXAMPLE 1.4. *The figure shows angles around a point. Find the value of x .*

Teacher’s Solution to an Unknown Angle Problem

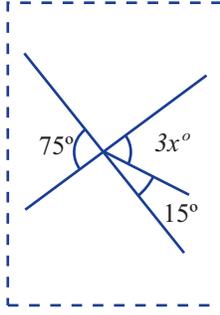


Diagram shows all needed information.

$$3x + 15 = 75$$

$$3x = 60$$

$$\therefore x = 20.$$

Each fact used is stated on the same line using our abbreviations.

vert. \angle s.

A new line for each step.

Arithmetic and algebra done one step at a time (no reasons needed).

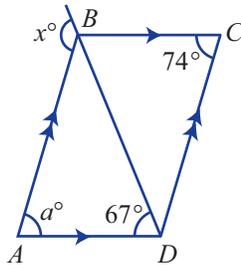
Answer clearly stated on the last line.

This solution is short and clear, yet displays all the reasoning. It always begins with a picture showing all points, lines, and angles used in the solution, and it always ends with a clear answer to the question asked.

Notice what happens with the degree signs. The angles in the picture have degree signs, so 75, 15 and $3x$ are all numbers. Thus we can drop the degree signs in the equations. This saves work and makes the solution clearer. Degree signs are handled in the same way in the next two examples.

EXAMPLE 1.5. *The figure shows a parallelogram. Find the value of x .*

Teacher’s Solution:



$$a = 74 \quad \text{opp. } \angle\text{s of } //\text{-ogram}$$

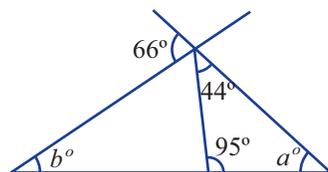
$$x = a + 67 \quad \text{ext. } \angle \text{ of } \triangle ABD$$

$$\therefore x = 74 + 67$$

$$= 141.$$

EXAMPLE 1.6. *Find the values of a and b in the following figure.*

Teacher’s Solution:



$$a + 44 + 95 = 180 \quad \angle \text{ sum of } \triangle$$

$$a + 139 = 180,$$

$$\therefore a = 41.$$

$$a + b = 66 \quad \text{ext. } \angle \text{ of } \triangle$$

$$41 + b = 66,$$

$$\therefore b = 25.$$

Elementary students are not usually asked to record reasons in the manner done in these examples. But in the New Elementary Mathematics textbook, middle school students are expected to present solutions exactly as above — middle school students use the same format as elementary school teachers!

This Teacher’s Solution format is used by

- *Teachers* in elementary school,
- *Teachers and students* in middle school.

Unknown angle problems in grades 5-7 also help introduce algebra. These geometry problems expose students to the idea of using letters to stand for numbers (notice that the whole alphabet, not just x and y , are used!). They often require solving simple linear equations, as you see in Example 1.4. This is a sneaky trick: giving students practice in algebra as they do geometry. Solving for unknown angles also gives students a visual way to understand what it means to “solve for x ” and appreciate why one would want to.

Homework Set 10

1. (*Study the Textbook!*) Give one-sentence answers to Exercises 1.1, 1.2, and 1.3 in this section.
2. (*Study the Textbook!*) Turn to page 87 of Primary Math 5A.
 - Do all of the problems on pages 87 and 88. Write your answers as a single list of numbers, separated by commas, but *not* labeled by the problem numbers.
 - Note the variation: on pages 87 and 88, how many different letters are used to stand for numbers?
3. On page 109 of Primary Math 6B, answer Problems 35, 36, and 37.
4. Answer Problem 2 on page 264 of NEM1. Show your reasoning.
5. For Problem 3 on page 264 of NEM 1, write Teacher’s Solutions to parts b, d, e, g, and j.
6. For Problem 4 on the next page (page 265) write Teacher’s Solutions to parts b, c, f and g.
7. (*Study the Textbook!*) Compare the grade 6 and grade 7 problems you did in this HW Set. Name two new features required in solving the grade 7 problems that are not present in the grade 6 problems.

