High-order Central Schemes for Hyperbolic Systems of Conservation Laws

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Joint work with E. Tadmor and C.C. Wu

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Outline

- central schemes for hyperbolic conservation laws: overview and implementation
- central schemes and MHD equations: the $\nabla \cdot \mathbf{B} = 0$ constraint
- some examples: Euler equations of Gas Dynamics and Ideal MHD equations

Hyperbolic Conservation Laws

We consider hyperbolic conservation laws in general

In one space dimension:

$$u_t + f(u)_x = 0,$$

and two space dimensions:

$$u_t + f(u)_x + g(u)_y = 0,$$

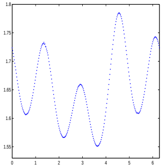
with some initial data

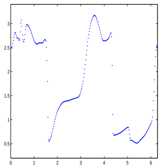
$$u(x, y, 0) = u_0(x, y),$$

where the Jacobian matrices $\frac{\partial f}{\partial u}$ and $\frac{\partial g}{\partial u}$ are diagonizable with real eigen values.

Challenges

- discontinuous solutions: even when the initial conditions are smooth, they evolve into steep gradients





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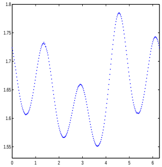

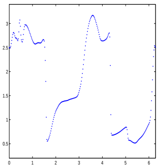

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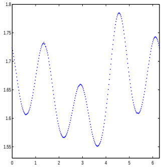

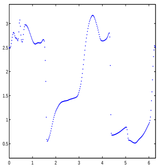

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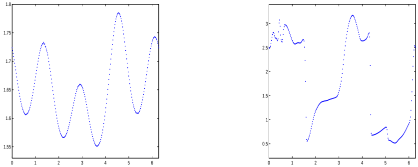
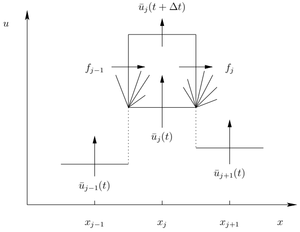


- onset of spurious oscillations
- additional challenges may come from the specific problem, e.g., for MHD equations, we need to solve a large system with an additional constraint
- we seek *efficient* numerical schemes capable of handling these challenges

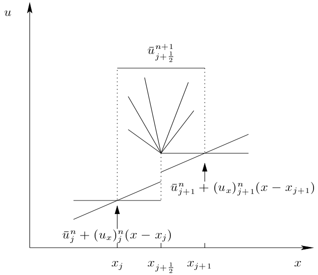
What do central schemes offer?

simplicity: no Riemann solvers

Upwind Scheme



Central Scheme



requires a Riemann solver to distinguish from right- and left-going waves

evolves solution over staggered grid, no Riemann solver is needed, but staggering requires smaller time step

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- straight forward extension to higher space dimensions
- highly adaptable implementation: minor changes required to solve different problems
- easy to parallelize: sequential function calls $\longrightarrow$ concurrent function calls

Fully-discrete Central Schemes – One Dimension

We begin by integrating the conservation law

$$u_t + f(u)_x = 0$$

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$$\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} u_t \; dt \; dx = -\frac{1}{\Delta x}\int_{t^n}^{t^{n+1}}\int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} f(u)_x \; dt \; dx$$

over the control volume $[x_j, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]$,

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over the control volume $[x_j, x_{j+1}] \times [t^n, t^{n+1}]$, this leads the equivalent cell average formulation

$$\bar{u}^{n+1}_{j+\frac{1}{2}} = \bar{u}^{n}_{j+\frac{1}{2}} - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \Big[f(u(x_{j+1}, t)) - f(u(x_j, t)) \Big] dt$$

We now proceed in two steps:

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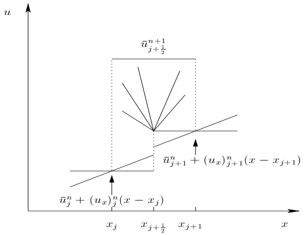

1. From the cell averages $\{\bar{u}^n_j\}$, a non-oscillatory polynomial reconstruction,

$$\tilde{u}(x, t^n) = \sum_j p_j(x, t^n) \cdot \mathbf{1}_{I_j},$$

is formed to recover $\{\bar{u}^n_{j+\frac{1}{2}}\}$; where $I_j = [x_j - \Delta x/2, x_j + \Delta x/2]$.

