(1) Derive conclusion from premises:

```
1 | ∀x(Q(x) → ∃yS(x,y))
2 | ¬∃xQ(x) → ∃x∃yR(x,y))
  :
  | ∃x∃y(R(x,y) ∨ S(x,y))
```

> STRATEGY:
> {∀x(Q(x) → ∃yS(x,y)), ¬∃xQ(x) → ∃x∃yR(x,y))}
> /∴ ∃x∃y(R(x,y) ∨ S(x,y))

(2) Our goal is an existential sentence. We have no constant, so no particular instance is obviously easy to get to use for ∃Intro. We have no existential premise, which would lead us to reach our goal by ∃Elim. So as a last resort, assume the negation of the goal sentence. A contradiction will yield the double negation of the goal sentence. Then drop the double negation.

```
1 | ∀x(Q(x) → ∃yS(x,y))
2 | ¬∃xQ(x) → ∃x∃yR(x,y))
3 |  | ¬∃x∃y(R(x,y) ∨ S(x,y))
  :  :
  |  | ⊥                              ⊥ Intro
  | ¬¬∃x∃y(R(x,y) ∨ S(x,y))          ¬Intro
  | ∃x∃y(R(x,y) ∨ S(x,y))            ¬Elim
```

(3) The only negated sentence we have to work from is line 3, so our contradictory pair will probably consist of 3 and the sentence it negates.

```
1 | ∀x(Q(x) → ∃yS(x,y))
2 | ¬∃xQ(x) → ∃x∃yR(x,y))
3 |  | ¬∃x∃y(R(x,y) ∨ S(x,y))
  :  :
  |  | ∃x∃y(R(x,y) ∨ S(x,y))
  |  | ⊥                              ⊥ Intro
  | ¬¬∃x∃y(R(x,y) ∨ S(x,y))          ¬Intro
  | ∃x∃y(R(x,y) ∨ S(x,y))            ¬Elim
```

(4) Further strategizing requires us to think about what provisional assumption might be useful. Assuming '∃xQ(x)' will enable us to use the first premise. Getting a contradiction under that PA would yield '¬∃xQ(x)', which we could use with line 2. So this is promising for our next stage.

```
1 | ∀x(Q(x) → ∃yS(x,y))
2 | ¬∃xQ(x) → ∃x∃yR(x,y))
3 |  | ¬∃x∃y(R(x,y) ∨ S(x,y))
4 |  |  | ∃xQ(x)
  :  :  :
  :  : ¬∃xQ(x)
  :  : ∃x∃yR(x,y)
  :  :
  |  | ∃x∃y(R(x,y) ∨ S(x,y))
  |  | ⊥                              ⊥ Intro
  | ¬¬∃x∃y(R(x,y) ∨ S(x,y))          ¬Intro
  | ∃x∃y(R(x,y) ∨ S(x,y))            ¬Elim
```

```
1 | ∀x(Q(x) → ∃yS(x,y))
2 | ¬∃xQ(x) → ∃x∃yR(x,y))
3 |  | ¬∃x∃y(R(x,y) ∨ S(x,y))
4 |  |  | ∃xQ(x)
5 |  |  |  | [a] Q(a)
  :  :  :  :
  :  :  :
  :  : ¬∃xQ(x)
  :  : ∃x∃yR(x,y)
  :  :
  |  | ∃x∃y(R(x,y) ∨ S(x,y))
  |  | ⊥                              ⊥ Intro
  | ¬¬∃x∃y(R(x,y) ∨ S(x,y))          ¬Intro
  | ∃x∃y(R(x,y) ∨ S(x,y))            ¬Elim
```

⇐ (5) The only way we will be able to use line 4 is by applying the ∃ Elim rule to it. So we start a subproof for that, using a boxed constant that does not occur outside this subproof.

⇒ ⇒ **CONTINUE** ⇒ ⇒

(CONTINUED)

6 ⇒

To get ⊥ under line 3, aim for ⊥ under line 4, then apply ¬Intro. Get ⊥ under line 4 by ∃Elim.