3.2 Finding Angles Using Parallel Lines

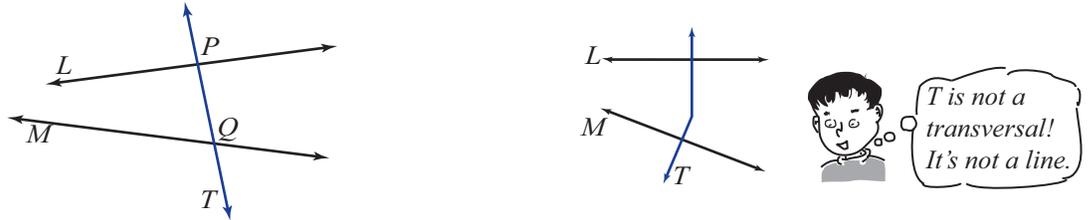
After completing Primary Mathematics 5 and 6, students enter grade 7 with two years' experience solving unknown angle problems. They understand that geometry is a game in which one uses a few simple facts to find relations among the lengths and angles in figures. The grade 7 textbook (New Elementary Mathematics 1) revisits the material learned earlier, putting it in a condensed, structured form. It then moves on to new ideas.

The revision begins with lines and angles (Chapter 9 of NEM1). Folding and measuring exercises are no longer primary; the discussion is built on facts about parallel lines and congruent triangles, and is broadened to include area. The transition is subtle: the spirit of the subject is unchanged and student work remains focused on solving short geometric puzzles.

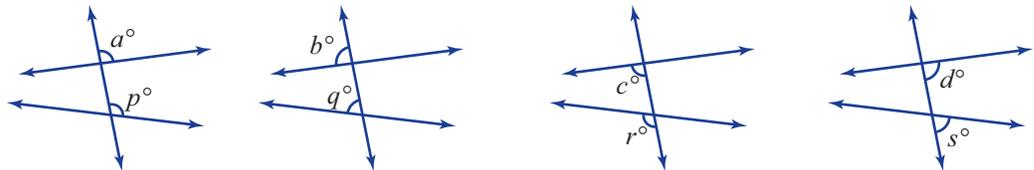
This section describes the basic facts about parallel lines. It ends with discussions of two sources of student confusion: the distinction between a statement and its converse, and the technique of drawing auxiliary lines.

Transversals and Parallel Lines

In the figure below, T is a *transversal* to the lines L and M . More precisely: Given a pair of lines L and M in a plane, a third line T is a **transversal** if it intersects L at a single point P and intersects M in a different point Q .

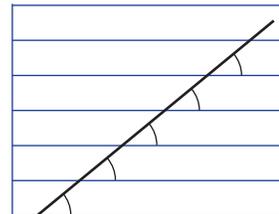


A transversal forms eight angles with the two lines. Pairs of angles are named according to their relative positions. Angles on the same side of the transversal and on the same side of the lines are called **corresponding angles**. Four pairs of corresponding angles are shown below.



The key fact about parallel lines is that *when a transversal intersects parallel lines, corresponding angles are equal*. Here is an activity introducing the idea.

*A sheet of lined paper has many parallel lines.
Draw a slanted line.
Compare the corresponding angles.
Are they equal?*

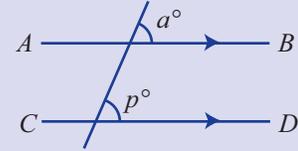


This lined paper activity provides experimental evidence, but it is not the kind of logical argument required in geometry. Instead, the value of the activity is psychological: it serves to make the stated fact clear and believable to students.

If a transversal intersects two parallel lines, then corresponding angles are equal, i.e.,

$$\text{if } \overline{AB} \parallel \overline{CD}, \text{ then } a = p.$$

(Abbreviation: **corr. \angle s**, $\overline{AB} \parallel \overline{CD}$.)



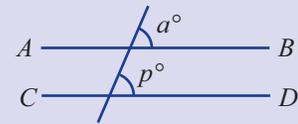
In the blue box, the same fact is stated three times: first in words, then as a labeled picture, and then again as an abbreviation. This presentation gives students three ways to understand and remember the fact.

Notice that the abbreviation says “corr. \angle s” and then identifies which pair of parallel lines we are using. Asking students to name the parallel lines is important for clarity. It reminds them that this fact requires parallel lines. It is also a courtesy to the teacher who is trying to follow the student’s reasoning.

The statement in the box above has a partner called its converse. For the converse, we consider a transversal intersecting two lines that are not necessarily parallel. We then measure corresponding angles. If these are equal, then we can conclude that the lines are parallel.

If $a = p$, then $\overline{AB} \parallel \overline{CD}$.

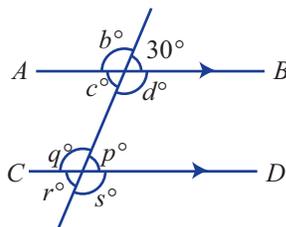
(Abbreviation: **corr. \angle s converse**.)