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The fully discrete approximation reads:

- predictor:

$$u_j^{n+\frac{1}{2}} := \bar{u}_j^n - \frac{\lambda}{2} f_j', \quad \lambda = \frac{\Delta t}{\Delta x},$$

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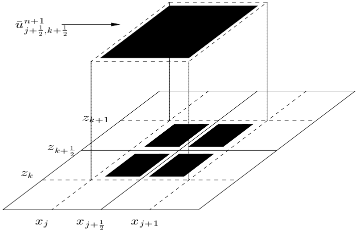
- corrector:

$$\bar{u}_{j+\frac{1}{2}}^{n+1} = \frac{1}{2}[\bar{u}_j^n + \bar{u}_{j+1}^n] + \frac{1}{8}[u_j' - u_{j+1}'] - \lambda\left[f(u_{j+1}^{n+\frac{1}{2}}) - f(u_j^{n+\frac{1}{2}})\right].$$

Fully-discrete Central Schemes – Two Dimensions



The staggered scheme can be extended to two space dimensions

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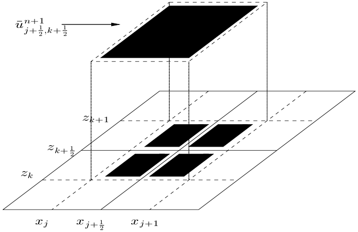

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- predictor

$$u^{n+\frac{1}{2}}_{j,k} := \bar{u}^{n}_{j,k} - \frac{\lambda}{2} f'_{j,k} - \frac{\mu}{2} g^{\backprime}_{j,k},$$

where $\lambda = \frac{\Delta t}{\Delta x}$ and $\mu = \frac{\Delta t}{\Delta z}$

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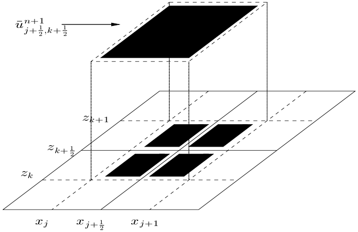

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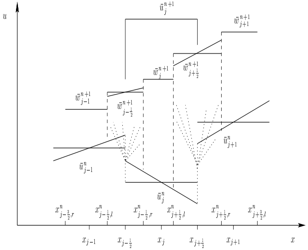
- corrector

$$\begin{aligned}
\bar{u}^{n+1}_{j+\frac{1}{2},k+\frac{1}{2}} &= \frac{1}{4}(\bar{u}^n_{j,k} + \bar{u}^n_{j+1,k} + \bar{u}^n_{j,k+1} + \bar{u}^n_{j+1,k+1}) + \frac{1}{16}(u'_{j,k} - u'_{j+1,k}) \\
&- \frac{\lambda}{2}\left[f(u^{n+\frac{1}{2}}_{j+1,k}) - f(u^{n+\frac{1}{2}}_{j,k})\right] + \frac{1}{16}(u'_{j,k+1} - u'_{j+1,k+1}) - \frac{\lambda}{2}\left[f(u^{n+\frac{1}{2}}_{j+1,k+1}) - f(u^{n+\frac{1}{2}}_{j,k+1})\right] \\
&+ \frac{1}{16}(u^{\backprime}_{j,k} - u^{\backprime}_{j,k+1}) - \frac{\mu}{2}\left[g(u^{n+\frac{1}{2}}_{j,k+1}) - g(u^{n+\frac{1}{2}}_{j,k})\right] \\
&+ \frac{1}{16}(u^{\backprime}_{j+1,k} - u^{\backprime}_{j+1,k+1}) - \frac{\mu}{2}\left[g(u^{n+\frac{1}{2}}_{j+1,k+1}) - g(u^{n+\frac{1}{2}}_{j+1,k})\right]
\end{aligned}$$

Semi-discrete Central Schemes – One Dimension

Modified central differencing (Kurganov and Tadmor, 2000)



- Using the information provided by the local speed of propagation,

$$a^n_{j+\frac{1}{2}} = \max_{u \in \mathcal{C}(u^-_{j+\frac{1}{2}}, u^+_{j+\frac{1}{2}})} \rho\left(\frac{\partial f}{\partial u}(u)\right),$$

where

$$u^+_{j+\frac{1}{2}} := p_{j+1}(x_{j+\frac{1}{2}}) \ \text{ and } \ u^-_{j+\frac{1}{2}} := p_j(x_{j+\frac{1}{2}}),$$