To get ⊥ under line 5, aim for a contradiction with line 3.

```
1 | ∀x(Q(x) → ∃yS(x,y))
2 | ¬∃xQ(x) → ∃x∃yR(x,y))
  |-----------------------
3 | | ¬∃x∃y(R(x,y) ∨ S(x,y))
  | |-----------------------
4 | | | ∃xQ(x)
  | | |-------
5 | | | | [a] Q(a)
  | | | |---------
: : : :
  | | | | ∃x∃y(R(x,y) ∨ S(x,y))
  | | | | ⊥
  | | | ⊥
  | | ¬∃xQ(x)
  | | ∃x∃yR(x,y)
: :
  | | ∃x∃y(R(x,y) ∨ S(x,y))
  | | ⊥                                  ⊥ Intro
  | ¬¬∃x∃y(R(x,y) ∨ S(x,y))              ¬Intro
  | ∃x∃y(R(x,y) ∨ S(x,y))                ¬Eelim
```

7 To make use of 'Q(a)', apply ∀E rule to line 1. That gives us '∃yS(a,y)' by →Elim.

```
1 | ∀x(Q(x) → ∃yS(x,y))
2 | ¬∃xQ(x) → ∃x∃yR(x,y))
  |-----------------------
3 | | ¬∃x∃y(R(x,y) ∨ S(x,y))
  | |-----------------------
4 | | | ∃xQ(x)
  | | |-------
5 | | | | [a] Q(a)
  | | | |---------
6 | | | | Q(a) → ∃yS(a,y)       ∀Elim:1
7 | | | | ∃yS(a,y)              →Elim:5,6
: : : :
  | | | | ∃x∃y(R(x,y) ∨ S(x,y))
  | | | | ⊥                     ⊥ Intro
  | | | ⊥                       ∃Elim
  | | ¬∃xQ(x)
  | | ∃x∃yR(x,y)
: :
  | | ∃x∃y(R(x,y) ∨ S(x,y))
  | | ⊥                         ⊥ Intro
  | ¬¬∃x∃y(R(x,y) ∨ S(x,y))     ¬Intro
  | ∃x∃y(R(x,y) ∨ S(x,y))       ¬Elim
```

8 From '∃yS(a,y)', it should be easy to get '∃x∃y(R(x,y) ∨ S(x,y))' by ∃Elim:

```
1  | ∀x(Q(x) → ∃yS(x,y))
2  | ¬∃xQ(x) → ∃x∃yR(x,y))
   |-----------------------
3  | | ¬∃x∃y(R(x,y) ∨ S(x,y))
   | |-----------------------
4  | | | ∃xQ(x)
   | | |-------
5  | | | | [a] Q(a)
   | | | |---------
6  | | | | Q(a) → ∃yS(a,y)
7  | | | | ∃yS(a,y)
8  | | | | | [b] S(a,b)
   | | | | |-----------
9  | | | | | R(a,b) ∨ S(a,b)
10 | | | | | ∃y(R(a,y) ∨ S(a,y))
11 | | | | | ∃x∃y(R(x,y) ∨ S(x,y))
12 | | | | ∃x∃y(R(x,y) ∨ S(x,y))
13 | | | ∃x∃y(R(x,y) ∨ S(x,y))
14 | | | ⊥
15 | | ¬∃xQ(x)
16 | | ∃x∃yR(x,y)
   : :
   | | ∃x∃y(R(x,y) ∨ S(x,y))
   | | ⊥
   | ¬¬∃x∃y(R(x,y) ∨ S(x,y))
   | ∃x∃y(R(x,y) ∨ S(x,y))
```