Recall the principle that when two lines cross, each of the angles formed determines the other three. The above fact gives a similar principle about the eight angles formed by a transversal that intersects two parallel lines: *any one angle determines all 8*. In fact, among the eight angles, four have the same measure as the given angle, and the other four have the supplementary measure.

EXAMPLE 2.1. In the figure below, $\overline{AB} \parallel \overline{CD}$. Find $b, c, d, p, q, r,$ and s .

Teacher’s Solution:



$$30 + b = 180, \quad \angle\text{s on a line}$$

$$b = 150.$$

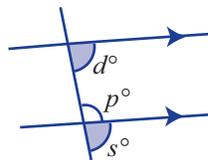
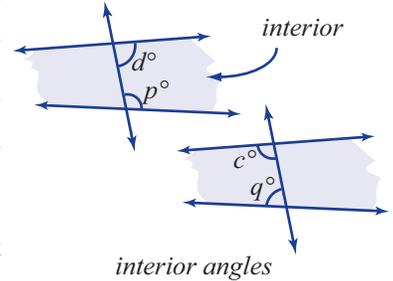
$$\left. \begin{array}{l} c = 30 \\ d = 150 \end{array} \right\} \text{vert. } \angle\text{s.}$$

$$\left. \begin{array}{l} p = 30 \\ q = 150 \\ r = 30 \\ s = 150 \end{array} \right\} \text{corr. } \angle\text{s, } \overline{AB} \parallel \overline{CD}.$$

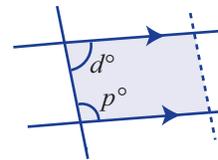
The reasoning in Example 2.1 combines corresponding angles with familiar facts about vertical and supplementary angles. To avoid having to determine all eight angles each time, it is convenient to restate the “corresponding angles are equal” fact in two ways. Each restatement begins by naming a type of angle pair.

First Restatement: Two angles are a pair of **interior angles along a transversal** if they are between the two lines and on the same side of a transversal.

One sees that *if the lines are parallel, the interior angles are supplementary*. The pictures below show two classroom explanations: one using corresponding angles, the other using the grade 5 fact about adjacent interior angles in a parallelogram. In contrast, in the pictures on the right, the lines are not parallel and the interior angles are not supplementary.



The shaded angles are equal.
 $p + s = 180$.
 Therefore $p + d = 180$.

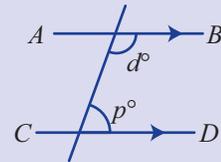


Draw a line to make a parallelogram.
 Then $p + d = 180$.

If a transversal intersects two parallel lines, then the interior angles on the same side of the transversal are supplementary, i.e.,

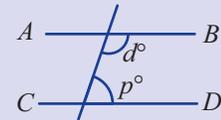
$$\text{if } \overline{AB} \parallel \overline{CD}, \text{ then } d + p = 180.$$

(Abbreviation: **int. \angle s, $\overline{AB} \parallel \overline{CD}$** .)

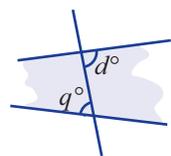


Conversely, if $d + p = 180$, then $\overline{AB} \parallel \overline{CD}$.

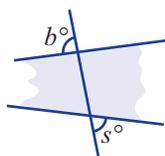
(Abbreviation: **int. \angle s converse**.)



Second Restatement: Pairs of angles on the opposite sides of a transversal are called **alternate angles**. The left-hand pictures below show two pairs of alternate angles. The example on the right shows why, when a transversal intersects parallel lines, alternate interior angles are equal.

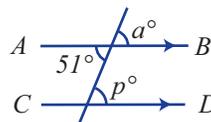


alternate interior angles



alternate exterior angles

In the figure, $AB \parallel CD$. Find p .

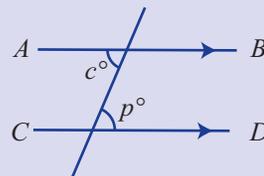


$$\begin{aligned} a &= 51 && \text{vert. } \angle\text{s,} \\ p &= 51 && \text{corr. } \angle\text{s, } \overline{AB} \parallel \overline{CD}. \end{aligned}$$

If a transversal intersects two parallel lines, then alternate interior angles are equal, i.e.,

$$\text{if } \overline{AB} \parallel \overline{CD}, \text{ then } c = p.$$

(Abbreviation: **alt. \angle s**, $\overline{AB} \parallel \overline{CD}$.)



Conversely, if $c = p$, then $\overline{AB} \parallel \overline{CD}$.

(Abbreviation: **alt. \angle s converse**.)



