Global L^∞ Estimates for a Class of Quasi-linear Parabolic Equations

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Abstract

In this paper we study quasi-linear parabolic equations which admit solutions which lie in a certain prescribed (possibly infinite) interval. These equations are of the form $u_t - \nabla c'(u)\nabla u = f$ together with appropriate boundary conditions, where $c : (a, b) \to \mathrm{R}$ satisfies certain conditions. We derive sufficient conditions for the solution u to lie in the domain of c for all times $t > 0$.

Keywords Quasi-linear parabolic equations, uniform a priori bounds.

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1 Introduction

Many physical systems can be described by a quasilinear parabolic initial-value problem of the form

$$u_t - \nabla c'(u)\nabla u = f \qquad \forall (x,t) \in \Omega \times \mathrm{R}^+, \tag{1}$$

$$u(x,0) = u_0(x) \qquad \forall x \in \Omega, \tag{2}$$

together with flux boundary conditions of the form

$$\frac{\partial u}{\partial \nu} = -A\,(u - u_\Gamma) \qquad \forall (x,t) \in \partial\Omega \times \mathrm{R}^+, \tag{3}$$

or with Dirichlet boundary conditions

$$u = g(x,t) \qquad \forall (x,t) \in \partial\Omega \times \mathrm{R}^+. \tag{4}$$

Here Ω is a domain in R^n, $(n = 2,3)$ with smooth boundary $\Gamma = \partial\Omega$. We are interested in the case when

$$c : (a,b) \to \mathrm{R} \tag{5}$$

is a smooth inreasing function. The interval (a,b) is either finite or of the form (a,∞).

The energy balances in several papers about phase field equations have been of this form (see, e.g. [4, 5, 6, 7, 9]). In these cases the interval $(0,\infty)$ was the domain of c.

To get global smooth solutions to the initial-value problem it is necessary to show that

$$u(x,t) \in (a,b) \tag{6}$$

for all $(x,t) \in \Omega \times \mathrm{R}^+$. This has been done for the specific examples mentioned above, but to the best of our knowledge, this has never been done in a more general setting.

The intent of this paper is to generalize the techniques used in [4, 5, 6, 7]. In particular, we will use Moser-type estimates like the ones introduced by Alikakos in [1], to obtain sufficient conditions for global existence.

If c satisfies

$$c : \mathrm{R}^+ \to \mathrm{R}^-$$

with

$$c'(\xi) \geq K_0 \xi^{-\alpha}. \tag{7}$$

for all $\xi > 0$ and some $K_0 > 0$ and $\alpha > 1$, we have the following theorem for the case of the flux boundary condition.

Theorem 1.1 *Suppose that $f \in L^\infty(0, T; L^p(\Omega))$ for some $p \geq 2$, and that*

$$\frac{p}{p-1} \leq \alpha \leq 2\frac{p(p+1)}{p-1}, \tag{8}$$

$$c'(\xi)\xi^{p-1} \qquad \textit{is non-decreasing for} \qquad \xi > 0. \tag{9}$$

Furthermore, assume that there are numbers $0 < a_0 \leq b_0 < \infty$ and $0 < a_\Gamma \leq b_\Gamma < \infty$ such that

$$a_0 \leq u_0(x) \leq b_0 \qquad \forall x \in \Omega \tag{10}$$

$$a_\Gamma \leq u_\Gamma(x, t) \leq b_\Gamma \qquad \forall (x, t) \in \partial\Omega \times \mathrm{R}^+. \tag{11}$$

Then for every $T > 0$ there exist constants $0 < a_T \leq b_T < \infty$ such that

$$a_T \leq u(x, t) \leq b_T \qquad \forall (x, t) \in \Omega_T = \Omega \times [0, T]. \tag{12}$$

The proof of this theorem can be found in section 2 of this paper. Observe that we do not require $f > 0$. The equation satisfies an implicit maximum (and minimum) principle.

