

## Math 350 — 2. Exam — Solutions

**Name:** \_\_\_\_\_

Complete at least five of the following six problems. Extra work will give you extra credit. Show all your work. Closed books and notes, calculators are ok. If you find spelling mistakes you may keep them. Good luck.

1. (20 pts.) Consider the set  $\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$ . Show that this set is compact.

**Solution 1:** Let  $\{O_\alpha\}$  be an arbitrary open cover of  $A$ . Then there exists an open set  $O_0 \in \{O_\alpha\}$  such that  $0 \in O_0$ . Since this set is open there exists a  $\delta > 0$  such that  $(0 - \delta, 0 + \delta) \subset O_0$ . For this  $\delta$  there exists a  $N$  such that  $\frac{1}{n} \in (0 - \delta, 0 + \delta) \subset O_0$  for all  $n \geq N$ . Next for  $n = 1, \dots, N$  let  $O_n \in \{O_\alpha\}$  be an open set such that  $\frac{1}{n} \in O_n$ . Then  $A \subset O_0 \cup O_1 \cup \dots \cup O_N$ , and  $\{O_0, \dots, O_N\}$  form a finite subcover of the original open cover. Hence,  $A$  is compact.

**Solution 2:**  $A \subset [0, 1]$  and therefore  $A$  is bounded. Furthermore, all elements of  $A$  are boundary points of  $A$ , and 0 is the only limit point of  $A$ . Therefore  $A$  is closed. By the Heine-Borel Theorem  $A$  is compact.

2. (20 pts.) Consider the set  $\mathbb{Q}$  of all rational numbers. Prove that the interior of  $\mathbb{Q}$  is empty, and that the closure of  $\mathbb{Q}$  is  $\mathbb{R}$ .

**Solution:** For every  $x \in \mathbb{R}$  and every  $\delta > 0$  the interval  $(x - \delta, x + \delta)$  contains both rational and irrational numbers. This implies that  $x$  cannot be an interior point of  $\mathbb{Q}$ , and that  $x$  is a boundary point of  $\mathbb{Q}$ . Therefore the interior of  $\mathbb{Q}$  is empty and the boundary of  $\mathbb{Q}$  is  $\mathbb{R}$ . Thus  $\overline{\mathbb{Q}} = b(\mathbb{Q}) \cup \mathbb{Q} = \mathbb{R} \cup \mathbb{Q} = \mathbb{R}$ .

3. (20 pts.) Let

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ x, & x \notin \mathbb{Q} \end{cases}$$

Prove that  $f$  is continuous at 0 and nowhere else.

**Solution:** To show that  $f$  is continuous at 0 observe that  $|x| \geq |f(x)| \geq 0$  and by the squeeze theorem  $\lim_{x \rightarrow 0} f(x) = 0 = f(0)$ . Thus,  $f$  is continuous at 0. Now let  $x \neq 0$  if  $x \notin \mathbb{Q}$  there exists a sequence of rational numbers  $\{q_n\}$  such that  $q_n \rightarrow x$ . Then  $f(q_n) \rightarrow 0 \neq x = f(x)$ . If  $x \in \mathbb{Q}$  there is a sequence of irrational numbers  $\{p_n\}$  such that  $p_n \rightarrow x$ , Then  $\lim_{n \rightarrow \infty} p_n \neq 0 = f(x)$ . Thus  $f$  is not continuous at any  $x \neq 0$ .

4. (20 pts.) Let  $A$  be a compact set and  $B$  a closed set. Show that  $A \cap B$  is compact.

**Solution:**  $A$  is compact and therefore closed and bounded.  $A \cap B \subset A$  and is also bounded.  $A \cap B$  is the intersection of two closed sets and therefore closed. So  $A \cap B$  is closed and bounded and therefore compact.

5. (20 pts.) Use the definition of continuity to show that  $f(x) = \sqrt{x}$  is continuous on  $[0, \infty)$ . Is this function uniformly continuous on this set. Justify your answer.

**Solution:** Let  $\epsilon > 0$ , and  $x_0$  be given. Choose  $\delta = \sqrt{x_0}\epsilon$  then

$$|\sqrt{x} - \sqrt{x_0}| = \frac{|x - x_0|}{\sqrt{x} + \sqrt{x_0}} \leq \frac{|x - x_0|}{\sqrt{x_0}} < \frac{\epsilon\sqrt{x_0}}{\sqrt{x_0}} = \epsilon,$$

for all  $x \in (x_0 - \delta, x_0 + \delta) \cap [0, \infty)$ . Therefore  $f$  is continuous on  $[0, \infty)$ . Now  $f$  is also uniformly continuous on  $[0, \infty)$ . To see this, we know that  $f$  is uniformly continuous on  $[0, 1]$ , since this set is compact. On  $[1, \infty)$  we may use  $\delta = \epsilon$  for all  $x_0$ . Thus  $f$  is uniformly continuous on  $[1, \infty)$ . Now for  $\epsilon > 0$  let  $\delta_1$  be the number we obtain from the uniform continuity of  $f$  on  $[0, 1]$  and  $\delta_2 = \epsilon$  (which works on  $[1, \infty)$ ). Then  $\delta = \min\{\delta_1, \delta_2\}$  works for all  $x_0 \in [0, \infty)$  and thus  $f$  is uniformly continuous.

6. (20 pts.) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function and let  $U \subset \mathbb{R}$  be an open set. Show that  $f^{-1}(U)$  is an open subset of  $\mathbb{R}$ .

**Solution:** Let  $x_0 \in f^{-1}(U)$ . We need to show that there is a  $\delta > 0$  such that  $(x_0 - \delta, x_0 + \delta) \subset f^{-1}(U)$ . To do this observe that  $f(x_0) \in U$  and since  $U$  is open there exists an  $\epsilon > 0$  such that  $(f(x_0) - \epsilon, f(x_0) + \epsilon) \subset U$ . Since  $f$  is continuous there exists a  $\delta > 0$  such that  $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$  for all  $x \in (x_0 - \delta, x_0 + \delta)$ . Therefore,  $(x_0 - \delta, x_0 + \delta) \subset f^{-1}[(f(x_0) - \epsilon, f(x_0) + \epsilon)] \subset f^{-1}(U)$ . Hence,  $f^{-1}(U)$  is open.