EXERCISE 2.2. Read Class Activity 4 (pages 245-246) of *New Elementary Mathematics 1*. What terms are introduced in this activity? On page 246, match the statements in the blue box with the statements in the blue boxes above.

Converses

Each fact in this section is paired with its converse. All students should be able to distinguish a statement from its converse. The distinction is a simple point of logic that arises in many contexts, but is rarely taught outside of geometry. A complete explanation requires only a few sentences and some examples.

Every “if... then” statement has a **converse**. The converse of the statement “If A is true then B is true” is the statement obtained by reversing the roles of A and B , namely “If B is true then A is true”. Here are three simple examples.

Statement: If an animal is a woman then it is human. (TRUE)

Converse: If an animal is human then it is a woman. (FALSE)

Statement: If two angles are vertically opposite, then they have equal measure. (TRUE)

Converse: If two angles have equal measure, then they are vertically opposite. (FALSE)

Statement: If a triangle is isosceles then it has two equal sides. (TRUE)

Converse: If a triangle has two equal sides then it is isosceles. (TRUE)

In the first two cases, the statement is true but its converse is false. In the third case, both the statement and its converse are true. Thus one should not confuse a statement with its converse. Knowing that one is true tells us nothing about the other.

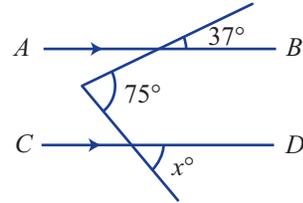
The blue boxes in this section give three statements about parallel lines, each paired with its converse. For all three facts, both the statement and its converse are true.

Auxiliary Lines

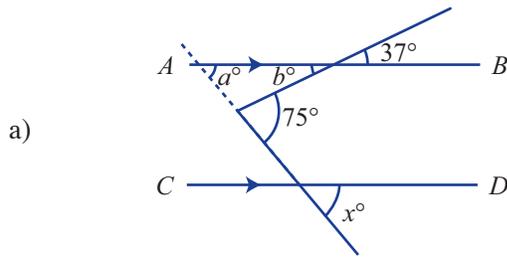
Some geometry problems can be solved easily after modifying the given figure. If you are stuck, it may help to extend an existing line or to add a new line; both are called *auxiliary lines*. You can also give letter names to angles not named in the original figure.

Hint: Look for an auxiliary line that forms a new triangle, or one that is parallel or perpendicular to an existing line.

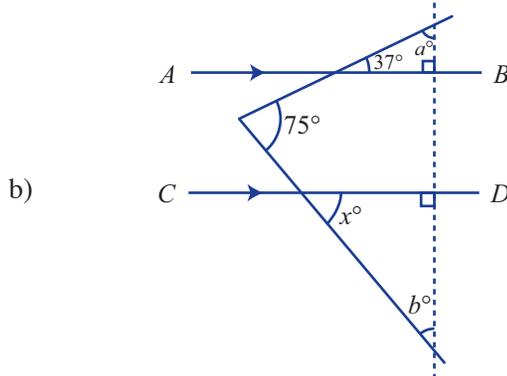
EXAMPLE 2.3. In the figure, $\overline{AB} \parallel \overline{CD}$. Find the value of x .



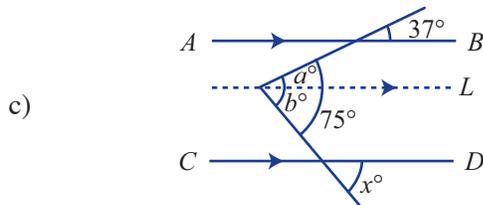
Below are three student solutions. Each uses an auxiliary line (the dotted line).



$$\begin{aligned} a &= x, & \text{corr. } \angle\text{s, } \overline{AB} \parallel \overline{CD}, \\ b &= 37, & \text{vert. } \angle\text{s,} \\ a + b &= 75, & \text{ext. } \angle\text{s of a } \Delta, \\ x + 37 &= 75, \\ \therefore x &= 38. \end{aligned}$$



$$\begin{aligned} a + 37 &= 90, & \angle \text{ sum of rt. } \Delta, \\ \therefore a &= 53. \\ a + b + 75 &= 180, & \angle \text{ sum of } \Delta, \\ 53 + b + 75 &= 180, \\ b + 128 &= 180, \\ \therefore b &= 52. \\ x + 52 &= 90, & \angle \text{ sum of rt. } \Delta, \\ \therefore x &= 38. \end{aligned}$$



$$\begin{aligned} a &= 37, & \text{corr. } \angle\text{s, } \overline{AB} \parallel L. \\ 37 + b &= 75, & \angle\text{s add,} \\ \therefore b &= 38. \\ \therefore x &= 38, & \text{corr. } \angle\text{s, } L \parallel \overline{DC}. \end{aligned}$$

Caution: In c), the line L was drawn parallel to \overline{AB} . It does not bisect the center angle. (One could draw the actual angle bisector, but then it would not be parallel to \overline{AB} or \overline{CD}).

