To show a statement is truth-functionally true, start proof with no premises and end with the statement:

STRATEGY to show that
(F ∨ G) → (¬F → G)
is a tautology
(is truth-functionally true).

1

```
|
⋮
| (F ∨ G) → (¬F → G)
```

2 We are aiming for a conditional, so we set up a subproof to get it by → Intro.

```
1 |
2 |  | (F ∨ G)__
  ⋮  ⋮
  |  | ¬F → G
  | (F ∨ G) → (¬F → G)        →Intro
```

3 Same strategy as in step 2:

```
1 |
2 |  | (F ∨ G)__
3 |  |  | ¬F__
  ⋮  ⋮  ⋮
  |  |  | G
  |  | ¬F → G
  | (F ∨ G) → (¬F → G)        →Intro
```

4 Now what we are aiming for is a single letter, so we look to see how we can get it from earlier steps.

A. Can we get it by an easy rule? (∧ Elim, → Elim, ↔ Elim)
B. Can we get it from any other complex sentence containing ‘G’, or if there are non with that, with ‘¬G’?

Here we find a ‘G’ in line 1, which is a disjunction. The only thing we can do with that is use ∨Elim, so we set that up.

```
1 |
2 |  | F ∨ G__
3 |  |  | ¬F__
4 |  |  |  | F__
  ⋮  ⋮  ⋮  ⋮
  |  |  |
  |  |  |  | G__
  |  |  | G                         ∨Elim
  |  | ¬F → G                      →Intro
  | (F ∨ G) → (¬F → G)             →Intro
```

5 We see that it is easy to get from ‘G’ to ‘G’, so the only work left is to get from ‘F’ to ‘G’.

To get ‘G’ under ‘F’, we notice that we already have '¬F' and 'F' available in this subproof. So we can use ⊥ Intro and then ⊥ Elim ‘G’.

```
1  |
2  |  | F ∨ G__
3  |  |  | ¬F__
4  |  |  |  | F__
5  |  |  |  | ⊥                    ⊥ Intro: 3,4
6  |  |  |  | G                     ⊥ Elim: 5
   |  |  |
7  |  |  |  | G__
8  |  |  | G                  ∨ Elim: 2,3-6,7-7
9  |  | ¬F → G                   →Intro: 3-8
10 | (F ∨ G) → (¬F → G)          →Intro: 2-9
```

To show 2 sentences are tautologically equivalent: From null (empty) set of premises, derive biconditional.

(1)

```
| 
:
| ¬(A ∨ B) ↔ (¬A ∧ ¬B)
```

STRATEGY to show ¬(A ∨ B) and (¬A ∧ ¬B) are tautologically equivalent.

(2) Start 2 subproofs to reach goal by ↔I:

```
|
|  | ¬(A ∨ B)
|  |------------
:  :
|  | ¬A ∧ ¬B
|
|  | ¬A ∧ ¬B
|  |------------
:  :
|  | ¬(A ∨ B)
| ¬(A ∨ B) ↔ (¬A ∧ ¬B)
```

(3) To get (¬A ∧ ¬B), aim for each conjunct.

```
|
|  | ¬(A ∨ B)
|  |------------
:  : ¬A
:  :
:  : ¬B
|  | ¬A ∧ ¬B
|
|  | ¬A ∧ ¬B
|  |------------
:  :
|  | ¬(A ∨ B)
| ¬(A ∨ B) ↔ (¬A ∧ ¬B)
```

(4)

```
1  |
2  |  | ¬(A ∨ B)
3  |  |  | A
4  |  |  | A ∨ B          ∨ Intro: 2
5  |  |  | ⊥              ⊥ Intro: 1,3
6  |  | ¬A                ¬Intro: 2-4
7  |  |  | B
8  |  |  | A ∨ B          ∨ Intro: 7
9  |  |  | ⊥              ⊥ Intro: 1,8
10 |  | ¬B                ¬Intro: 7-9
11 |  | ¬A ∧ ¬B           ∧Intro: 6,10

12 |  | ¬A ∧ ¬B
   :  :
   |  | ¬(A ∨ B)
   | ¬(A ∨ B) ↔ (¬A ∧ ¬B)
```

(4) To get ¬A, assume A, then use ¬Intro. We already have a negation, '¬(A ∨ B)', on line 1, so that will be one member of our contradictory pair. Get '(A ∨ B)' directly from A by ∨Intro for the desired contradiction. The same strategy gives us ¬B:

(5) Our goal in the second subproof is a negation. We can't get it directly by easy rules like ∧Elim or →Elim. So we assume '(A ∨ B)'.

```
1  |
2  |  | ¬(A ∨ B)
3  |  |  | A
4  |  |  | A ∨ B      ∨ Intro, 3
5  |  |  | ⊥          ⊥ Intro: 2,4
6  |  | ¬A            ¬Intro: 3-5
7  |  |  | B
8  |  |  | A ∨ B      ∨ Intro: 7
9  |  |  | ⊥          ⊥ Intro: 2,8
10 |  | ¬B            ¬Intro: 7-9
11 |  | ¬A ∧ ¬B       ∧In: 6,10

12 |  | ¬A ∧ ¬B
13 |  |  | A ∨ B
   :  :  :
   :  :  :
   |  | ¬(A ∨ B)
   | ¬(A ∨ B) ↔ (¬A ∧ ¬B)
```

(6) We must use ∨Elim. We need a sentence and its negation. Take negated conjunct from 11. Get unnegated statement by ∨Elim.

```
1  |
2  |  | ¬(A ∨ B)
3  |  |  | A
4  |  |  | A ∨ B      ∨ Intro, 3
5  |  |  | ⊥          ⊥ Intro: 2,4
6  |  | ¬A            ¬Intro: 3-5
7  |  |  | B
8  |  |  | A ∨ B      ∨ Intro: 7
9  |  |  | ⊥          ⊥ Intro: 2,8
10 |  | ¬B            ¬Intro: 7-9
11 |  | ¬A ∧ ¬B       ∧In: 6,10

12 |  | ¬A ∧ ¬B
13 |  |  | A ∨ B
14 |  |  |  | A
15 |  |  |  | B
   :  :  :  :
   |  |  |  | A
   |  |  | A
   |  |  | ¬A
   |  |  | ⊥
   |  | ¬(A ∨ B)
   | ¬(A ∨ B) ↔ (¬A ∧ ¬B)
```

(7) We already have a contradiction. Use it for ⊥Intro. Then use ⊥Elim to get A again so we can pull it out by ∨Elim for a contradiction under 'A ∨ B'

```
1  |
2  |  | ¬(A ∨ B)
3  |  |  | A
4  |  |  | A ∨ B                ∨ Intro, 3
5  |  |  | ⊥                    ⊥ Intro: 2-4
6  |  | ¬A                      ¬Intro: 3-5
7  |  |  | B
8  |  |  | A ∨ B                ∨ Intro: 7
9  |  |  | ⊥                    ⊥ Intro: 2,8
10 |  | ¬B                      ¬Intro: 7-9
11 |  | ¬A ∧ ¬B                 ∧ Intro: 6,10
12 |  | ¬A ∧ ¬B
13 |  |  | A ∨ B
14 |  |  |  | A
15 |  |  |  | B
16 |  |  |  | ¬B               ∧Elim: 12
17 |  |  |  | ⊥                ⊥ Intro: 15,16
18 |  |  |  | A                ⊥ Elim: 17
19 |  |  | A          ∨Elim: 13,14-14,15-18
20 |  |  | ¬A                   ∧ Elim: 12
21 |  |  | ⊥                    ⊥ Intro: 19,20
22 |  | ¬(A ∨ B)                ¬ Intro: 13-21
23 | ¬(A ∨ B) ↔ (¬A ∧ ¬B)
                        ↔Intro: 2-11,12-22
```

STRATEGY
Show that (¬A v B) and A → B are tautologically equivalent.

(1) Derive biconditional from the null (empty) set of premises. As usual, plan to build biconditional by ↔Intro.

```
1 | | ¬A v B
  | |_______
  | | ⋮
  | | A →  B
  |
  | | A → B
  | |______
  | | ⋮
  | | ¬A v B
  | (¬A v B) ↔ (A→ B)
```

(2) ⇐ In the first subproof, build « the usual way, by ↔Intro. Within this subproof, get ‘B’ by vElim.

```
1 | | ¬A v B
  | |_______
2 | | | A
3 | | | | ¬A
  | | | | ⋮
  | | | | B
  | | | | ⋮
  | | | | B
  | | | B
  | | A→ B
  | | A → B
  | |______
  | | ⋮
  | | ¬A v B
  | (¬A v B) ↔ (A→ B)
```