Remark: The theorem can be easily extended to the case

$$c : (a, \infty) \to (-\infty, d) \tag{13}$$

by using the linear transformtions

$$\begin{aligned} w &= u - a, \\ \tilde{c}(\xi) &= c(\xi - a) - d, \end{aligned}$$

where w and $\tilde{(}\xi)$ satisfy the hypotheses of the theorem.

In section 3 we extend this result to the situation, where c is a mapping

$$c : (a, b) \to \mathrm{R}$$

with

$$c'(\xi) \geq K_a(\xi - a)^{-\alpha} + K_b(b - \xi)^{-\alpha} \tag{14}$$

for all $\xi > 0$ and some $K > 0$ and $\alpha > 1$. Then the following theorem holds in the case of the flux boundary conditions.

Theorem 1.2 *Suppose that $f \in L^\infty(0, T; L^p(\Omega))$ for some $p \geq 2$, and that $\alpha > 1$. Furthermore, assume that there are numbers $a < a_0 \leq b_0 < b$ and $a < a_\Gamma \leq b_\Gamma < b$ such that*

$$a_0 \leq u_0(x) \leq b_0 \qquad \forall x \in \Omega \tag{15}$$

$$a_\Gamma \leq u_\Gamma(x, t) \leq b_\Gamma \qquad \forall (x, t) \in \partial\Omega \times \mathrm{R}^+. \tag{16}$$

Then for every $T > 0$ there exist constants $a < a_T \leq b_T < b$ such that

$$a_T \leq u(x, t) \leq b_T \qquad \forall (x, t) \in \Omega_T = \Omega \times [0, T]. \tag{17}$$

There are fewer restrictions on α in this case than in Theorem 1.1.

Remarks:

- Both theorems can be extended to the case of Dirichlet boundary conditions (4). In this case there is a restriction on $g(x,t)$:

$$a_\Gamma \le g(x,t) \le b_\Gamma \tag{18}$$

- Throughout the paper we assume local existence of solutions to these equations. Global existence of solutions follows from local existence together with the $L^\infty-$ bounds proven in this paper (see, e.g. [2, 3]).

2 Proof of Theorem 1.1

We restrict ourselves to the flux boundary condition case. The proof for the Dirichlet boundary conditions is very similar. We assume that the equation admits a unique local solution. The crucial part of the proof of this theorem is to show that there exist uniform a priori bounds on u. To do this we need the following technical lemma, which is an extension of the lemma proven by Ph. Laurençot (Lemma 4.1.5. of [9], and [6]).

Lemma 2.1 *Let $a > 1$, $b \ge 0$, $c \in \mathrm{R}$, $c_0 \ge 1$ and $\eta_0 > 0$ be given numbers and $\{\eta_k\}_{k\ge0}$, $\{\gamma_k\}_{k\ge0}$, and $\{\alpha_k\}_{k\ge0}$ be sequences of positive real numbers such that*

$$\eta_0 + \frac{c}{a-1} > 0 \tag{19}$$

$$\eta_{k+1} = a\eta_k + c, \quad \forall k \in \mathrm{N} \tag{20}$$

$$\gamma_0 \le c_1^{\eta_0} \tag{21}$$

$$\alpha_k = a(1+r_k) \qquad \text{such that} \qquad \Pi_{k=0}^{\infty}(1+r_k) < \infty \tag{22}$$

$$\gamma_{k+1} \le c_0 \eta_{k+1}^b \max(c_1^{\eta_k}, \gamma_k^{\alpha_k}). \tag{23}$$

Then

$$\gamma_k^{\frac{1}{\eta_k}} \le M, \quad \forall k \in \mathrm{N}, \tag{24}$$

where M is a positive constant which depends on c_0, c_1, a, b, η_0, and c.