In many figures there are several possibilities for auxiliary lines and it may not be immediately clear which is best. Often it doesn't matter: many problems can be solved in several ways, using different auxiliary lines. Some approaches may be more efficient than others. For the solutions in Example 2.3, the auxiliary line in b) leads to a complicated solution, while the one in c) yields a short efficient solution.

The existence of different routes to the same conclusion is one of the joys of geometry. As a teacher, it is important to give your students opportunities to compare approaches. When solving unknown angle problems, students can learn just as much from seeing a second and a third solution as they can from the first. There is pleasure in watching different approaches unfold and end with exactly the same conclusion. Students, working alone or in small groups, can enjoy a friendly competition to come up with the simplest solution, and can enjoy showing off their solutions to the class.

Homework Set 11

1. Write the converse to the following statements. State whether the converse is true or false.
 - a) If a baby is hungry, then it cries.
 - b) If it rained, the grass is wet.
 - c) If a triangle has 3 congruent sides, then it has 3 congruent angles.
2. Convert to an “if-then” statement, then give the converse:
 - a) All tall men play basketball.
 - b) An angle measuring less than 90° is acute.
 - c) A square is a rhombus.
3. Write a Teacher's Solution for Problem 2 on page 248. Your first line should be “Draw line $L \parallel$ to \overline{BA} ”.
4. Answer Problem 7 on page 249. Include a reason for each pair of parallel lines.
5. Teacher's Solutions for parts (a) through (f) on page 250. Be sure to include a picture for each.
6. Find x for Problems (g) through (j) on the same page (no need for a Teacher's Solution).
7. On page 251, write Teacher's Solutions to Problems 9a, 9b and 10d.

Open NEM1 to page 247. Carefully read the solution to parts (a) and (b) of Worked Example 4. Notice that these follow the Teacher's Solution format, and that the solutions require

- *adding a new line to the figure.*
- *giving letter names to some angles that are not present or not named in the original figure.*

Look over pages 262–263 of NEM1. These quickly review the triangle facts students learned in grade 5 and give examples — in Teacher's Solution format — that use these facts to solve multi-step unknown angle problems.

8. On page 264–265, write Teacher's Solutions to Problems 3f, 4d, 4i, 4j, 4m, 4n and 4p.
9. Give a Teacher's Solution to Problem 1 on page 286 of NEM1.

3.3 Angles of a Polygon

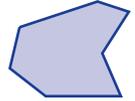
Children first encounter polygons in pre-school or kindergarten when they learn to name triangles, squares, rectangles, pentagons, etc. Polygons remain central objects of study throughout the K-12 geometry curriculum. Curiously, as we will explain, the word “polygon” has two different meanings in school mathematics, and the meaning shifts depending on context.

The main focus of this section is on finding the sum of the interior and exterior angles of an n -sided polygon, and applying the results. This topic is included in most middle school

curricula. It is a simple example of “building new facts from known ones”, and the facts learned can be used to solve interesting unknown angle problems. Here, for the first time, students learn to make statements about n -sided polygons without specifying the number n . It is an ideal place in the curriculum for discussing the distinction between inductive and deductive reasoning.

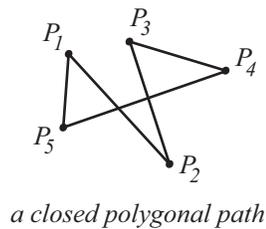
In the early grades, polygons are usually regarded as regions in the plane whose boundary is a union of straight segments. Students are given the school definition below, and lots of examples (as described in Section 2.2). In the school definition, polygons have interiors, so the meaning of “interior angles” is clear.

School Definition: A region enclosed by 3 straight segments is a triangle.
A region enclosed by n straight segments is an n -sided **polygon**.



Children also learn to use the correct mathematical definition, in which a polygon is a collection of line segments. The definition is based on the same idea as connect-the-dots puzzles: draw a segment from the first point to the second, from the second to the third, etc., and end by connecting the last to the first point, thereby “closing up” the figure. The definition is clearest when given in two parts.

DEFINITION 3.1. Given $n \geq 3$ points P_1, P_2, \dots, P_n , all different, the corresponding **closed polygonal path** is the collection of segments $\overline{P_1P_2}, \overline{P_2P_3}, \dots, \overline{P_nP_1}$. The points are called the **vertices** and the segments are called the **sides** of the polygonal path.