(3) ⇒ ‘B’ is already in the second subproof. In the first, use the standard strategy we apply when we already have or can easily get a contradiction: use ⊥Intro and ⊥Elim.

```
1 | | ¬A v B
2 | | | A
3 | | | | ¬A
4 | | | | ⊥            ⊥ Intro: 2,3
5 | | | | B            ⊥ Elim: 4
6 | | | | B
7 | | | B        v Elim:1,3-5,6-6
8 | | A → B          →Intro: 2-7
  | | A → B
  | | ⋮
  | | ¬A v B
  | (¬A v B) ↔ (A → B)
```

(4) ⇐ Our goal is a disjunction. Neither disjunct is easy to get. We have no earlier disjunction to which we can apply vElim. So assume the opposite. Use the common strategy for getting a contradiction when one of the things we have to work from is the negation of a disjunction. Here we assume ‘¬A’ because its opposite will be useful in combination with line 9.

```
 1 | | ¬A v B
 2 | | | A
 3 | | | | ¬A
 4 | | | | ⊥           ⊥ Intro: 2,3
 5 | | | | B           ⊥ Elim: 4
 6 | | | | B
 7 | | | B        vElim:1,3-5,6-6
 8 | | A → B        →Intro:2-7
 9 | | A → B
10 | | | ¬(¬A v B)
11 | | | | ¬A
12 | | | | ¬A v B    vIntro: 11
13 | | | | ⊥         ⊥Intro:10,12
14 | | | ¬¬A         ¬Intro:11-13
15 | | | A           ¬Elim: 14
   | | ⋮
   | | ¬A v B
   | (¬A v B) ↔ (A → B)
```

(5) ⇒ In keeping with that standard strategy, get the other disjunct, and build the disjunction again. Fill in the justifications to complete the proof.

```
 1 |
 2 | | ¬A v B
 3 | | | A
 4 | | | | ¬A
 5 | | | | ^                 ⊥ Intro: 3,4
 6 | | | | B                 ⊥ Elim: 5
 7 | | | | B
 8 | | | B            v Elim: 2,4-6,7-7
 9 | | A → B              →Intro: 3-8
10 | | A → B
11 | | | ¬(¬A v B)
12 | | | | ¬A
13 | | | | ¬A v B            v Intro: 12
14 | | | | ⊥                 ⊥ Intro: 11,13
15 | | | ¬¬A                 ¬ Intro: 12-14
16 | | | A                   ¬ Elim: 15
17 | | | B                   →Elim: 10,16
18 | | | ¬A v B              v Intro: 17
19 | | | ⊥                   ⊥ Intro: 11,18
20 | | ¬¬(¬A v B)            ¬Intro: 11-19
21 | | ¬A v B                ¬¬Elim: 20
22 | (¬A v B) ↔ (A → B)
                     ↔ Intro: 2-9,10-21
```

To show that a set is tautologically inconsistent, derive a contradiction from the members of the set.

STRATEGY
Show that this set is tautologically inconsistent:
{A ∧ (B ↔ C), A → B, (A ∧ B) → D, A → (C → ¬D)}

1

1	A ∧ (B ↔ C)	P
2	A → B	P
3	(A ∧ B) → D	P
4	<u>A → (C → ¬D)</u>	P
	⋮	
	⊥	

2 The only '¬' anywhere in this set is in '¬D' inside line 4. So we'll aim for '¬D' and for 'D', then apply ⊥ Intro.

1	A ∧ (B ↔ C)	P
2	A → B	P
3	(A ∧ B) → D	P
4	<u>A → (C → ¬D)</u>	P
	⋮	
	¬D	
	⋮	
	D	
	⊥	⊥ Intro

⇐ **3** We just saw that '¬D' comes from line 4. To start breaking line 4 apart, we need 'A'. We notice that we can get it from 1 by ∧Elim.

1	A ∧ (B ↔ C)	P
2	A → B	P
3	(A ∧ B) → D	P
4	<u>A → (C → ¬D)</u>	P
5	A	∧ Elim: 1
6	C → ¬D	→Elim: 4,5
	⋮	
	¬D	
	⋮	
	D	
	⊥	⊥ Intro

4 ⇒

We also need 'C'. That, too will have to come from breaking apart 1, but not directly. First we pull out 'B ↔ C'.

