# MHD, the $\nabla \cdot \mathbf{B}$ Constraint and Central Schemes

Jorge Balbás

Dept. of Mathematics, University of Michigan, Ann Arbor
jbalbas@umich.edu

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Joint work with E. Tadmor and C.C. Wu

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- equation of state:

$$e = \frac{1}{2}\rho v^2 + \frac{1}{2}B^2 + \frac{p}{\gamma - 1}$$

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- If the numerical scheme fails to satisfy $\nabla \cdot \mathbf{B} = 0$, the solution becomes unstable,
  - The Lorentz force in the momentum flux involves terms proportional to $\nabla \cdot \mathbf{B}$

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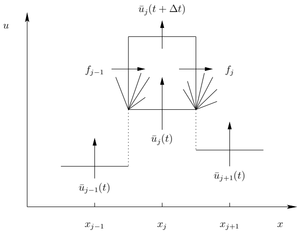
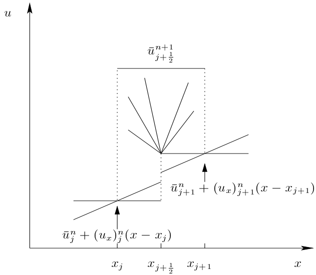
$$\mathbf{B} \cdot F = \mathbf{B} \cdot \left[ \nabla \cdot \left( \frac{1}{2} B^2 \mathbb{I}_{3\times 3} - \mathbf{B}\mathbf{B}^\top \right) \right] = 0$$

  - In non-smooth regions, the order of convergence of numerical schemes decreases (to first order), the error in $\nabla \cdot \mathbf{B}$ grows, and builds over time.

# What to Do – Discontinuous Solutions

A common approach consists on adapting an existing scheme from gas dynamics, e.g., Godunov-type scheme (in one space dimension)

| **Upwind Scheme** | **Central Scheme** |
|---|---|
|  |  |
| requires a Riemann solver to distinguish from right- and left-going waves | evolves solution over staggered grid, no Riemann solver is needed, but staggering requires smaller time step |

# What to Do – The Constraint $\nabla \cdot \mathbf{B} = 0$

- Hodge Projection (Brackbill and Barnes, 1980):
  - After updating the solution form $t$ to $t + \Delta t$, the magnetic field, $\mathbf{B}$, is reprojected onto its divergenge free component, by solving

$$\Delta\phi = -\nabla \cdot \mathbf{B}$$

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- Eight Wave Formulation (Powell et. al., 1994):
  - A source term proportional to $\nabla \cdot \mathbf{B}$ is added to the momentum, energy and transport equations
  - This approach keeps $\nabla \cdot \mathbf{B}$ small (to the order of the scheme), but it follows from the non conservative formulation of MHD equations

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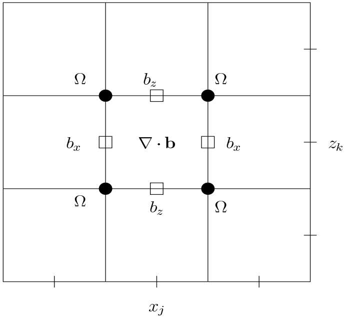
Constrained Transport (Evans and Hawley, 1988):




- This method takes advantage of the fact that (in the *xz*-plane)

$$\frac{\partial B^x}{\partial t} = -\frac{\partial \Omega}{\partial z}, \qquad \frac{\partial B^z}{\partial t} = \frac{\partial \Omega}{\partial x}$$

where $\Omega = -\mathbf{v} \times \mathbf{B}$ is the $y$ component of the electric field, to evolve a magnetic field centered at the cell interfaces as

$$b^{x,n+1}_{j+\frac{1}{2},k} = b^{x,n}_{j+\frac{1}{2},k} - \frac{\Delta t}{\Delta z}\left(\Omega^{n+\frac{1}{2}}_{j+\frac{1}{2},k+\frac{1}{2}} - \Omega^{n+\frac{1}{2}}_{j+\frac{1}{2},k-\frac{1}{2}}\right)$$

$$b^{z,n+1}_{j,k+\frac{1}{2}} = b^{z,n}_{j,k+\frac{1}{2}} + \frac{\Delta t}{\Delta x}\left(\Omega^{n+\frac{1}{2}}_{j+\frac{1}{2},k+\frac{1}{2}} - \Omega^{n+\frac{1}{2}}_{j-\frac{1}{2},k+\frac{1}{2}}\right)$$