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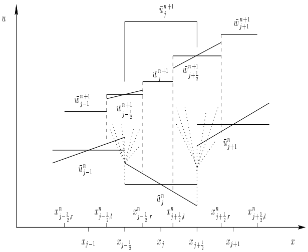

- Using the information provided by the local speed of propagation,

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$$u_{j+\frac{1}{2}}^+ := p_{j+1}(x_{j+\frac{1}{2}}) \text{ and } u_{j+\frac{1}{2}}^- := p_j(x_{j+\frac{1}{2}}),$$

- we distinguish between the regions where the solution remains smooth – no Riemann fans, and regions where discontinuities propagate

Semi-discrete Central Schemes – One Dimension

- two sets of evolved values are calculated:
  - staggered values over non-smooth regions $\{\bar{w}^{n+1}_{j+\frac{1}{2}}\}$
  - non-staggered evolution over smooth regions $\{\bar{w}^{n+1}_{j}\}$

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- or one can take the limit as $\Delta t \rightarrow 0$ to arrive at the semi-discrete formulation:

$$\frac{d}{dt}\bar{u}_j(t) = \lim_{\Delta t \rightarrow 0} \frac{\bar{u}_j^{n+1} - \bar{u}_j^n}{\Delta t} = -\frac{H_{j+\frac{1}{2}}(t) - H_{j-\frac{1}{2}}(t)}{\Delta x},$$

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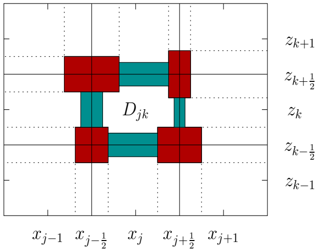
- where $H_{j+\frac{1}{2}}(t) := \frac{f(u^+_{j+\frac{1}{2}}(t))+f(u^-_{j+\frac{1}{2}}(t))}{2} - \frac{a_{j+\frac{1}{2}}(t)}{2}\left[u^+_{j+\frac{1}{2}}(t) - u^-_{j+\frac{1}{2}}(t)\right]$

Semi-discrete Central Schemes – Two Dimensions



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1

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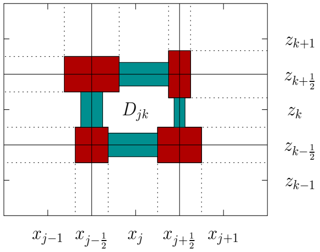
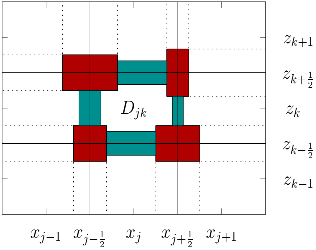


Similarly, in two space dimensions, we apply:

- staggered evolution over red cells

Semi-discrete Central Schemes – Two Dimensions



Similarly, in two space dimensions, we apply:

- staggered evolution over red cells
- staggered evolution in one direction over green strips

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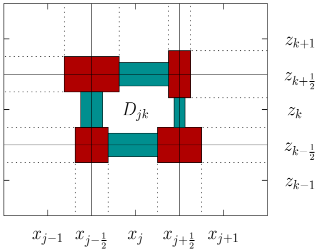

Similarly, in two space dimensions, we apply:

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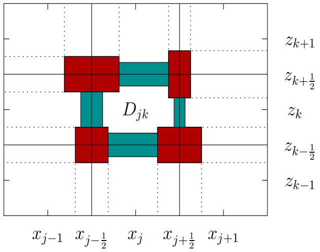

Similarly, in two space dimensions, we apply:

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- non-staggered evolution over $D_{j,k}$, and

- reprojecting over original cells and taking the limit as $\Delta t \to 0$, we arrive at:

$$\frac{d}{dt}\bar{u}_j(t) = -\frac{H^x_{j+\frac{1}{2},k}(t) - H^x_{j-\frac{1}{2},k}(t)}{\Delta x} - \frac{H^z_{j,k+\frac{1}{2}}(t) - H^z_{j,k-\frac{1}{2}}(t)}{\Delta z}$$