1	A ∧ (B ↔ C)	P
2	A → B	P
3	(A ∧ B) → D	P
4	<u>A → (C → ¬D)</u>	P
5	A	∧ Elim: 1
6	C → ¬D	→Elim: 4,5
7	B ↔ C	∧ Elim: 1
	⋮	
	C	
	¬D	
	⋮	
	D	
	⊥	⊥ Intro

5 All that is missing to get 'C' is 'B'. We can get that from 2 and 5 by →Elim.

1	A ∧ (B ↔ C)	P
2	A → B	P
3	(A ∧ B) → D	P
4	<u>A → (C → ¬D)</u>	P
5	A	∧ Elim: 1
6	C → ¬D	→Elim: 4,5
7	B ↔ C	∧ Elim: 1
8	B	→Elim: 2,5
9	C	↔ Elim: 7,8
10	¬D	→Elim: 6,9
	⋮	
	D	
	⊥	⊥ Intro

6 ⇒

Now we have all we need to justify '¬D'. Next we work on getting 'D'. It must come from line 3. To get it, we must first have 'A ∧ B', the antecedent of the conditional. We already have both conjuncts separately, so we can build 'A ∧ B' by ∧Intro. That gives us all the steps we need, so all that's left is completing the justifications.

1	A ∧ (B ↔ C)	P
2	A → B	P
3	(A ∧ B) → D	P
4	<u>A → (C → ¬D)</u>	P
5	A	∧ Elim: 1
6	C → ¬D	→Elim: 4,5
7	B ↔ C	∧ Elim: 1
8	B	→Elim: 2,5
9	C	↔ Elim: 7,8
10	¬D	→Elim: 6,9
11	A ∧ B	∧ Intro: 5,8
12	D	→Elim: 3,11
13	⊥	⊥ Intro: 10,12

Show that
{J ∨ K, J → (L ∧ ¬N), K ↔ (N ∧ ¬L), ¬(L ∨ N)}
is tautologically inconsistent

(1) To show that a set of sentences is tautologically inconsistent, derive a contradiction from the members of the set.

```
1 | J ∨ K           P
2 | J → (L ∧ ¬N)    P
3 | K ↔ (N ∧¬L)     P
4 | ¬(L ∨ N)        P
  ⋮
  | ⊥
```

(2) ⇒ Look for a negated sentence you already have or can easily get as one member of the contradictory pair. Here, if we try for 'L ∨ N' to contradict the sentence on line 4, we use the common strategy of applying ∨Elim to one disjunction to get another.

```
1 | J ∨ K
2 | J → (L ∧ ¬N)
3 | K ↔ (N ∧ ¬L)
4 | ¬(L ∨ N)
  | | J
  ⋮ ⋮
  | | L ∨ N
  |
  | | K
  ⋮ ⋮
  | | L ∨ N
  | L ∨ N
  | ⊥
```

```
1 | J ∨ K                  P
2 | J → (L ∧ ¬N)           P
3 | K ↔ (N ∧ ¬L)           P
4 | ¬(L ∨ N)               P
5 | | J
6 | | L ∧ ¬N     →Elim:2,5
  ⋮ ⋮
  | | L ∨ N
  |
  | | K
  ⋮ ⋮
  | | L ∨ N
  | L ∨ N
  | ⊥
```

⇐(3) For the first subderivation, we notice that 'J' is the antecedent in line 2, and the first disjunct in 'L ∨ N' is in the consequent, so we apply →Elim to line 2.

(4) ⇒ To get 'L ∨ N', we just need to separate 'L' from 6 by ∧Elim.

```
1 | J ∨ K                     P
2 | J → (L ∧ ¬N)              P
3 | K ↔ (N ∧ ¬L)              P
4 | ¬(L ∨ N)                  P
5 | | J
6 | | L ∧ ¬N        →Elim:2,5
7 | | L             ∧ Elim: 6
8 | | L ∨ N         ∨ Elim: 7
  |
  | | K
  ⋮ ⋮
  | | L ∨ N
  | L ∨ N
  | ⊥
```

```
1  | J ∨ K                          P
2  | J → (L ∧ ¬N)                   P
3  | K ↔ (N ∧ ¬L)                   P
4  | ¬(L ∨ N)                       P
5  | | J
6  | | L ∧ ¬N          → Elim:2,5
7  | | L                 ∧ Elim: 6
8  | | L ∨ N             ∨ Elim: 7
   |
9  | | K
10 | | N ∧ ¬L          ↔Elim: 3,9
11 | | N                 ∧Elim: 10
12 | | L ∨ N            ∨Intro: 11
13 | L ∨ N      ∨Elim: 1,5-8,9-12
14 | ⊥                ⊥Intro: 4,13
```

⇐(5) Completion of the second subderivation is similar. Apply ↔Elim to 3, then separate one of the desired disjuncts by ∧Elim.