Proof: The proof is very similar to the proof of the Lemma by Laurençot, which can be found in the references. We only sketch it here. To start let $A_k = \max(c_1^{\eta_{k-1}}, \gamma_k)$, and $\tau_k = \max(\eta_k/\eta_{k-1}, \alpha_k)$. Then τ_k has the same properties as α_k and

$$A_{k+1} \le c_0 \eta_{k+1}^b A_k^{\tau_k}$$

We get

$$A_{k+1} \le c_0^{\Pi_{j=1}^k \tau_j} \eta_1^{b\Pi_{j=2}^k \tau_j} \eta_2^{b\Pi_{j=3}^k \tau_j} \cdots \eta_{k+1}^b A_0^{\Pi_{j=0}^k \tau_j}$$

Taking the η_{k+1} root on both sides we see that the right-hand-side stays bounded as $k \to \infty$ and the result follows immediately.

Q.E.D.

Lemma 2.2 *For every $T > 0$ there exists a constant b_T such that*

$$\|u\|_{L^\infty(\Omega_T)} \le b_T \tag{25}$$

Proof: We start by multiplying the equation by u^{p-1}, where p is the norm of f ($p < \infty$). We obtain after integration by parts:

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + (p-1)\int_\Omega c'(u)u^{p-2}(\nabla u)^2\,dx$$
$$+A\int_{\partial\Omega} c'(u)\left(u^p - u_\Gamma u^{p-1}\right)\,d\gamma = \int_\Omega f u^{p-1}\,dx.$$

Using (7) and Hölder's inequality yield:

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + (p-1)K\int_\Omega u^{p-2-\alpha}(\nabla u)^2\,dx$$
$$+A\int_{\partial\Omega} c'(u)\left(u^p - u_\Gamma u^{p-1}\right)\,d\gamma \le \|f\|_{L^p(\Omega)}\|u\|_{L^p(\Omega)}^{p-1}.$$

Note that

$$(p-1)K\int_\Omega u^{p-2-\alpha}(\nabla u)^2\,dx = K\frac{4(p-1)}{(p-\alpha)^2}\int_\Omega\left(\nabla u^{\frac{p-\alpha}{2}}\right)^2\,dx.$$

For fixed t the boundary can be decomposed into two parts as follows:

$$\begin{aligned}\Gamma_+(t) &= \{z\in\partial\Omega : u(z,t)\ge u_\Gamma(z,t)\}\\ \Gamma_-(t) &= \{z\in\partial\Omega : u(z,t)< u_\Gamma(z,t)\}.\end{aligned}$$

This implies that the boundary term on the left is positive on $\Gamma_+(t)$ and negative on $\Gamma_-(t)$. Since $c'(\xi)\xi^{p-1}$ is monotonically increasing, we have

$$\int_{\Gamma_-(t)} c'(u)u^{p-1}u_\Gamma\,d\gamma \le \int_{\Gamma_-(t)} c'(u_\Gamma)u_\Gamma^p\,d\gamma \le \int_{\partial\Omega} c'(u_\Gamma)u_\Gamma^p\,d\gamma \le C.$$

Using this we can drop the boundary terms and get

$$\frac{1}{p}\frac{d}{dt}\|u\|_p^p + K\frac{4(p-1)}{(p-\alpha)^2}\int_\Omega\left(\nabla u^{\frac{p-\alpha}{2}}\right)^2\,dx \le \|f\|_{L^p(\Omega)}\|u\|_{L^p(\Omega)}^{p-1} + C.$$

Applying Gronwall's Lemma to this last inequality, we arrive at:

$$\sup_{0\le s\le T}\|u\|_{L^p(\Omega)} \le K_1, \tag{26}$$
$$\int_\Omega \left\|u^{\frac{p-\alpha}{2}}\right\|_{H^1(\Omega)}^2\,dx \le K_2. \tag{27}$$

To continue let P_n be a sequence given by

$$\begin{aligned}P_0 &= p\\ P_n &= 2P_{n-1}+\alpha,\end{aligned}$$

which implies that $P_n\ge p$ for all $n\ge 0$. We multiply the equation by u^{P_n-1} to get

$$\frac{1}{P_n}\frac{d}{dt}\|u\|_{P_n}^{P_n} + (P_n-1)\int_\Omega c'(u)u^{P_n-2}(\nabla u)^2\,dx$$
$$+A\int_{\partial\Omega} c'(u)\left(u^{P_n}-u_\Gamma u^{P_n-1}\right)\,d\gamma = \int_\Omega f u^{P_n-1}\,dx.$$