9 ⇒ ⇒ ⇒ ⇒ ⇒ ⇒ ⇒ ⇒

Similarly, it should be easy to get '∃x∃y(R(x,y) ∨ S(x,y))' from '∃x∃yR(x,y)' by ∃Elim.

```
1  | ∀x(Q(x) → ∃yS(x,y))
2  | ¬∃xQ(x) → ∃x∃yR(x,y))
   |-----------------------
3  | | ¬∃x∃y(R(x,y) ∨ S(x,y))
   | |-----------------------
4  | | | ∃xQ(x)
   | | |-------
5  | | | | [a] Q(a)
   | | | |---------
6  | | | | Q(a) → ∃yS(a,y)               ∀Elim: 1
7  | | | | ∃yS(a,y)                      → Elim: 5,6
8  | | | | | [b] S(a,b)
   | | | | |-----------
9  | | | | | R(a,b) ∨ S(a,b)             ∨ Intro: 8
10 | | | | | ∃y(R(a,y) ∨ S(a,y))         ∃ Intro: 9
11 | | | | | ∃x∃y(R(x,y) ∨ S(x,y))       ∃ Intro: 10
12 | | | | ∃x∃y(R(x,y) ∨ S(x,y))         ∃Elim:7,8-11
13 | | | ∃x∃y(R(x,y) ∨ S(x,y))           ∃Elim: 4,5-12
14 | | | ⊥                               ⊥ Intro: 3,13
15 | | ¬∃xQ(x)                           ¬ Intro: 4-13
16 | | ∃x∃yR(x,y)                        → Elim: 2,15
17 | | | [c] ∃yR(c,y)
   | | |-------------
18 | | | | [d] R(c,d)
   | | | |-----------
19 | | | | R(c,d) ∨ S(c,d)               ∨ Intro: 18
20 | | | | ∃y(R(c,y) ∨ S(c,y))           ∃ Intro: 19
21 | | | | ∃x∃y(R(x,y) ∨ S(x,y))         ∃ Intro: 20
22 | | | ∃x∃y(R(x,y) ∨ S(x,y))           ∃Elim:17,18-21
23 | | ∃x∃y(R(x,y) ∨ S(x,y))             ∃Elim:16,17-22
24 | | ⊥                                 ⊥ Intro: 3,23
25 | ¬¬∃x∃y(R(x,y) ∨ S(x,y))             ¬Intro: 3-24
26 | ∃x∃y(R(x,y) ∨ S(x,y))               ¬Elim: 25
```

(1)

⋮	
$\neg\exists x F(x) \to \forall x(F(x) \to G(x))$	

To show that a sentence is FO-valid, derive it from the null set of premises.

> STRATEGY to show:
> $\neg\exists x F(x) \to \forall x(F(x) \to G(x))$
> is first-order valid.

(2) Since the main operator in this sentence is the arrow, we try to build it by →Intro. So we assume the antecedent as PA and aim for the consequent.

1		
2	$\neg\exists x Fx$	
⋮	⋮	
	$\forall x(F(x) \to G(x))$	
	$\neg\exists x F(x) \to \forall x(F(x) \to G(x))$	→Intro

(3) To get '$\forall x(F(x) \to G(x))$', we begin a subproof for ∀Intro with a new boxed constant an easy choice since no constant has appeared yet. We need an instance of the universal with that constant. Plan to use →Intro, then ∀Intro.

1		
2	$\neg\exists x F(x)$	
3	[a]	
4	$F(a)$	
⋮	⋮	
	$G(a)$	
	$F(a) \to G(a)$	→Intro
	$\forall x(F(x) \to G(x))$	∀Intro
	$\neg\exists x F(x) \to \forall x(F(x) \to G(x)))$	→Intro

(4) There is no way we can use line 2 or line 3 directly right now. The only way we can use '$\neg\exists x F(x)$' is as a member of a contradictory pair. Since '$\neg\exists x F(x)$' will be one member of our contradictory pair, the other will be '$\exists x F(x)$'.