Central Schemes – Reconstruction

Example of a non-oscillatory reconstruction:
second order *minmod* reconstruction (Van Leer, 1979)

$$p_{j,k}(x,z) = \bar{u}^n_{j,k} + u'_{j,k}\frac{(x-x_j)}{\Delta x} + u^{\grave{}}_{j,k}\frac{(z-z_k)}{\Delta z}$$

where

$$u'(j,k) = minmod(\alpha\Delta_{+,x}\ \bar{u}^n_{j,k}, \frac{1}{2}\Delta_{0,x}\ \bar{u}^n_{j,k}, \alpha\Delta_{-,x}\ \bar{u}^n_{j,k}),$$

$$u^{\grave{}}(j,k) = minmod(\alpha\Delta_{+,z}\ \bar{u}^n_{j,k}, \frac{1}{2}\Delta_{0,z}\ \bar{u}^n_{j,k}, \alpha\Delta_{-,z}\ \bar{u}^n_{j,k}),$$

with $1 \le \alpha < 2$, and

$$minmod(x_1, x_2, \ldots, x_n) = \begin{cases} \min x_i, & \text{if} \quad x_i > 0 \quad \forall i \\ \max x_i, & \text{if} \quad x_i < 0 \quad \forall i \\ 0 & \text{otherways} \end{cases}$$

Semi-discrete Central Schemes – Time Evolution

Solution evolved with SSP RK Schemes (Gottlieb et. al., 2001),

Example: Third-order scheme

$$
\begin{aligned}
u^{(1)} &= u^{(0)} + \Delta t C[u^{(0)}], \\
u^{(2)} &= u^{(1)} + \frac{\Delta t}{4}(-3C[u^{(0)}] + C[u^{(1)}]), \\
u^{n+1} &:= u^{(3)} = u^{(2)} + \frac{\Delta t}{12}(-C[u^{(0)}] - C[u^{(1)}] + 8\ C[u^{(2)}]),
\end{aligned}
$$

where

$$
C[w(t)] = -\frac{H^x_{j+\frac{1}{2},k}(w(t)) - H^x_{j-\frac{1}{2},k}(w(t))}{\Delta x} - \frac{H^z_{j,k+\frac{1}{2}}(w(t)) - H^z_{j,k-\frac{1}{2}}(w(t))}{\Delta z}
$$

Ideal MHD Equations

- conservation of mass:

$$\rho_t = -\nabla \cdot (\rho \mathbf{v}),$$

- conservation of momentum:

$$(\rho \mathbf{v})_t = -\nabla \cdot [\rho \mathbf{v}\mathbf{v}^\top + (p + \frac{1}{2}B^2)\mathbb{I}_{3\times 3} - \mathbf{B}\mathbf{B}^\top],$$

- conservation of energy:

$$e_t = -\nabla \cdot \left[\left(\frac{\gamma}{\gamma - 1} p + \frac{1}{2}\rho v^2 \right) \mathbf{v} - \left(\mathbf{v} \times \mathbf{B} \right) \times \mathbf{B} \right],$$

- transport equation:

$$\mathbf{B}_t = \nabla \times (\mathbf{v} \times \mathbf{B})$$

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- solenoidal constraint:

$$\nabla \cdot \frac{\partial \mathbf{B}}{\partial t} = \nabla \cdot [\nabla \times (\mathbf{v} \times \mathbf{B})] \quad \Rightarrow \quad \frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0$$

Solenoidal Constraint

- Numerical results indicate central schemes maintain $\nabla \cdot \mathbf{B}$ small

Solenoidal Constraint

- Numerical results indicate central schemes maintain $\nabla \cdot \mathbf{B}$ small
- Also, using the constraint transport approach and notation (Evans and Hawley, 1988), it can be shown that the magnetic field, as evolved by the second order fully-discrete staggered scheme (JT), can be written as

$$B^{x,n+1}_{j+\frac{1}{2},k+\frac{1}{2}} = \frac{1}{2}\left(b^{x,n+1}_{j,k+\frac{1}{2}} + b^{x,n+1}_{j+1,k+\frac{1}{2}}\right)$$

with

$$b^{x,n+1}_{j,k+\frac{1}{2}} = \tilde{b}^{x,n}_{j,k+\frac{1}{2}} - \frac{\Delta t}{\Delta z}\left(\Omega^{n+\frac{1}{2}}_{j,k+\frac{1}{2}} - \Omega^{n+\frac{1}{2}}_{j+1,k+\frac{1}{2}}\right),$$