# What to Do – The Constraint $\nabla \cdot \mathbf{B} = 0$

- The magnetic field $\mathbf{B}^{n+1}$ is then recovered as the average

$$B^{x,n+1}_{j,k} = \frac{1}{2}(b^{x,n+1}_{j+\frac{1}{2},k} + b^{x,n+1}_{j-\frac{1}{2},k}),$$

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- And the divergence is conserved in the sense

$$(\nabla \cdot \mathbf{b})^{n+1}_{j,k} = \frac{b^{x,n+1}_{j+\frac{1}{2},k} - b^{x,n+1}_{j-\frac{1}{2},k}}{\Delta x} + \frac{b^{z,n+1}_{j,k+\frac{1}{2}} - b^{z,n+1}_{j,k-\frac{1}{2}}}{\Delta z} = (\nabla \cdot \mathbf{b})^{n}_{j,k}$$

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$$\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} u_t \; dt \; dx = -\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} f(u)_x \; dt \; dx$$

over the control volume $[x_j, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]$,

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$$\bar{u}^{n+1}_{j+\frac{1}{2}} = \bar{u}^{n}_{j+\frac{1}{2}} - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \Big[ f(u(x_{j+1}, t)) - f(u(x_j, t)) \Big] dt$$

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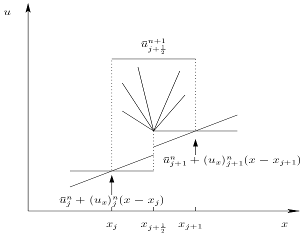


1. From the cell averages $\{\bar{u}^n_j\}$, a non-oscillatory polynomial reconstruction,

$$\tilde{u}(x, t^n) = \sum_j p_j(x, t^n) \cdot \mathbf{1}_{I_j},$$

is formed to recover $\{\bar{u}^n_{j+\frac{1}{2}}\}$; where $I_j = [x_j - \Delta x/2, x_j + \Delta x/2]$.

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$$\bar{u}_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}[\bar{u}_j^n + \bar{u}_{j+1}^n] + \frac{1}{8}[u_j' - u_{j+1}'] - \lambda\Big[f(u_{j+1}^{n+\frac{1}{2}}) - f(u_j^{n+\frac{1}{2}})\Big].$$

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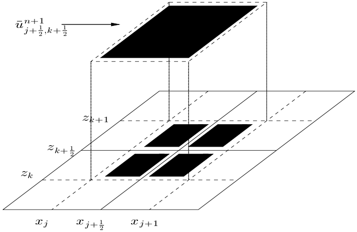


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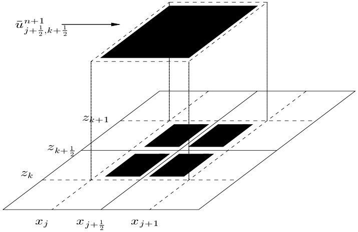


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$$u^{n+\frac{1}{2}}_{j,k} := \bar{u}^{n}_{j,k} - \frac{\lambda}{2} f'_{j,k} - \frac{\mu}{2} g^{\backprime}_{j,k},$$

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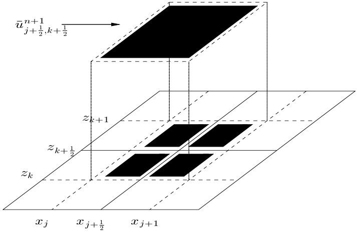


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- corrector

$$\begin{aligned}
\bar{u}^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} &= \frac{1}{4}(\bar{u}^n_{j,k} + \bar{u}^n_{j+1,k} + \bar{u}^n_{j,k+1} + \bar{u}^n_{j+1,k+1}) + \frac{1}{16}(u'_{j,k} - u'_{j+1,k}) \\
&- \frac{\lambda}{2}\left[f(u^{n+\frac{1}{2}}_{j+1,k}) - f(u^{n+\frac{1}{2}}_{j,k})\right] + \frac{1}{16}(u'_{j,k+1} - u'_{j+1,k+1}) - \frac{\lambda}{2}\left[f(u^{n+\frac{1}{2}}_{j+1,k+1}) - f(u^{n+\frac{1}{2}}_{j,k+1})\right] \\
&+ \frac{1}{16}(u^{\backprime}_{j,k} - u^{\backprime}_{j,k+1}) - \frac{\mu}{2}\left[g(u^{n+\frac{1}{2}}_{j,k+1}) - g(u^{n+\frac{1}{2}}_{j,k})\right] \\
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\end{aligned}$$