DEFINITION 3.2. An n -sided **polygon** or n -gon is a closed polygonal path in a plane with $n \geq 3$ vertices such that

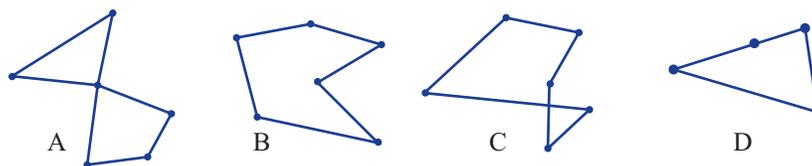
- (i) the sides intersect only at their endpoints and
- (ii) no adjacent sides are collinear.

Conditions (i) and (ii) may seem awkward, but they are needed to make this definition compatible with the school definition above.

Polygons separate the plane into two regions, the interior and the exterior. A polygon together with its interior is called a **polygonal region**; the school definition is actually the definition of a polygonal region. Condition (ii) ensures that the count of sides and vertices is the same as the count obtained from the school definition.



EXERCISE 3.3. Which of the figures below is a polygon? Which violates the requirement that the vertices be distinct? Is figure D a triangle or a quadrilateral?



Definition 3.2 is different from the school definition, yet the two coexist through elementary and middle school. Sometimes the word “polygon” refers to a union of segments, and sometimes it means a region. In most textbooks, including the always-careful Primary Mathematics and New Elementary Mathematics books, the meaning of words like “triangle”, “rectangle” and “polygon” shifts according to the topic being covered.

- When discussing area, “polygons” are regions.
- When finding unknown angles, polygons are unions of segments.
- For some topics, the distinction is not important. One example is the topic on the next page: the sum of interior angles.

Teachers should be alert to possible confusion. When clarity is needed, both teachers and students should speak of “triangular regions” and “polygonal regions”. This long-winded terminology becomes tiresome in studying topics, such as area, where one is *always* considering regions. In such situations, it is fine to say “triangle” and “quadrilateral” instead of “triangular region” and “quadrilateral region” provided that all students are aware that the words refer to regions.

The parts of a polygon are named using terms students learned when studying triangles and quadrilaterals: vertex, side, diagonal, interior angle, and exterior angle. Look at page 271 in New Elementary Mathematics 1 to see how the terms (*vertex*, *side*, *diagonal*, *interior angle*, and *exterior angle*) are reviewed for 7th grade students simply by drawing a single picture.

Two other terms frequently enter discussions of polygons. The first gives a name to the most commonly-seen examples of polygons.

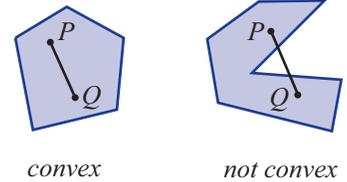
DEFINITION 3.4. A polygon is **regular** if (i) all sides have equal length, and (ii) all angles have equal measure.

Most children are familiar with the regular polygons below. In fact, teachers should be sure that students realize that words like “pentagon” do not automatically refer to a regular pentagon.



EXERCISE 3.5. A regular pentagon has rotational symmetry. What is an angle of rotation for this symmetry?

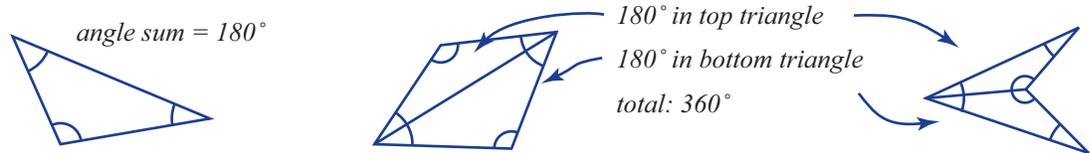
A polygon is **convex** if, for any two points P and Q in the interior of the polygon, the segment \overline{PQ} lies completely in the interior of the polygon. Any polygon with an interior angle greater than 180° is not convex.



Sum of the Interior Angles

We already know that the sum of the interior angles of a triangle is 180° (“one straight angle”). One can ask the analogous question for polygons: What is the sum of the interior angles of an n -sided polygon? This question is a standard middle school topic, and it has an elegant answer.

As a first step, notice that any quadrilateral has a diagonal that splits its interior into two triangles. Hence the angle sum of a quadrilateral is “two straight angles”, which is 360° .