The boundary terms are treated as before, and the lower bound on $c'(u)$ is used. After applying Hölder's inequality to the right-hand-side we obtain

$$\frac{1}{P_n}\frac{d}{dt}\|u\|_{P_n}^{P_n} + K\frac{4(P_n-1)}{(P_n-\alpha)^2}\int_\Omega\left(\nabla u^{\frac{P_n-\alpha}{2}}\right)^2\,dx$$
$$\le \|f\|_{L^p(\Omega)}\left(\int_\Omega u^{(P_n-1)q}\,dx\right)^{\frac{1}{q}} + \int_{\Gamma_-(t)} c'(u_\Gamma)u_\Gamma^{P_n}\,d\gamma$$
$$\le C_1\left(\int_\Omega u^{(P_n-1)q}\,dx\right)^{\frac{1}{q}} + C_2.$$

Next since $\alpha < 2p-3$ and $P_{n-1} \geq p$, we have that $\frac{(P_n-1)q}{P_{n-1}} < 4$, and we can estimate the term on the right-hand-side as follows:

$$\left(\int_\Omega u^{(P_n-1)q}\, dx\right)^{\frac{1}{q}} = \left(\int_\Omega \left(u^{P_{n-1}}\right)^{\frac{(P_n-1)q}{P_{n-1}}}\, dx\right)^{\frac{1}{q}} \tag{28}$$

$$\leq C_3 \left\| u^{P_{n-1}} \right\|_{L^4(\Omega)}^{2+\frac{\alpha-1}{P_{n-1}}} \tag{29}$$

To continue we apply the Nirenberg-Gagliardo inequality followed by Young's inequality

$$\left\| u^{P_{n-1}} \right\|_{L^4(\Omega)}^{2+\frac{\alpha-1}{P_{n-1}}} \leq C \left\| u^{P_{n-1}} \right\|_{H^1(\Omega)}^{\frac{9}{14}(2+\frac{\alpha-1}{P_{n-1}})} \left\| u^{P_{n-1}} \right\|_{L^1(\Omega)}^{\frac{5}{14}(2+\frac{\alpha-1}{P_{n-1}})} \tag{30}$$

$$\leq \delta_n \left\| u^{P_{n-1}} \right\|_{H^1(\Omega)}^{2} + C_1 \frac{1}{\delta_n} \|u\|_{L^{P_{n-1}}(\Omega)}^{P_{n-1}Q_n}, \tag{31}$$

where

$$Q_n = \frac{5}{14}(2+\frac{\alpha-1}{P_{n-1}})\frac{28P_{n-1}}{10P_{n-1}-9(\alpha-1)} = \frac{2(10P_{n-1}+5(\alpha-1))}{10P_{n-1}-9(\alpha-1)}.$$

Observe, that these last two steps are always possible for the choice of α and p in the theorem. Choosing δ_n small enough, we can move the H^1-term to the left with the derivative term. Applying the previous Lemma with $P_n = \eta_n$ and $Q_n = \alpha_n$, and $a = 2$ we have

$$\|u(t)\|_{L^P(\Omega)} \leq b_T \qquad \forall t \in [0,T], \quad \forall P \in [p,\infty), \tag{32}$$

where the constant b_T is independent of P. This immediately implies the Lemma.

Q.E.D.

Remark: In the first part of this proof we used the H^1-norm of $u^{\frac{p-\alpha}{2}}$. This is only correct, if $p \neq 2$ and $\alpha \neq p$. However, the term

$$(p-1)K \int_\Omega u^{p-2-\alpha}(\nabla u)^2\, dx$$

is always positive (independent of p and α) and we do not need the H^1-norm later. It follows that the proof will work independently of the size of p and α, as long as they satisfy the restrictions of the theorem.