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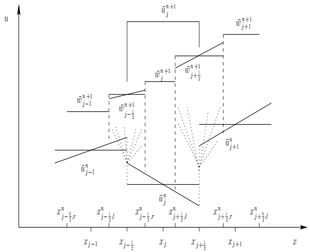
Modified central differencing (Kurganov and Tadmor, 2000)



- Using the information provided by the local speed of propagation,

$$a^n_{j+\frac{1}{2}} = \max_{u \in \mathcal{C}(u^-_{j+\frac{1}{2}},\, u^+_{j+\frac{1}{2}})} \rho\left(\frac{\partial f}{\partial u}(u)\right),$$

where

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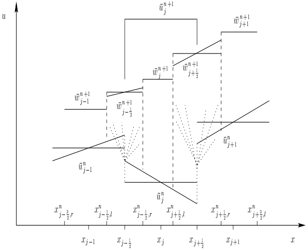

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- we distinguish between the regions where the solution remains smooth – no Riemann fans, and regions where discontinuities propagate

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- two sets of evolved values are calculated:
  - staggered values over non-smooth regions $\{\bar{w}_{j+\frac{1}{2}}^{n+1}\}$
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$$\frac{d}{dt}\bar{u}_j(t) = \lim_{\Delta t \to 0} \frac{\bar{u}_j^{n+1} - \bar{u}_j^n}{\Delta t} = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x},$$

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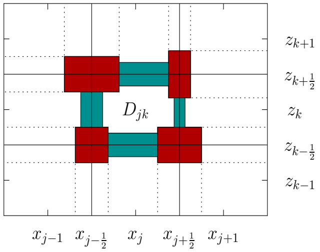
- where $H_{j+\frac{1}{2}}(t) := \frac{f(u^{+}_{j+\frac{1}{2}}(t)) + f(u^{-}_{j+\frac{1}{2}}(t))}{2} - \frac{a_{j+\frac{1}{2}}(t)}{2}\left[u^{+}_{j+\frac{1}{2}}(t) - u^{-}_{j+\frac{1}{2}}(t)\right]$

# Semi-discrete Central Schemes – Two Dimensions

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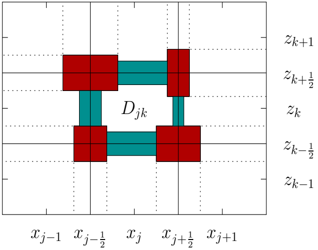


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- staggered evolution over red cells

1

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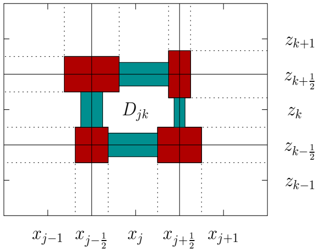


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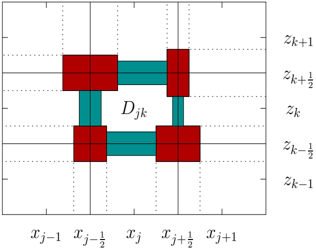


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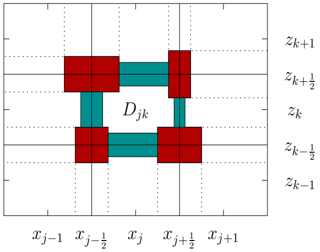


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- staggered evolution in one direction over green strips
- non-staggered evolution over $D_{j,k}$, and
- reprojecting over original cells and taking the limit as $\Delta t \to 0$, we arrive at:

$$\frac{d}{dt}\bar{u}_{j,k}(t) = -\frac{H^x_{j+\frac{1}{2},k}(t) - H^x_{j-\frac{1}{2},k}(t)}{\Delta x} - \frac{H^z_{j,k+\frac{1}{2}}(t) - H^z_{j,k-\frac{1}{2}}(t)}{\Delta z}$$

1

# Central Schemes – Reconstruction

Examples of non-oscillatory reconstructions:

- second order *minmod* reconstruction (Van Leer, 1979)

$$p_{j,k}(x,z) = \bar{u}^n_{j,k} + u'_{j,k}\frac{(x - x_j)}{\Delta x} + u^{\backprime}_{j,k}\frac{(z - z_k)}{\Delta z}$$