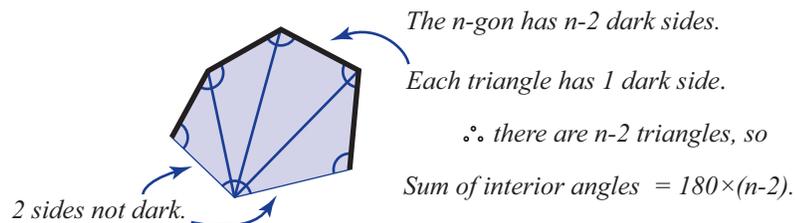


EXERCISE 3.6. Find the sum of the interior angles of a pentagon by splitting it into triangles.

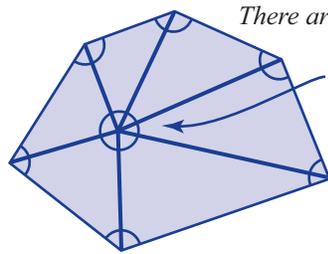
One can continue on to 6-gons, 7-gons, and so forth, until a pattern emerges. This is the approach taken in *New Elementary Mathematics 1* on page 272. It is an example of inductive reasoning. In general, **inductive reasoning** is the type of reasoning in which one uses a limited number of cases to draw a conclusion or make a prediction. Inductive reasoning supports a conclusion, but does not guarantee that it is correct because not all cases have been examined.

Geometry is based on **deductive reasoning** — reasoning in which a conclusion follows logically from previously known facts. For the case of the angle sum of an n -gon, an argument using deductive reasoning gives the definitive answer for all n . Here are two deductive arguments for showing the sum of interior angles in an n -gon is $180(n - 2)$. In both, we assume that the polygon is convex.

Vertex Method. Choose one vertex. Draw lines from that vertex to each of the other vertices, thereby decomposing the n -gon into triangles. Darken the opposite side (with respect to the vertex) of each triangle to help count triangles.



Interior Point Method. Choose a point in the interior. Draw lines from that interior point to every vertex, thereby decomposing the n -gon into n triangles.



There are 180° in each triangle.

Center angles total 360° .

The remaining angles form the interior angles of the polygon.

$$\begin{aligned} \therefore \text{Sum of interior angles} &= 180 \times n - 360 \\ &= 180 \times (n-2). \end{aligned}$$

The case of non-convex polygons is more complicated. But by drawing pictures like the one below, you should be able to convince yourself that every non-convex polygonal region with n sides can be partitioned into $(n - 2)$ triangular regions — in fact there are often many ways of doing this.



A 7-gon decomposed into 5 triangles (in two different ways).

The sum of the interior angles of an n -gon is $180(n - 2)$ degrees.

(Abbreviation: \angle **sum of n -gon.**)

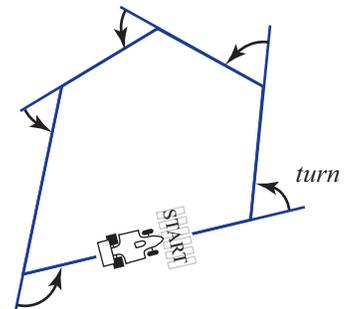
EXERCISE 3.7. Do Problem 1 in Class Activity 3 of NEMI on page 272.

- What do you conclude about the sum of interior angles of an n -sided polygon?
- Is this activity an example of building a fact inductively or deductively?

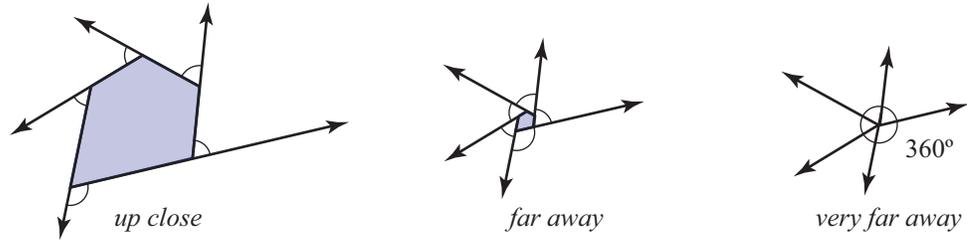
Sum of the Exterior Angles

At each vertex of a convex polygon there is an interior angle and also two exterior angles. There is a formula for the sum of the exterior angles that is analogous to the one for interior angles, but simpler. Here are three classroom explanations.