1		
2	$\neg\exists x F(x)$	
3	[a]	
4	$F(a)$	
⋮	⋮	
	$\exists x F(x)$	
	$\bot$	
⋮	⋮	
	$G(a)$	
	$F(a) \to G(a)$	→Intro
	$\forall x(F(x) \to G(x))$	∀Intro
	$\neg\exists x F(x) \to \forall x(F(x) \to G(x))$	→Intro

(5) We already have all we need to justify 'G(a)'. All that is missing is the rules and the line numbers.

1		
2	$\neg\exists x F(x)$	
3	[a]	
4	$F(a)$	
5	$\exists x F(x)$	∃ Intro: 4
6	$\bot$	⊥Intro: 2,5
7	$G(a)$	⊥Elim: 6
8	$F(a) \to G(a)$	→Intro: 4-7
9	$\forall x(F(x) \to G(x))$	∀Intro: 3-8
10	$\neg\exists x F(x) \to \forall x(F(x) \to G(x))$	→Intro: 2-9

STRATEGY to show that $\forall x \forall y((J(x,y) \rightarrow x \neq y) \rightarrow \neg \exists x J(x,x)$ is first-order valid

(1) Derive the sentence from the null (empty) set of premises:

```
1 |
  :  :
  | ∀x∀y((J(x,y) → x≠y) → ¬∃xJ(x,x)
```

(2) Goal sentence is a conditional, so start a subderivation for →Intro:

```
1 |
2 |  | ∀x∀y((J(x,y) → x≠y)
  :  :
  |  | ¬∃xJ(x,x)
  | ∀x∀y((J(x,y) → x≠y) → ¬∃xJ(x,x)
```

(3) Our current goal is a negation, so we start a subderivation for ¬Intro:

```
1 |
2 |  | ∀x∀y((J(x,y) → x≠y)
3 |  |  | ∃xJ(x,x)
  :  :  :
  |  |  | ⊥
  |  | ¬∃xJ(x,x)
  | ∀x∀y((J(x,y) → x≠y) → ¬∃xJ(x,x)
```

(4) We have an existential sentence to work with now, so we start a subderivation for ∃Elim, aiming for our current goal of ⊥.

```
1 |
2 |  | ∀x∀y((J(x,y) → x≠y)
3 |  |  | ∃xJ(x,x)
4 |  |  |  | [a] J(a,a)
  :  :  :  :
  |  |  |  | ⊥
  |  |  | ⊥
  |  | ¬∃xJ(x,x)
  | ∀x∀y((J(x,y) → x≠y) → ¬∃xJ(x,x)
```

(5) Removing the universal quantifiers from line 1 will give us a conditional whose antecedent we already have on line 4:

```
1 |
2 |  | ∀x∀y((J(x,y) → x≠y)
3 |  |  | ∃xJ(x,x)
4 |  |  |  | [a] J(a,a)
5 |  |  |  | ∀y((J(a,y) → a≠y)
6 |  |  |  | J(a,a) → a ≠ a
  :  :  :  :
  |  |  |  | ⊥
  |  |  | ⊥
  |  | ¬∃xJ(x,x)
  | ∀x∀y((J(x,y) → x≠y) → ¬∃xJ(x,x)
```

(6) ⇓ Applying →Elim to lines 4 and 6 gives us 'a≠a'. We can get a contradiction immediately by applying =Intro:

```
1  |
2  |  | ∀x∀y((J(x,y) → x≠y)
3  |  |  | ∃xJ(x,x)
4  |  |  |  | [a] J(a,a)
5  |  |  |  | ∀y(J(a,y) → a≠y)                  ∀Elim:1
6  |  |  |  | J(a,a) → a ≠ a                    ∀Elim: 5
7  |  |  |  | a ≠ a                            →Elim:4,6
8  |  |  |  | a = a                               =Intro
9  |  |  |  | ⊥                              ⊥Intro: 7,8
10 |  |  | ⊥                                ∃Elim: 3,4-9
11 |  | ¬∃xJ(x,x)                           ¬Intro: 3-10
12 | ∀x∀y((J(x,y) → x≠y) → ¬∃xJ(x,x)       →Intro: 3-11
```

(1) To show a sentence is first-order valid (quantificationally true), derive it from the null set of premises.