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- third order CWENO reconstruction (Kurganov and Levy, 2000) direction-by-direction

$$p_{j,k}(x,z_k) = w_{\mathrm{L}} P_{\mathrm{L}}(x,z_k) + w_{\mathrm{C}} P_{\mathrm{C}}(x,z_k) + w_{\mathrm{R}} P_{\mathrm{R}}(x,z_k), \quad x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$$

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- fourth order genuinely two-dimensional reconstruction (Levy et. al., 2002):

$$p_{j,k}(x,z) = \sum_{r,s=-1}^{1} w_{r,s} P_{r,s}(x,z)$$

# Semi-discrete Central Schemes – Time Evolution

Solution evolved with SSP RK Schemes (Shu, 1988, S. Gottlieb et. al., 2001),

Example: Third-order scheme

$$\begin{aligned} u^{(1)} &= u^{(0)} + \Delta t C[u^{(0)}], \\ u^{(2)} &= u^{(1)} + \frac{\Delta t}{4}(-3C[u^{(0)}] + C[u^{(1)}]), \\ u^{n+1} &:= u^{(3)} = u^{(2)} + \frac{\Delta t}{12}(-C[u^{(0)}] - C[u^{(1)}] + 8\ C[u^{(2)}]), \end{aligned}$$

where

$$C[w(t)] = -\frac{H^x_{j+\frac{1}{2},k}(w(t)) - H^x_{j-\frac{1}{2},k}(w(t))}{\Delta x} - \frac{H^z_{j,k+\frac{1}{2}}(w(t)) - H^z_{j,k-\frac{1}{2}}(w(t))}{\Delta z}$$

# Central Schemes – Solenoidal Constraint

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# Central Schemes – Solenoidal Constraint

How do we enforce $\nabla \cdot \mathbf{B} = 0$?

- We don't do anything!
- Numerical results indicate central schemes maintain $\nabla \cdot \mathbf{B}$ small ($\sim 10^{-13}$)
- Also, using the constraint transport approach and notation, it can be shown that the magnetic field, as evolved by the second order fully-discrete staggered scheme (JT), can be written as

$$B^{x,n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{2}\left(b^{x,n+1}_{j,k+\frac{1}{2}} + b^{x,n+1}_{j+1,k+\frac{1}{2}}\right)$$

with

$$b^{x,n+1}_{j,k+\frac{1}{2}} = \tilde{b}^{x,n}_{j,k+\frac{1}{2}} - \frac{\Delta t}{\Delta z}\left(\Omega^{n+\frac{1}{2}}_{j,k+\frac{1}{2}} - \Omega^{n+\frac{1}{2}}_{j+1,k+\frac{1}{2}}\right),$$

and a similar expression for $B^{z,n+1}_{j+\frac{1}{2},k+\frac{1}{2}}$

# Central Schemes – Solenoidal Constraint

This result allows us to write

$$(\nabla \cdot \bar{\mathbf{B}})^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = (\nabla \cdot \bar{\mathbf{B}})^{n}_{j+\frac{1}{2},k+\frac{1}{2}}$$

where $\bar{\mathbf{B}}^{n}_{j+\frac{1}{2},k+\frac{1}{2}}$ is the reconstructed cell average of the magnetic field at the vertex $(j+\frac{1}{2}, k+\frac{1}{2})$ (not the cell center) at time $t = t^n$

# Brio-Wu Rotated Shock Tube

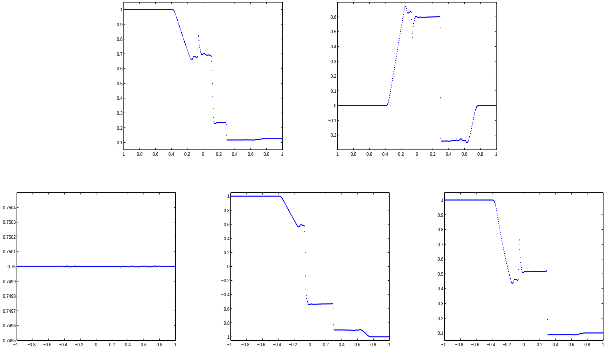
- One-dimensional Riemann problem with initial states given by

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, p)^\top = \begin{cases} (1, 0, 0, 0, 0.75, 0, 1, 1)^\top & \text{for } x < 0 \\ (0.125, 0, 0, 0, 0.75, 0, -1, 0.1)^\top & \text{for } x > 0 \end{cases}$$

- Solved over a two dimensional domain with the direction of the flow rotated 45°
- Solution computed up to $t = 0.2$, $x \in [-1, 1]$, with $600 \times 600$ grid points, $\gamma = 2$.

