An elastic membrane with an attached nonlinear thermoelastic rod

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Abstract

We study a model of a membrane whose slow movements are controlled by the motions of an attached shape memory wire. A quasi-static model is used. We include the elastic feed-back of the membrane on the rod. This results in investigating an elliptic boundary value problem in a domain $\Omega \in \mathbb{R}^2$ with a cut, coupled with non-linear equations for the vertical motions cut and the temperature on the cut. These non-linear equations are formally identical to the ones previously studied by J. Sprekels, S. Zheng and others, however, the inhomogeneity has significantly less regularity, which complicates the problem. We prove the existence of a unique global solution to this problem using a fixed point argument.

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1 Introduction

In this note we investigate a model where one or more thin linear rods of shape memory alloy are attached to an elastic membrane. Heating these rods can be used to change their shape and in turn deform the membrane. We assume that the reference domain for the membrane is a bounded domain $\Omega \subset \mathbb{R}^2$. We will also assume that the motions of the rod are slow compared to vibrations of the membrane. In this case it is sufficient to look at a quasi-static model, i. e. we will model the membrane statically and the rod dynamically. However, the membrane itself will act on the actuator via the elastic force. This feed-back of the structure on the actuator will change the dynamics of the entire system. This model was previously introduced in [6]. In the present paper we will present a thorough investigation of the analytic properties of this model.

The rod could be either attached to the boundary or in the interior of the membrane. The boundary case is less complicated and its mathematical properties follow directly from the case when the actuating rod is attached in the interior. The case of finitely many rods is analogous to the case when there is only one rod. We will also assume that the membrane itself is not conducting heat.

In section 2 of this paper contains a comprehensive description of the model at hand. We will introduce the necessary terminology and state the major results, a local existence and uniqueness theorem as well as a global version of the same. Since this situation involves the solution of an elliptic boundary value problem in a domain containing a cut the solutions will have less regularity than the known results for one dimensional models of shape memory alloys (see, for example, [1, 4, 15]). It will therefore be necessary to consider a weak formulation of the problem.

In section 3 we will prove the local existence and uniqueness of weak solutions. This proof was sketched in [6] and will be done in a more detailed way here.

In section 4 we will prove uniform apriori estimates to extend the local solutions to global solutions.

Throughout the article we will use the same model as in [1, 4, 15]. We refer the reader to these papers and the works cited therein for a derivation of this model and its properties. The basic techniques of the present paper are also following the techniques of [1, 4, 15], however, in the present situation we have to do many of the steps with significantly less regularity because of the low regularity of the feedback term.

Several authors have studied related questions. A one-dimensional model of an adaptive structure was introduced in [3]. Two-dimensional models are found in the works of K.-H. Hoffmann, I. Pawłow and A. Zochowski [7, 14, 17]. However, these models differ from the present one in that they use visco-elastic shape memory materials, and that they do not consider the feed-back of the structure (in our case the membrane) on the actuator.

2 Description of the model

In this section we will discuss the situation with an actuator in the interior of the domain. We assume that $\Omega \in \mathbf{R}^2$ is a bounded connected domain with a C^1 boundary Γ. Let Q be a line segment which lies in the interior of Ω. We will assume that $Q = \{\mathbf{x} \in \Omega : 0 \leq x \leq 1, y = 0\}$. Furthermore, we assume that we can extend Q to a smooth non-selfintersecting curve γ which intersects Γ transversally in two points and divides Ω into two subdomains Ω_+ and Ω_- with Lipschitz boundaries as indicated in the figure below. This will allow us to apply Green's formula type arguments.

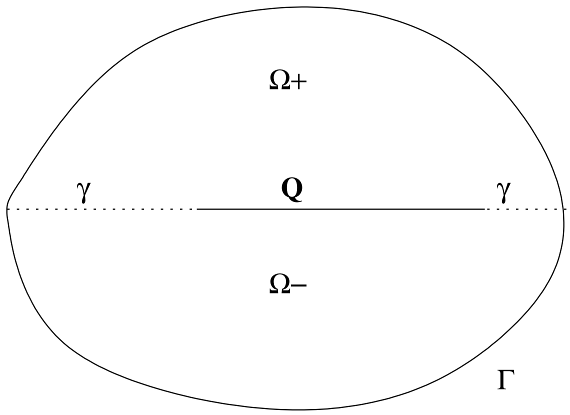

Figure 1: The domain Ω

The vertical displacement U of the membrane satisfies

$$-\Delta U \quad = \quad f(\mathbf{x}, t) \quad \text{for} \quad \mathbf{x} = (x, y) \in \Omega, \tag{1}$$

$$U(\mathbf{x},t)|_Q = v(x,t), \quad (2)$$

$$U(\mathbf{x},t)|_\Gamma = w(\mathbf{x},t), \quad (3)$$

for all $0 \leq t \leq T$.We will also assume that $f \in C^1([0,T];H^1(\Omega))$ and $w \in C^1([0,T];H^{\frac{5}{2}}(\Gamma))$