To get the lower bound for u in Theorem 1.1 let

$$w = \frac{1}{u} \tag{33}$$

Then w satisfies the following boundary-value problem:

$$\frac{w_t}{w^2} - \nabla \frac{c'(1/w)}{w^2}\nabla w = -f \tag{34}$$

$$\frac{c'(1/w)}{w^2}\frac{\partial w}{\partial \nu} = A\left(\frac{1}{w} - u_\Gamma\right) \qquad \text{for all} \qquad (x,t) \in \partial\Omega \times [0,T]. \tag{35}$$

For this new variable, we have:

Lemma 2.3 *For every $T > 0$ there is positive constant C_T such that*

$$\max_{0 \leq s \leq T} \left\| w^{\frac{\alpha}{2}}(s) \right\|_{L^\infty(\Omega)} \leq C_T \tag{36}$$

Proof: The proof of this assertion again relies on a Moser-type argument. To start we multiply the equation (34) by w^{r+1} for some $r > 0$ and integrate over Ω to get

$$\frac{1}{r}\frac{d}{dt}\|w\|_{L^r(\Omega)}^r + (r+1)\int_\Omega w^r \frac{c'(1/w)}{w^2}|\nabla w|^2\, dx$$
$$-A\int_{\partial\Omega}\left(w^r - u_\Gamma w^{r+1}\right) d\gamma = -\int_\Omega f w^{r+1}\, dx$$

For the boundary terms observe that $u_\Gamma \geq a_\Gamma$ and therefore

$$\begin{aligned} Aa_\Gamma \left\| w^{\frac{r+\alpha}{2}} \right\|_{L^{\frac{2(r+1)}{r+\alpha}}(\partial\Omega)}^{\frac{2(r+1)}{r+\alpha}} &= Aa_\Gamma \int_{\partial\Omega} w^{r+1}\, d\gamma \\ &\leq A \int_{\partial\Omega} u_\Gamma w^{r+1}\, d\gamma. \end{aligned}$$

For the second boundary term we have

$$\begin{aligned} A\int_{\partial\Omega} w^r\, d\gamma &= A\,\|w\|_{L^r(\partial\Omega)}^r \\ &\leq A \left\| w^{\frac{r+\alpha}{2}} \right\|_{L^{\frac{2r}{r+\alpha}}(\partial\Omega)}^{\frac{2r}{r+\alpha}} \\ &\leq A\delta \frac{r+1}{r} \left\| w^{\frac{r+\alpha}{2}} \right\|_{L^{\frac{2r}{r+\alpha}}(\partial\Omega)}^{\frac{2(r+1)}{r+\alpha}} + \frac{A}{\delta(r+1)}. \end{aligned}$$

Choosing δ sufficiently small and using the bounds on $c'(1/w)$ we get

$$\begin{aligned} &\frac{1}{r}\frac{d}{dt}\|w\|_{L^r(\Omega)}^r + \frac{4K(r+1)}{(r+\alpha)^2}\int_\Omega \left|\nabla w^{\frac{r+\alpha}{2}}\right|^2 dx + c_2 \left\| w^{\frac{r+\alpha}{2}} \right\|_{L^{\frac{2(r+1)}{r+\alpha}}(\partial\Omega)}^{\frac{2(r+1)}{r+\alpha}} \\ &\quad\leq \frac{A}{\delta(r+1)} + \|f\|_{L^p(\Omega)} \left\| w^{r+1} \right\|_{L^q(\Omega)} \\ &\quad\quad \leq \frac{A}{\delta(r+1)} + \|f\|_{L^p(\Omega)} \left\| w^{\frac{r+\alpha}{2}} \right\|_{L^{\frac{2q(r+1)}{r+\alpha}}(\Omega)}^{\frac{2(r+1)}{r+\alpha}} \end{aligned}$$

