PUMP SEMINAR 1 – October 8

Assignments

The following list contains 12 problems. Each student should select one of these problems to prepare and give a approx 20 minute presentation of the results. The student will also provide a written solution using LaTeX. (More about LaTeX in class.) The written solutions are due on November 19. You should work in class together on these problems, help each other to solve them and then one person presents it.

1. (Presentation on October 15) Construct a Möbius transform that maps the real line into the unit circle (i.e. the circle of radius 1 centered at the origin). Use this to construct a Möbius transform that maps any given line into the unit circle.

2. (October 15) The transform $T(z) = -\frac{z+1}{z-1}$ maps the imaginary axis into the unit circle. Show that the region $\Im z < 0$ is mapped to the interior of the circle. Then find the images of lines which are parallel to the x-axis under this map. Do the same for lines which are parallel to the y-axis.

3. (October 22) The transform $T(z) = -\frac{z+1}{z-1}$ maps the imaginary axis into the unit circle. Find the inverse of this map. Compute the images of the concentric cirles $|z| = r$ under this inverse. Are these images circles or lines?

4. (October 22) Consider the function $f(z) = z^2$. Show that this function maps the first quadrant into the upper half plane (i.e. the first two quadrants). Use this to construct a function which maps the first quadrant into the unit circle.

5. (October 29) Let f be a function such that $f(z_1 + z_2) = f(z_1)f(z_2)$ for all $z_1, z_2 \in \mathbb{C}$. Furthermore let f be differetiable at $z = 0$. Show that f is everywhere differentiable, and that $f'(z) = f'(0)f(z)$.

6. (October 29) Let f be a function such that $f(z_1 + z_2) = f(z_1)f(z_2)$ for all $z_1, z_2 \in \mathbb{C}$. Show that $f(z) = f(1))^z$ for all $z \in \mathbb{C}$.

7. (November 5) Let f be a function such that $f(z_1 + z_2) = f(z_1)f(z_2)$ for all $z_1, z_2 \in \mathbb{C}$. Furthermore let f be differetiable at $z = 0$. Show that the function

has infinitely many derivatives and $f^{(n)}(z) = (f'(0))^n f(z)$. Show that the Taylor Series

$$\sum_{k=1}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

converges for every $z \in \mathbb{C}$.

8. (November 5) Consider the series

$$f(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}.$$

Compute the real and imaginary part of $f(ix)$. Conclude that $f(ix) = \cos x + i \sin x$.

9. (November 12) The Cauchy formula for $f(z) = e^z$ states:

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{e^z}{z} \, dz = 1.$$

Use this to compute the integrals

$$\frac{1}{2\pi} \int_0^{2\pi} e^{\cos t} \cos(\sin t) \, dt, \qquad \frac{1}{2\pi} \int_0^{2\pi} e^{\cos t} \sin(\sin t) \, dt$$

10. (November 12) The Cauchy formula says

$$f^{(n)}(0) = \frac{n!}{2\pi i} \int_{|z|=R} \frac{f(z)}{z^{n+1}} \, dz$$

Assume that $|f(z)| \leq M$ for all $|z| \leq R$. Show that

$$|f^{(n)}(0)| \leq \frac{n! M}{R^n}$$

11. (November 19) Use the result of the previous problem to show that if $|f(z)| < M$ for all $z \in \mathbb{C}$, then $f^{(n)}(0) = 0$ for all $n \geq 1$. Show that f must be a constant function.

12. (November 19) Let $P(z)$ be a non-constant polynomial. Use the previous problem to show that there is a $z_0 \in \mathbb{C}$ such that $P(z_0) = 0$. Hint: Suppose it is not the case, then show that $f(z) = \frac{1}{P(z)}$ is bounded and conclude that it is constant, therfore $P(z)$ is also constant. This result is known as the Fundamental Theorem of Algebra.