and a similar expression for $B^{z,n+1}_{j+\frac{1}{2},k+\frac{1}{2}}$

MHD: Brio-Wu Rotated Shock Tube

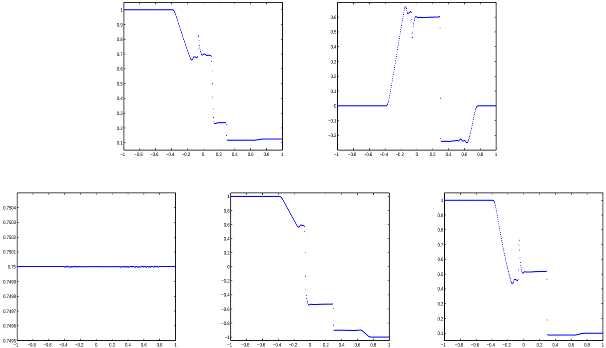
- One-dimensional Riemann problem with initial states given by

$$(\rho, v_x, v_y, v_z, B_x, B_y, B_z, p)^\top = \begin{cases} (1, 0, 0, 0, 0.75, 0, 1, 1)^\top & \text{for } x < 0 \\ (0.125, 0, 0, 0, 0.75, 0, -1, 0.1)^\top & \text{for } x > 0 \end{cases}$$

- Solved over a two dimensional domain with the direction of the flow rotated 45°
- Solution computed up to $t = 0.2$, $x \in [-1, 1]$, with 600×600 grid points, $\gamma = 2$.

MHD: Brio-Wu Rotated Shock Tube

Solution at $t = 0.2$



From top to bottom and from left to right: density, transverse velocity, transverse magnetic filed, parallel magnetic field, and pressure. The divergence of the reconstructed polynomial $\sim 10^{-13}$. Results computed with Jacobian free formulation of 2nd order JT scheme.

MHD: Shock – Cloud Interaction

- Disruption of a high density cloud by a strong shock

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- Initial conditions

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high density cloud – $\rho = 10$, $p = 1$ – centered at $x = 0.8$, $y = 0.5$, with radius 0.15,

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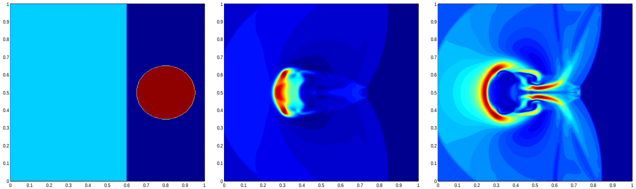
- Solved up to $t = 0.06$, $(x, z) \in [0, 1] \times [0, 1]$, with 256×256 grid points, CFL number 0.5 and $\gamma = 5/3$

MHD: Shock – Cloud Interaction



Solution of shock-cloud interaction, left: density at t=0, center: density at t=0.06, right: magnetic field lines at t=0.06. Results compputed with 3rd order semi-discrete scheme.

Euler Equations of Gas Dynamics

- conservation of mass:

$$\rho_t = -\nabla \cdot (\rho \mathbf{v}),$$

- conservation of momentum:

$$(\rho \mathbf{v})_t = -\nabla \cdot (\rho \mathbf{v}\mathbf{v}^\top + p\ \mathbb{I}_{3\times 3}),$$

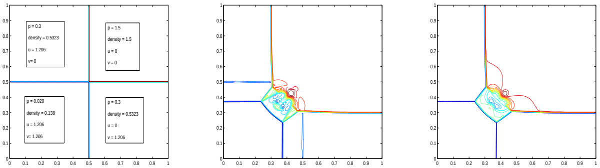
- conservation of energy:

$$e_t = -\nabla \cdot \left[\left(\frac{\gamma}{\gamma - 1} p + \frac{1}{2}\rho v^2 \right) \mathbf{v} \right],$$

- equation of state:

$$p = (\gamma - 1)\left[e - \frac{1}{2}\rho v^2 \right]$$

Euler Equations: 2d Riemann Problem



Solution of a 2d Riemann problem, left: density at t=0 and initial conditions, center: density at t=0.3 $(S^{\leftarrow}_{21}, S^{\leftarrow}_{32}, S^{\leftarrow}_{34}, S^{\leftarrow}_{41})$, right: pressure at t=0.3. Results comppputed with 3rd order semi-discrete scheme using 400 $\times$ 400 grid cells.