# Brio-Wu Rotated Shock Tube

## Solution at $t = 0.2$



From top to bottom and from left to right: density, transverse velocity, transverse magnetic filed, parallel magnetic field, and pressure. The divergence of the reconstructed polynomial $\sim 10^{-13}$. Results computed with Jacobian free formulation of 2nd order JT scheme.

# Orszag-Tang Vortex System

- This test problem considers the evolution of a compressible vortex system with several interacting shock waves
- The initial data is given by

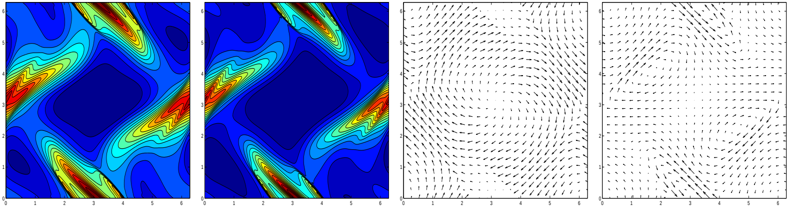
$$\begin{aligned} \rho(x,z,0) &= \gamma^2, \quad v_x(x,y,0) = -\sin z, \quad v_z(x,z,0) = \sin x, \\ p(x,z,0) &= \gamma, \quad B_x(x,z,0) = -\sin z, \quad B_z(x,z,0) = \sin 2x, \end{aligned}$$

  where $\gamma = 5/3$.

- The problem is solved in $[0, 2\pi] \times [0, 2\pi]$, with periodic boundary conditions in both $x$- and $z$-directions using a uniform grid with $288 \times 288$ cells. Results computed with 3rd order semi-discrete scheme, using Kurganov and Levy's CWENO reconstruction.

# Orszag–Tang Vortex System

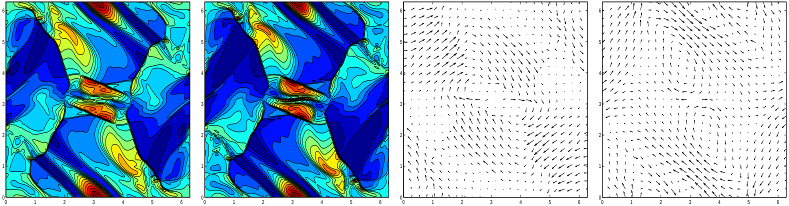
## Solution at $t = 1.0$



Orszag–Tang MHD turbulence problem with a $288 \times 288$ uniform grid. There are 16 contours for density (left) and pressure (second from left). Red–high value, blue–low value. Second from the right: velocity field and right: magnetic filed.

# Orszag–Tang Vortex System

## Solution at $t = 3.0$



Orszag–Tang MHD turbulence problem with a $288 \times 288$ uniform grid. There are 16 contours for density (left) and pressure (second from left). Red–high value, blue–low value. Second from the right: velocity field and right: magnetic filed.

# Shock – Cloud Interaction

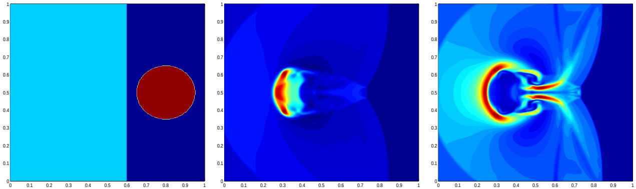
- Disruption of a high density cloud by a strong shock
- Initial conditions

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, p)^\top = \begin{cases} (3.86, 0, 0, 0, 0, -2.18, 2.18, 167.34)^\top & \text{for } x < 0.6 \\ (1, -11.25, 0, 0, 0, 0.564, 0.564, 1)^\top & \text{for } x > 0.6 \end{cases}$$

high density cloud – $\rho = 10$, $p = 1$ – centered at $x = 0.8, y = 0.5$, with radius 0.15,

- Solved up to $t = 0.06$, $(x, z) \in [0, 1] \times [0, 1]$, with $256 \times 256$ grid points, CFL number 0.5 and $\gamma = 5/3$

# Shock – Cloud Interaction



Solution of shock-cloud interaction, left: density at t=0, center: density at t=0.06, right: magnetic field lines at t=0.06. Results compputed with 3rd order semi-discrete scheme.