The remaining boundary term and the term with the gradient can be combined in the H^1-norm to get:

$$\begin{aligned} &\frac{1}{r}\frac{d}{dt}\|w\|_{L^r(\Omega)}^r + C_1(r) \left\| w^{\frac{r+\alpha}{2}} \right\|_{H^1(\Omega)}^2 \\ &\quad\quad \leq \frac{A}{\delta(r+1)} + \|f\|_{L^p(\Omega)} \left\| w^{\frac{r+\alpha}{2}} \right\|_{L^{\frac{2q(r+1)}{r+\alpha}}(\Omega)}^{\frac{2(r+1)}{r+\alpha}} \end{aligned}$$

Observe that $\frac{2q(r+1)}{r+\alpha} \leq 4$ and $\frac{r+1}{r+\alpha} < 1$. We can therefore use the Sobolev imbedding theorem and Young's inequality to get

$$\|f\|_{L^p(\Omega)} \left\| w^{\frac{r+\alpha}{2}} \right\|_{L^{\frac{2q(r+1)}{r+\alpha}}(\Omega)}^{\frac{2(r+1)}{r+\alpha}} \leq \|f\|_{L^p(\Omega)} \left\| w^{\frac{r+\alpha}{2}} \right\|_{H^1(\Omega)}^{\frac{2(r+1)}{r+\alpha}}, \tag{37}$$

$$\leq \delta_1 \frac{r+1}{r+\alpha} \left\| w^{\frac{r+\alpha}{2}} \right\|_{H^1(\Omega)}^2 + \frac{1}{\delta_1}\frac{\alpha-1}{r+\alpha}\|f\|_{L^p(\Omega)}^{\frac{r+\alpha}{\alpha-1}}. \tag{38}$$

Choosing δ_1 sufficiently small and integrating over $[0,t]$ yields

$$\begin{aligned} &\|w(t)\|_{L^r(\Omega)}^r + C(r)\int_0^t \left\| w^{\frac{r+\alpha}{2}}(s) \right\|_{H^1(\Omega)}^2 ds \\ &\quad\quad \leq \frac{Ar}{\delta(r+1)}t + C_0 + rt\frac{1}{\delta_1}\frac{\alpha-1}{r+\alpha}\|f\|_{L^p(\Omega)}^{\frac{r+\alpha}{\alpha-1}}. \end{aligned}$$

Note that all the terms on the right can be bounded by a polynomials in r as $r \to \infty$. Taking the supremum on both sides for $0 \leq t \leq T$ and the r^{th} root we get.

$$\sup_{0\leq t\leq T} \|w(t)\|_{L^r(\Omega)} \leq (C_0 + TP(r))^{\frac{1}{r}}.$$

Since the right-hand side of this inequality stays bounded by a constant C_T as $r \to \infty$, it follows immediately that

$$\|w\|_{L^\infty(\Omega_T)} \leq C_T.$$

Q.E.D.

Choosing $a_T = 1/C_T$ we get that

$$u(x,t) \geq a_T \qquad \forall (x,t) \in \Omega_T.$$

Remark: The proof of Lemma 2.3 has very few restrictions on the size of α, only in (37-38) we need $\alpha > 1$.

3 Proof of Theorem 1.2

We prove the case when $a = 0$ and $b = 1$. The general case easily follows using the linear transformation

$$w = \frac{u-a}{b-a}$$

The proof consists of changing the dependent variables and applying Lemma 2.3 to the resulting system. To start let

$$u = \frac{\phi}{1+\phi},$$

then

$$\frac{du}{d\phi} = \frac{1}{1+\phi^2},$$

and ϕ satisfies the parabolic equation

$$\frac{1}{1+\phi^2}\phi_t - \nabla\left(\frac{dc}{d\phi}\nabla\phi\right) = f, \tag{39}$$

and the boundary condition

$$\frac{dc}{d\phi}\frac{\partial\phi}{\partial\nu} = -A\left(\frac{\phi}{1+\phi} - u_\Gamma\right). \tag{40}$$