> STRATEGY to show:
> ∃x∀y(Q(y) → R(x,y)) → (∀xQ(x) → ∃xR(x,x))
> is first-order valid.

(2) Since our goal sentence is a conditional, try to build it by →I. Start a subproof with the antecedent, and aim for the consequent.

```
1  |
2  |  | ∃x∀y(Q(y) → R(x,y))
   |  |  ⋮
   |  | ∀xQ(x) → ∃xR(x,x)
   | ∃x∀y(Q(y) → R(x,y)) → (∀xQ(x) → ∃xR(x,x))
```

(3) Our new goal is another conditional. Set up subproof for another →Intro.

```
1  |
2  |  | ∃x∀y(Q(y) → R(x,y))
3  |  |  | ∀xQ(x)
   |  |  |  ⋮
   |  |  | ∃xR(x,x)
   |  | ∀xQ(x) → ∃xR(x,x)
   | ∃x∀y(Q(y) → R(x,y)) → (∀xQ(x) → ∃xR(x,x))
```

(4) To use line 2, we must apply the ∃Elim rule. As soon as we recognize that, we begin the subproof this rule requires.

We need a sentence without the boxed constant introduced in the first line of the subproof. We can move that sentence out of the subproof by ∃Elim. So we aim for our next goal, '∃xR(x,x)'.

```
1  |
2  |  | ∃x∀y(Q(y) → R(x,y))
3  |  |  | ∀xQ(x)
4  |  |  |  | [a] ∀y(Q(y) → R(x,y))
   |  |  |  |  ⋮
   |  |  |  | ∃xR(x,x)
   |  |  | ∃xR(x,x)
   |  | ∀xQ(x) → ∃xR(x,x)
   | ∃x∀y(Q(y) → R(x,y)) → (∀xQ(x) → ∃xR(x,x))
```

(5) Substituting 'a' for the variable in both universal sentences gives us what we need to get 'R(a,a)'. That in turn lets us reach the goal of '∃xR(x,x)'. All we need to add is line numbers and justifications.

```
1  |
2  |  | ∃x∀y(Q(y) → R(x,y))
3  |  |  | ∀xQ(x)
4  |  |  |  | [a] ∀y(Q(y) → R(x,y))
5  |  |  |  | Q(a)                                        ∀Elim: 3
6  |  |  |  | Q(a) → R(a,a)                               ∀Elim: 4
7  |  |  |  | R(a,a)                                      →Elim: 5,6
8  |  |  |  | ∃xR(x,x)                                    ∃Intro: 7
9  |  |  | ∃xR(x,x)                                       ∃Elim: 2,4-8
10 |  | ∀xQ(x) → ∃xR(x,x)                                 →Intro: 3-9
11 | ∃x∀y(Q(y) → R(x,y)) → (∀xQ(x) → ∃xR(x,x))           →Intro: 1-10
```

To show 2 sentences are first-order equivalent, derive biconditional from the null (empty) set of premises. Use standard strategy: build biconditional by ↔Intro.

STRATEGY:
Show that ∀x(B(x) ↔ ¬C(x)) is first-order equivalent to ¬∃x[(B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x))]

(1)

```
1 |
2 |  | ∀x(B(x) ↔ ¬C(x))
  :  :
  |  | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))

  |  | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
  :  :
  |  | ∀x(B(x) ↔ ¬C(x))
  | ∀x(B(x) ↔ ¬C(x)) ↔ ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
```

(2) Since the goal in the first subproof is a negation, we'll probably get it by ¬Intro. So we assume the sentence that it negates.

```
1 |
2 |  | ∀x(B(x) ↔ ¬C(x))
3 |  |  | ∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
  :  :  :
  |  | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
  :  :
```

