Math 350 — Practice Questions for Final

1. Let $0 < a < 1$ and $\{a_n\} = \{a^{\frac{1}{n}}\}$. Show that this sequence converges and find its limit. Do the same for $a > 1$.

2. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences such that $a_n \leq b_n$ for all but finitely many values of n. Prove that $\overline{\lim} a_n \leq \overline{\lim} b_n$ and $\underline{\lim} a_n \leq \underline{\lim} b_n$.

3. Show that a bounded sequence that does not converge has more than one subsequential limit point.

4. Show that a compact set contains its supremum and infimum.

5. Let f be a continuous function and $C \subset \mathcal{D}(f)$ be compact. Show that $f(C)$ is compact.

6. Let f and g be continuous functions such that $f(q) = g(q)$ for every $q \in \mathbb{Q} \cap \mathcal{D}(f)$. Show that $f(x) = g(x)$ for all $x \in \mathcal{D}(f)$.

7. Show that if f is continuous on $[a, b]$ then $f([a, b])$ is either a single point or a closed and bounded interval.

8. Suppose that f and g have n derivatives on (a, b). Let $h = fg$ Show that for $x \in (a, b)$,

$$h^{(n)}(x) = \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) f^{(k)}(x) g^{(n-k)}(x).$$

9. Let f be continuous on $[a, b]$ and differentiable on (a, b) with $f'(x) > 0$ for all $x \in (a, b)$. Show that $f(a) \leq f(x) \leq f(b)$ for all $x \in (a, b)$.

10. Suppose that $f''(x)$ exists and is continuous on (a, b). Show that for $x_0 \in (a, b)$

$$f\text{"}(x_0) = \lim \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}.$$

11. Suppose f is differentiable on an open interval containing $[a, b]$. Define

$$g(x) = \begin{cases} \frac{f(x) - f(a)}{x - a}, & x \neq a \\ f'(a), & x = a \end{cases}$$

Show that $g(x)$ attains every value between $f'(a)$ and $\frac{f(b)-f(a)}{b-a}$,

12. Let P be a partition of $[a,b]$ and Q be a refinement of P. Show that $\overline{S}(f;P) \geq \overline{S}(f;Q)$.

13. Let f be a continuous function on $[0,1]$. Show that

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

14. Give an example of a function f that is not Riemann integrable, but $|f|$ is Riemann integrable.

15. Let f and g be Riemann integrable on $[a,b]$. Define $h(x) = \max\{f(x), g(x)\}$. Prove that h is Riemann integrable.

16. Let $L(x) = \int_1^x \frac{dt}{t}$. Prove that $L(ab) = L(a) + L(b)$.

17. Let f be differentiable on an open interval containing $[a,b]$ and $f(a) = 0$. Show that

$$\int_a^b |f(x)|\,dx \leq (b-a)\int_a^b |f'(x)|\,dx$$

18. For any positive integer n define

$$a_n = \left(\sum_{k=1}^{n} \frac{1}{k}\right) - \ln n.$$

Show that this sequence is a bounded monotone sequence and converges to a limit. This limit is known as Euler's constant.

19. Let f and g be Riemann integrable on $[a,b]$. Show that

$$\left|\int_a^b f(x)g(x)\,dx\right| \leq \left(\int_a^b f^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_a^b g^2(x)\,dx\right)^{\frac{1}{2}}$$

20. Let f be Riemann integrable on $[a,b]$. Show that $\int_a^b |f(x)|\,dx \leq \sup_{x\in[a.b]} |f(x)|(b-a)$.