(1) To show that a set of sentences is tautologically inconsistent, derive a contradiction from the set's members.

> Strategy:
> Show that this set is tautologically inconsistent:
> {¬(P ∧ Q), R → (P ↔ S), S ∧ R, S → (¬Q → ¬R)}

```
1 | ¬(P ∧ Q)
2 | R → (P ↔ S)
3 | S ∧ R
4 | S → (¬Q → ¬R)
  :
  | ⊥
```

(2) ⇒ Look for a negation we already have as one member of the contradictory pair. Here, try for 'P ∧ Q' to contradict the sentence on line 1.

```
1 | ¬(P ∧ Q)
2 | R → (P ↔ S)
3 | S ∧ R
4 | S → (¬Q → ¬R)
  :
  | P ∧ Q
  | ⊥
```

⇐ (3) To get 'P ∧ Q', we need each conjunct. First get 'P'. 'P' occurs only in lines 1 and 2. We can't break apart line 1, so we must use line 2. We need to separate out its consequent.

```
1 | ¬(P ∧ Q)
2 | R → (P ↔ S)
3 | S ∧ R
4 | S → (¬Q → ¬R)
  :
  | P ↔ S
  :
  | P
  :
  | Q
  | P ∧ Q
  | ⊥
```

(4) ⇒ To get 'P ↔ S' from 2, we must first get the antecedent, 'R'. Get 'R' from 3 by ∧E.

```
1 | ¬(P ∧ Q)
2 | R → (P ↔ S)
3 | S ∧ R
4 | S → (¬Q → ¬R)
5 | R               ∧Elim: 3
6 | P ↔ S       → Elim: 2,5
  :
  | P
  :
  | Q
  | P ∧ Q
  | ⊥
```

(5) To get 'P' from 'P ↔ S', we need 'S', which we can also get from 3.

```
1 | ¬(P ∧ Q)
2 | R → (P ↔ S)
3 | S ∧ R
4 | S → (¬Q → ¬R)
5 | R               ∧ Elim: 3
6 | P ↔ S       → Elim: 2,5
7 | S               ∧ Elim: 3
8 | P              ↔Elim: 6,7
  :
  | Q
  | P ∧ Q
  | ⊥
```

(6) 'Q' appears above only in lines 1 & 4. We can't get 'Q' directly, so plan to get a contradiction. 'R' and '¬R' look promising. Assume '¬Q' to help us get '¬R' and '⊥'.

```
1 | ¬(P ∧ Q)
2 | R → (P ↔ S)
3 | S ∧ R
4 | S → (¬Q → ¬R)
5 | R                 ∧ Elim: 3
6 | P ↔ S         → Elim: 2,5
7 | S                 ∧ Elim: 3
8 | P               ↔ Elim: 6,7
9 |   | ¬Q
  :   :
  |   | ¬R
  |   | ⊥
  | ¬¬Q
  | Q
  | P ∧ Q
  | ⊥
```

(7) '¬Q' leads easily to '¬R'. With 'R' (line 5), we have a contradiction. This completes the proof.

```
 1 | ¬(P ∧ Q)
 2 | R → (P ↔ S)
 3 | S ∧ R
 4 | S → (¬Q → ¬R)
 5 | R                      ∧ Elim: 3
 6 | P ↔ S              → Elim: 2,5
 7 | S                      ∧ Elim: 3
 8 | P                    ↔ Elim: 6,7
 9 |   | ¬Q
10 |   | ¬Q → ¬R      →Elim: 4,7
11 |   | ¬R              →Elim: 9,10
12 |   | ⊥               ⊥ Intro: 5,11
13 | ¬¬Q               ¬ Intro: 9-12
14 | Q                    ¬¬Elim: 13
15 | P ∧ Q             ∧Intro: 8,14
16 | ⊥                 ⊥ Intro: 1,15
```