(3) To use line 3, plan on ∃Elim. Step 4 begins a subproof for ∃Elim. We need a contradiction under 3, so we aim for ⊥. We will move it left by ∃Elim to get ⊥ under line 3, where we really want it.

```
1 |
2 |  | ∀x(B(x) ↔ ¬C(x))
3 |  |  | ∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
4 |  |  |  | [a] (B(a) ∧ C(a)) ∨ (¬B(a) ∧ ¬C(a))
  :  :  :  :
  |  |  |  | ⊥
  |  |  | ⊥
  |  | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
  :
```

(4) We will need to use line 2, so substitute the boxed constant from line 4 for the x's in line 2.

```
1 |
2 |  | ∀x(B(x) ↔ ¬C(x))
3 |  |  | ∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
4 |  |  |  | [a] (B(a) ∧ C(a)) ∨ (¬B(a) ∧ ¬C(a))
5 |  |  |  | B(a) ↔ ¬C(a)
  :  :  :  :
  |  |  |  | ⊥
  |  |  | ⊥
```

```
1 |
2 |  | ∀x(B(x) ↔ ¬C(x))
3 |  |  | ∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
4 |  |  |  | [a] (B(a) ∧ C(a)) ∨ (¬B(a) ∧ ¬C(a))
5 |  |  |  | B(a) ↔ ¬C(a)
6 |  |  |  |  | B(a) ∧ C(a)
  :  :  :  :  :
  |  |  |  |
  |  |  |  |  | ¬B(a) ∧ ¬C(a)
  :  :  :  :  :
  |  |  |  | ⊥
  |  |  | ⊥
  |  | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
```

(5) ⇒ To use line 4, start two subderivations for ∨Elim.

⇒ ⇒ **CONTINUE** ⇒ ⇒

(6)

```
1  |
2  |  | ∀x(B(x) ↔ ¬C(x))
   |  |----------------
3  |  |  | ∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
   |  |  |-----------------------------------
4  |  |  |  | [a] (B(a) ∧ C(a)) ∨ (¬B(a) ∧ ¬C(a))
   |  |  |  |-------------------------------------
5  |  |  |  | B(a) ↔ ¬C(a)                        2,∀ Elim
6  |  |  |  |  | B(a) ∧ C(a)
   |  |  |  |  |------------
:  :  :  :  :  :
   |  |  |  |  | ⊥
   |  |  |  |
   |  |  |  |  | ¬B(a) ∧ ¬C(a)
   |  |  |  |  |--------------
:  :  :  :  :  :
   |  |  |  |  | ⊥                                 ⊥Intro
   |  |  |  | ⊥                                    ∨ Elim
   |  |  | ⊥                                       ∃ Elim
   |  | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))      ¬Intro
```

We need a contradiction under line 6. That will allow us to apply the ¬Intro rule to line 6. We can get contradictions in both subproofs for ∨ Elim. So we'll bring '⊥' out of both subproofs in our ∨Elim step.

(7)

```
1  |
2  |  | ∀x(B(x) ↔ ¬C(x))
   |  |----------------
3  |  |  | ∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
   |  |  |-----------------------------------
4  |  |  |  | [a] (B(a) ∧ C(a)) ∨ (¬B(a) ∧ ¬C(a))
   |  |  |  |-------------------------------------
5  |  |  |  | B(a) ↔ ¬C(a)                        2,∀ Elim
6  |  |  |  |  | B(a) ∧ C(a)
   |  |  |  |  |------------
7  |  |  |  |  | B(a)                             6,∧ Elim
8  |  |  |  |  | C(a)                             6,∧ Elim
9  |  |  |  |  | ¬C(a)                            5,7,↔ Elim
10 |  |  |  |  | ⊥                                8,9,⊥Intro
   |  |  |  |
11 |  |  |  |  | ¬B(a) ∧ ¬C(a)
   |  |  |  |  |--------------
12 |  |  |  |  | ¬C(a)                            11,∧ Elim
13 |  |  |  |  | B(a)                             5,12,↔ Elim
14 |  |  |  |  | ¬B(a)                            11,∧ Elim
15 |  |  |  |  | ⊥                                13,14,⊥Intro
16 |  |  |  | ⊥                          4,6-10, 11-15,∨ Elim
17 |  |  | ⊥                                      3,4-16, ∃ Elim
18 |  | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))      2-16,¬Intro
   |
19 |  | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
:  :  :
   |  | ∀x(B(x) ↔ ¬C(x))
   | ∀x(B(x) ↔ ¬C(x)) ↔ ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
```