We now show that $1+\phi$ satisfies the following a priori bound:

$$\|1+\phi\|_{L^\infty(\Omega_T)} \leq B_T, \tag{41}$$

for some constant A_T. To do this, multiply (39) by $(1+\phi)^{r+1}$ for an arbitrary large positive number r. We can proceed exactly as in the proof of Lemma 2.3., using the fact that

$$\frac{dc}{d\phi} \geq K_1(1+\phi)^{\alpha-2}.$$

We get:

$$\frac{1}{r}\frac{d}{dt}\|(1+\phi)\|^r_{L^r(\Omega)} + K(1+r)\int_\Omega (1+\phi)^{r+\alpha-2}|\nabla\phi|^2\,dx$$
$$+\int_{\partial\Omega}(1+\phi)^{r+1}\left(\frac{\phi}{1+\phi} - u_\Gamma\right)d\gamma \leq \int_\Omega f(1+\phi)^{r+1}\,dx$$

The boundary terms have to be treated differently. Let $\Gamma^+(t)$ be the subset of the boundary where

$$\frac{\phi}{1+\phi} \geq c = \frac{1+b_\Gamma}{2} \quad \text{i. e.} \qquad \phi \geq \frac{c}{1-c},$$

and $\Gamma^-(t)$ its complement. We have

$$A\int_{\partial\Omega}(1+\phi)^{r+1}\left(\frac{\phi}{1+\phi} - u_\Gamma\right)d\gamma$$
$$= A\int_{\Gamma^+(t)}(1+\phi)^{r+1}\left(\frac{\phi}{1+\phi} - u_\Gamma\right)d\gamma$$

$$+A\int_{\Gamma^-(t)}(1+\phi)^{r+1}\left(\frac{\phi}{1+\phi}-u_\Gamma\right)\,d\gamma$$

$$\geq A\frac{1-b_\Gamma}{2}A\int_{\Gamma^+(t)}(1+\phi)^{r+1}\,d\gamma$$

$$+A\int_{\Gamma^-(t)}(1+\phi)^{r+1}\left(\frac{\phi}{1+\phi}-u_\Gamma\right)\,d\gamma$$

To continue observe that

$$\left|A\int_{\Gamma^-(t)}(1+\phi)^{r+1}\left(\frac{\phi}{1+\phi}-u_\Gamma\right)\,d\gamma\right|\leq 2\mu(\Gamma^-(t))\left(1+\frac{c}{1-c}\right)^{r+1}=C_1,$$

and

$$\int_{\partial\Omega}(1+\phi)^{r+1}\,d\gamma\leq\int_{\Gamma^+(t)}(1+\phi)^{r+1}\,d\gamma+\mu(\Gamma^-(t))\left(1+\frac{c}{1-c}\right)^{r+1}$$

This provides us with a bound for the $L^{r+1}-$norm of $1+\phi$ on the boundary. From here one follows exactly the steps in the proof of Lemma 2.3. to obtain the desired result. This implies that ϕ is uniformly bounded by B_T. It follows that

$$u(x,t)\leq\frac{B_T}{1+B_T}=b_T<1\qquad\forall(x,t)\in\Omega_T. \tag{42}$$

To get the lower bound for u a different change of dependent variable is used. We let

$$u=\frac{1}{1+\psi}.$$

Then ψ satisfies

$$\frac{1}{1+\psi^2}\psi_t-\nabla\left(\frac{c'(1/(1+\psi))}{(1+\psi)^2}\nabla\psi\right)=-f, \tag{43}$$

and the boundary condition

$$\frac{c'(1/(1+\psi))}{(1+\psi)^2)}\frac{\partial\psi}{\partial\nu}=A\left(\frac{1}{1+\psi}-u_\Gamma\right). \tag{44}$$

This is the same equation as (34) for the variable $1+\psi$. We can apply Lemma 2.3 directly to get

$$\|1+\psi\|_{L^\infty(\Omega_T)}\leq A_T,$$

and it follows that

$$u(x,t)\geq\frac{1}{1+A_T}=a_T>0\qquad\forall(x,t)\in\Omega_T.$$

This concludes the proof of Theorem 1.2.

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