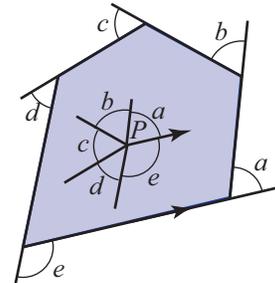
Racetrack Method. Imagine the polygon as a racetrack. A car starts on one side and moves around the track counter-clockwise. At the first vertex it turns left through an angle equal to the exterior angle at that vertex. When it gets to the second vertex it turns again, by an amount equal to the second exterior angle. When the car returns to its starting point it has completed one full turn — 360 degrees. Thus the sum of the exterior angles of the polygon is 360° .



Zoom-out Method. Moving around the polygon in one direction, extend each side to a ray. Then “zoom-out”, looking at the polygon from farther and farther away. From 1000 miles away, the figure looks like a point and the exterior angles clearly add to 360° .



Base Point Method. Fix a point P and draw segments at P parallel to the sides of the polygon as shown. This creates angles congruent to the exterior angles of the polygon whose sum is 360° .



The sum of the exterior angles, one at each vertex, of a convex polygon is 360° .

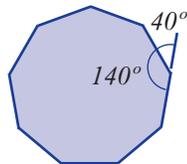
(Abbreviation: **ext. \angle s of polygon.**)

EXERCISE 3.8. In the above “Base Point Method” picture, why are the two angles labeled a congruent to each other?

Recall that there are two exterior angles at each vertex. In the above explanations, the exterior angles were chosen in a consistent manner (the ones that arise by “turning left” at each vertex). But there is no need to be consistent: because the two exterior angles at each vertex are congruent, we can arbitrarily choose one external angle at each vertex and still have a sum of 360° .

EXAMPLE 3.9. What is the measure of each interior angle of a regular 9-gon?

Teacher’s Solution:



Sum of exterior angles: 360 ,
 Each exterior angle: $360 \div 9 = 40$,
 Each interior angle: $180 - 40 = 140$.

\therefore Each interior angle of a regular 9-gon is 140° .

In your homework, you will consider whether the sum of the exterior angles is 360° for polygons that are not convex.

Which is Taught First?

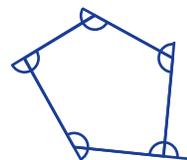
For convex polygons, we now have a number of ways of calculating the two numbers:

$$I = \text{sum of the interior angles,}$$

$$E = \text{sum of the exterior angles.}$$

In fact, the numbers I and E are related. At each vertex there is an interior and an exterior angle that, together, make a straight angle. Adding up the angles associated with each of the n vertices of an n -gon shows that

$$I + E = n \text{ straight angles} = 180n.$$



$$\begin{aligned} &5 \text{ interior angles} \\ &+ 5 \text{ exterior angles} \\ &= 5 \text{ straight angles} \\ &= 5 \times 180^\circ. \end{aligned}$$

This formula relates I and E : if we already know I we can use it to find E , and if we already know E we can use it to find I . For example, one might use the racetrack method to find that $E = 360^\circ$; the above relation then gives

$$I = 180n - 360 = 180(n - 2).$$

Some textbooks first focus on the sum of the interior angles, while others start with the sum of the exterior angles. Others give separate discussions, as if these were two unrelated facts. All of these approaches are correct, but teachers and students should be aware that the sum formulas for interior and exterior angles are two sides of the same coin.

Homework Set 12

- Read Problem 1 of Class Activity 3 on page 272 of NEM1. Fill in the table in the book (but not in your homework). What do you obtain for the sum of the angles of an n -gon? Is this an example of inductive or deductive reasoning?
- (Study the textbook!) Read pages 270-274 in NEM 1. Do Problems 1-11 on pages 274-5.
- Continuing, do parts (a), (b) and (c) of Problem 12. *Hint:* For (b), mark the angles a , b , etc. and then express other angles in the figure in terms of these. For example, you might label one angle $180 - a - b$.
- Continuing, turn the page and do Problems 13 and 14 (the given ratios are multiples of some unit; call that unit x).
- Find the measure of an interior angle of a regular polygon with
 - 5 sides
 - 6 sides
 - 20 sides
 - n sides
- A polygon has m sides. If one of its interior angles is 80 degrees and the other interior angles are each equal to 160 degrees, find the value of m .
- Give a Teacher's Solution to the following problem.

The figure is a regular pentagon. Find, in order, angles a and b , angle c , angle d , and angle e . Conclude that $c = d = e$, so the top interior angle is *trisected*.
- Draw a non-convex 10-sided polygon. Partition your polygon into triangles as in the example on page 70. How many triangles did you get?
- Here is a non-convex polygon. Is the sum of the measures of its exterior angles 360° ? Explain.
- * Solve problem 5 on page 287 of NEM1.
- * Solve problem 6 on page 287 of NEM1.