In the first subproof, get 'B(a)' by ∧ Elim. To get'¬B(a)', notice that we can easily get both 'C(a)' (by ∧ Elim) and '¬C(a)' (by ↔ Elim). To put this contradiction to work for us, apply ⊥Intro. Getting 'B(a)' and '¬B(a)' in the second subproof is easy, giving us ⊥ again.. Moving ⊥ out of the 2 subproofs by ∨ Elim gives us ⊥ under line 3. We can then move it to the left, under line 2, by ∃ Elim, justifying ¬Intro. This completes the first subproof.

⇒ ⇒ **CONTINUE** ⇒ ⇒

(CONTINUED)

(8) We are now ready to begin the second subproof.

```
1   | | ∀x(B(x) ↔ ¬C(x))
    : :
18  | | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))

19  | | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
    : :
    | | ∀x(B(x) ↔ ¬C(x))
    | ∀x(B(x) ↔ ¬C(x)) ↔ ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
```

(9) To get '∀x(B(x) ↔ ¬C(x))', we plan to use ∀Intro. So we start a subproof with a boxed constant that does not occur outside the subproof, 'b'. Aim for 'B(b) ↔ ¬C(b)', by ↔ Intro.

```
    :
19  | | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
20  | | | [b]
21  | | | | B(b)
    : : : :
    | | | | ¬C(b)

    | | | | ¬C(b)
    : : : :
    | | | | B(b)
    | | | B(b) ↔ ¬C(b)
    | | ∀x(B(x) ↔ ¬C(x))
```

(10) To get '¬C(b)' by ¬Intro, assume 'C(b)'. Conjoin with 'B(b). Use ∨ Intro and ∃ Intro to build the sentence that is negated in line 19, yielding a contradiction:

```
    :
19  | | ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
20  | | | [b]
21  | | | | B(b)
22  | | | | | C(b)
23  | | | | | B(b) ∧ C(b)                               ∧Intro: 21,22
24  | | | | | (B(b) ∧ C(b)) ∨ (¬B(b) ∧ ¬C(b))           ∨Intro: 23
25  | | | | | ∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))       ∃Intro: 24
26  | | | | | ⊥                                         ⊥Intro: 19,25
27  | | | | ¬C(b)                                       ¬Intro: 22-26

28  | | | | ¬C(b)
    : : : :
    | | | | B(b)
    | | | B(b) ↔ ¬C(b)
    | | ∀x(B(x) ↔ ¬C(x))
```

(11) To get 'B(b)' from '¬C(b)', use a strategy like the one for getting '¬C(b)' from 'B(b)'. This completes the proof.

```
    : : : : :
27  | | | | ¬C(b)                                       22-26,¬Intro

28  | | | | ¬C(b)
29  | | | | | ¬B(b)
30  | | | | | ¬B(b) ∧ ¬C(b)                             ∧ Intro: 28, 29
31  | | | | | (B(b) ∧ C(b)) ∨ (¬B(b) ∧ ¬C(b))           ∨ Intro: 30
32  | | | | | ∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))       ,∃ Intro: 31
33  | | | | | ⊥                                         ⊥ Intro: 19,32
34  | | | | ¬¬B(b)                                      ¬Intro: 29-33
35  | | | | B(b)                                        ¬ Elim: 34
36  | | | B(b) ↔ ¬C(b)                                  ↔Intro: 21-27, 28-35
37  | | ∀x(B(x) ↔ ¬C(x))                                ∀Intro: 20-36
38  | ∀x(B(x) ↔ ¬C(x)) ↔ ¬∃x((B(x) ∧ C(x)) ∨ (¬B(x) ∧ ¬C(x)))
                                                      ↔Intro: 2-18,19-37
```