The function $v(x,t)$ is the vertical displacement of the actuator. It satisfies the nonlinear system for shape memory rods given in [15].

$$v_{tt} - (\sigma(\theta, v_x))_x + Rv_{xxxx} = f_1, \quad (4)$$

$$\theta_t - \kappa\theta_{xx} - \theta\,(\sigma(\theta, v_x))_\theta\, v_{xt} = g. \quad (5)$$

Here θ is the absolute temperature of the rod and the function σ is given by

$$\sigma(\theta, v_x) = \gamma\,(\theta_1 - \theta)\, v_x - \beta v_x^3 + \alpha v_x^5. \quad (6)$$

θ_1, α, β, γ, κ and R are positive constants. We refer the reader to [15] for a detailed investigation of this model. v can be interpreted as either the tangential (as in [15]) or the normal displacement (as in [16]). We will interpret it as the normal displacement here. The function g represents an external heat source. As in previous papers (see, for example, [15]) we will assume that $g \in L^2(0,T;L^2(Q))$ and that $g(x,t) \geq 0$ on $Q \times [0,T]$. This positivity condition is necessary to apply the maximum principle to (5).

The function f_1 represents an external force on the rod. In our case we assume that the external force is only the elastic force acting on the rod from the deformation of the attached, commonly membrane. Following Hooke's law the elastic force is proportional to the normal derivative of the displacement

$$f_1 = c\left[\frac{\partial U}{\partial y}\right], \quad (7)$$

where

$$\left[\frac{\partial U}{\partial y}\right] = \left(-\lim_{y\to 0^+}\frac{\partial U}{\partial y} + \lim_{y\to 0^-}\frac{\partial U}{\partial y}\right). \quad (8)$$

To simplify notations we will assume that the constant $c = 1$.

The system of equations (4)–(5) is augmented by the following set of initial and boundary conditions.

$$\theta_x(0,t) = \theta_x(1,t) = 0, \tag{9}$$

$$v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = 0, \tag{10}$$

$$\theta(x,0) = \theta_0(x), \tag{11}$$

$$v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x). \tag{12}$$

These boundary conditions, especially $v(0,t) = v(1,t) = 0$ are rather restrictive. We assume that the actuator is glued to the membrane and moves with the membrane even if the actuator is not deformed. To model this let $U(x,t) = u(x,t) + \hat{u}(x)$ where $\hat{u}$ satisfies

$$-\Delta\hat{u} = f \quad \text{on} \quad \Omega \tag{13}$$

$$\left[\frac{\partial\hat{u}}{\partial y}\right]_Q = 0 \tag{14}$$

$$\hat{u}|_\Gamma = w, \tag{15}$$

and u satisfies

$$-\Delta u = 0 \quad \text{on} \quad \Omega \tag{16}$$

$$u|_Q = v \tag{17}$$

$$u|_\Gamma = 0. \tag{18}$$

The system for $\hat{u}$ is time independent and can therefore be treated separately. The existence and uniqueness of solutions of this is a classical result. In fact the boundary condition along Q can be ignored, since it is automatically satisfied by any smooth solution to the equation.

We will therefore just consider (16)–(18) coupled with the non-linear system (4)–(5) via the force

$$f_1 = \left[\frac{\partial u}{\partial y}\right]. \tag{19}$$

For any $v \in H^1(Q)$ the solutions to (16)–(18) are only in $H^1(\Omega)$, and therefore $f_1 \in (H^{\frac{1}{2}}(Q))'$. It is therefore neccessary to define f_1 as a linear functional. We follow the approach used in [5]. For this let $D = \Omega \cup Q$. Then for any $\phi \in H_0^1(D)$, define

$$\langle f_1, \phi \rangle = -\int_\Omega \nabla u \nabla \phi \, dx, \tag{20}$$

where u satisfies (16)–(18). Formally, this is Green's formula, and

$$\langle f_1, \phi \rangle = \int_Q \left[\frac{\partial u}{\partial y} \right] \phi \, dx.$$

The functional f_1 is supported on Q. We can therefore restrict it to functions $\hat{\phi} \in H^{\frac{1}{2}}(Q)$ as follows by taking $\phi \in H_0^1(D)$ an extension of $\hat{\phi}$ to D. We denote the action by f_1 on $H^{\frac{1}{2}}(Q)$ by

$$\langle f_1, \phi \rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)}.$$

In order to write a weak formulation of (4)–(5) we introduce the following spaces:

$$\begin{aligned} X_1(t) &= C(0,t; H^3(Q)) \cap C^1(0,t; H^1(Q)) && (21)\\ X_2(t) &= L^2(0,t; H^2(Q)) \cap C(0,t; H^1(Q)) \cap C^1(0,t; L^2(Q)). && (22) \end{aligned}$$

