Math 350 — Practice Questions for Final

1. Let $0 < a < 1$ and $\{a_n\} = \{a^{\frac{1}{n}}\}$. Show that this sequence converges and find its limit. Do the same for $a > 1$.

2. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n \leq b_n$ for all but finitely many values of $n$. Prove that $\lim a_n \leq \lim b_n$ and $\lim a_n \leq \lim b_n$.

3. Show that a bounded sequence that does not converge has more than one subsequential limit point.

4. Show that a compact set contains its supremum and infimum.

5. Let $f$ be a continuous function and $C \subset \mathcal{D}(f)$ be compact. Show that $f(C)$ is compact.

6. Let $f$ and $g$ be continuous functions such that $f(q) = g(q)$ for every $q \in \mathbb{Q} \cap \mathcal{D}(f)$. Show that $f(x) = g(x)$ for all $x \in \mathcal{D}(f)$.

7. Show that if $f$ is continuous on $[a, b]$ then $f([a, b])$ is either a single point or a closed and bounded interval.

8. Suppose that $f$ and $g$ have $n$ derivatives on $(a, b)$. Let $h = fg$ Show that for $x \in (a, b)$,

$$h^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x)g^{(n-k)}(x).$$

9. Let $f$ be continuous on $[a, b]$ and differentiable on $(a, b)$ with $f'(x) > 0$ for all $x \in (a, b)$. Show that $f(a) \leq f(x) \leq f(b)$ for all $x \in (a, b)$.

10. Suppose that $f''(x)$ exists and is continuous on $(a, b)$. Show that for $x_0 \in (a, b)$

$$f''(x_0) = \lim \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}.$$ 

11. Suppose $f$ is differentiable on an open interval containing $[a, b]$. Define

$$g(x) = \begin{cases} \frac{f(x) - f(a)}{x - a}, & x \neq a \\ f'(a), & x = a \end{cases}$$

Show that $g(x)$ attains every value between $f'(a)$ and $\frac{f(b) - f(a)}{b - a}$. 
12. Let $P$ be a partition of $[a, b]$ and $Q$ be a refinement of $P$. Show that $\mathcal{S}(f; P) \geq \mathcal{S}(f; Q)$.

13. Let $f$ be a continuous function on $[0, 1]$. Show that
\[
\int_0^1 f(x) \, dx = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n f \left( \frac{k}{n} \right).
\]

14. Give an example of a function $f$ that is not Riemann integrable, but $|f|$ is Riemann integrable.

15. Let $f$ and $g$ be Riemann integrable on $[a, b]$. Define $h(x) = \max\{f(x), g(x)\}$. Prove that $h$ is Riemann integrable.

16. Let $L(x) = \int_1^x \frac{dt}{t}$. Prove that $L(ab) = L(a) + L(b)$.

17. Let $f$ be differentiable on an open interval containing $[a, b]$ and $f(a) = 0$. Show that
\[
\int_a^b |f(x)| \, dx \leq (b - a) \int_a^b |f'(x)| \, dx.
\]

18. For any positive integer $n$ define
\[
a_n = \left( \sum_{k=1}^n \frac{1}{k} \right) - \ln n.
\]

Show that this sequence is a bounded monotone sequence and converges to a limit. This limit is known as Euler’s constant.

19. Let $f$ and $g$ be Riemann integrable on $[a, b]$. Show that
\[
\left| \int_a^b f(x)g(x) \, dx \right| \leq \left( \int_a^b f^2(x) \, dx \right)^{\frac{1}{2}} \left( \int_a^b g^2(x) \, dx \right)^{\frac{1}{2}}
\]

20. Let $f$ be Riemann integrable on $[a, b]$. Show that $\int_a^b |f(x)| \, dx \leq \sup_{x \in [a, b]} |f(x)|(b - a)$. 