⇒ ⇒ **Entire proof shown on the next page** ⇒ ⇒

In total, our proof showing that

$\forall x(B(x) \leftrightarrow \neg C(x))$

is first-order equivalent to

$\neg\exists x((B(x) \wedge C(x)) \vee (\neg B(x) \wedge \neg C(x)))$

looks like this:

1		
2	$\forall x(B(x) \leftrightarrow \neg C(x))$	
3	$\exists x((B(x) \wedge C(x)) \vee (\neg B(x) \wedge \neg C(x)))$	
4	$\boxed{a}$ $(B(a) \wedge C(a)) \vee (\neg B(a) \wedge \neg C(a))$	
5	$B(a) \leftrightarrow \neg C(a)$	$\forall$ Elim: 2
6	$B(a) \wedge C(a)$	
7	$B(a)$	$\wedge$ Elim: 6
8	$C(a)$	$\wedge$ Elim: 6
9	$\neg C(a)$	$\leftrightarrow$ Elim: 5,7
10	$\bot$	$\bot$ Intro: 8,9
11	$\neg B(a) \wedge \neg C(a)$	
12	$\neg C(a)$	$\wedge$Elim: 11
13	$B(a)$	$\leftrightarrow$Elim: 5,12
14	$\neg B(a)$	11,$\wedge$Elim: 11
15	$\bot$	13,14,$\bot$Intro: 13,14
16	$\bot$	$\vee$ Elim: 4,5-10,11-15
17	$\bot$	$\exists$ Elim: 3,4-16
18	$\neg\exists x((B(x) \wedge C(x)) \vee (\neg B(x) \wedge \neg C(x)))$	$\neg$ Intro: 3-17
19	$\neg\exists x((B(x) \wedge C(x)) \vee (\neg B(x) \wedge \neg C(x)))$	
20	$\boxed{b}$	
21	$B(b)$	
22	$C(b)$	
23	$B(b) \wedge C(b)$	$\wedge$ Intro: 21,22
24	$B(b) \wedge C(b) \vee (\neg B(b) \wedge \neg C(b))$	$\vee$ Intro: 23
25	$\neg\exists x((B(x) \wedge C(x)) \vee (\neg B(x) \wedge \neg C(x)))$	$\exists$ Intro: 24
26	$\bot$	$\bot$ Intro: 19,25
27	$\neg C(b)$	$\neg$ Intro: 22-26
28	$\neg C(b)$	
29	$\neg B(b)$	
30	$\neg B(b) \wedge \neg C(b)$	28,29,$\wedge$ Intro: 28,29
31	$(B(b) \wedge C(b)) \vee (\neg B(b) \wedge \neg C(b))$	$\vee$ Intro: 30
32	$\exists x((B(x) \wedge C(x)) \vee (\neg B(x) \wedge \neg C(x)))$	$\exists$ Intro: 31
33	$\bot$	$\bot$ Intro:19,32
34	$\neg\neg B(b)$	$\neg$ Intro: 29-33
35	$B(b)$	$\neg\neg$ Elim: 34
36	$B(b) \leftrightarrow \neg C(b)$	$\leftrightarrow$ Intro: 21-27,28-35
37	$\forall x(B(x) \leftrightarrow \neg C(x))$	$\forall$Intro: 20-36
38	$\forall x(B(x) \leftrightarrow \neg C(x)) \leftrightarrow \neg\exists x[(B(x) \wedge C(x)) \vee (\neg B(x) \wedge \neg C(x))]$	$\leftrightarrow$Intro: 2-18,19-37