Let $\|\cdot\|_{X_i(t)}$ denote the norms in these spaces.
We say the pair $(v, \theta) \in X_1(t) \times X_2(t)$ is a weak solution to (4)-(5) together with the initial and boundary conditions, if (v, θ) satisfy the initial conditions and

$$\int_0^t \left(\langle v_t, \phi_t \rangle + \langle \sigma, \phi_x \rangle - R \langle v_{xxx}, \phi_x \rangle \right) \, ds \tag{23}$$

$$= \langle v_1, \phi \rangle - \int_0^t \langle f_1, \phi \rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)} \, ds, \tag{24}$$

$$\int_0^t \left(\langle \theta_t, \psi \rangle + \kappa \langle \theta_x, \psi_x \rangle - \langle \sigma_\theta v_{xt} \theta, \psi \rangle \right) \, ds = \int_0^t \langle g, \psi \rangle \, ds, \tag{25}$$

for every pair $(\phi, \psi) \in \hat{X}_1(t) \times \hat{X}_2(t)$. Here

$$\begin{aligned} \hat{X}_1(t) &= \left\{ \phi \in H^1(0,t; H_0^1(Q)) : \phi(x,0) = 0 \right\} && (26)\\ \hat{X}_2(t) &= L^2(0,t; H^1(Q)). && (27) \end{aligned}$$

We can now state the main results of this paper. In Section 3 we will proof the following local existence theorem:

Proposition 2.1 *For sufficiently small $t > 0$ there exists a unique triple*

$$(u, v, \theta) \in C(0, t; H^1(\Omega)) \times X_1(t) \times X_2(t)$$

such that u solves (16)–(18), and (v, θ) satisfy the initial conditions and (24)–(25).

In Section 4 we will proof some uniform apriori estimates for the weak solutions which will give the following global existence result:

Proposition 2.2 *For any given $T > 0$ there exists a unique triple*

$$(u, v, \theta) \in C(0, T; H^1(\Omega)) \times X_1(T) \times X_2(T),$$

such that u solves (16)–(18), and (v, θ) satisfy the initial conditions and (24)–(25).

Remark: One could arrive at higher regularities by using weighted Sobolev space methods as is described in [8, 9, 10, 11]. In particular we have that the solution u to (16)–(18) satisfies

$$u \in V_\beta^{l+1}(\Omega)$$

for $|\beta - l| < \frac{1}{2}$. Its trace on the boundary $\partial\Omega$ (which includes Q) satisfies

$$u \in V_\beta^{l+\frac{1}{2}}(\partial\Omega).$$

In this situation the weighted Sobolev spaces $V_\beta^{l+1}(\Omega)$ are endowed with the following norm:

$$\|u\|^2_{V_\beta^l(\Omega)} = \|(1 - \zeta_1 - \zeta_2)u\|^2_{H^l(\Omega)} + \sum_{j=1}^{2} \int_\Omega \sum_{|\alpha| \leq l} |x - x_j|^{2(\beta - l + |\alpha|)} \left|D_x^\alpha(\zeta_j u)\right|^2 \, dx,$$

where x_1 and x_2 denote the endpoints of the rod Q and ζ_j are $C^\infty-$functions which are equal to one in a neighborhood of x_j and vanish outside a neighborhood of x_j. $\alpha = (\alpha_1, \alpha_2)$ is a multi-index. We refer the reader to [10] for further information on this subject.

3 Proof of Proposition 2.1

Before proving this result we need to get some preliminary results. To do this define the following linear pseudo-differential operator associated to the elliptic system (16)–(18).

$$\mathcal{F} : v \mapsto f_1 = \left[\frac{\partial u}{\partial y}\right]. \tag{28}$$

This operator involves the solution of (16)–(18). The next Lemma gives a result about the regularity of $\mathcal{F}$. For this let $(H^{\frac{1}{2}}(Q))'$ denote the usual dual space of $H^{\frac{1}{2}}(Q)$.

Lemma 3.1 *The operator*

$$\mathcal{F} : C^1(0,t; H^1(Q)) \to C^1(0,t; (H^{\frac{1}{2}}(Q))')$$

is bounded, i.e. there are constants C_1, C_2 which only depend on Ω such that:

$$\|\mathcal{F}(v)\|_{(H^{\frac{1}{2}}(Q))'} \leq C_1 \|v\|_{H^1(Q)}, \tag{29}$$

$$\|(\mathcal{F}(v))_t\|_{(H^{\frac{1}{2}}(Q))'} \leq C_3 \|v_t\|_{H^1(Q)}. \tag{30}$$

Proof: Elliptic regularity theory implies:

$$\|u\|_{H^1(\Omega)} \leq C \|v\|_{H^{\frac{1}{2}}(Q)}. \tag{31}$$

(see, for example, [12, 13]). We combine this estimate with the trace theorem and the compact inclusion $H^1(Q) \subset H^{\frac{1}{2}}(Q)$ and get the first estimate of the lemma. Next observe that u_t satisfies

$$-\Delta u_t = 0 \quad \text{on} \quad \Omega \tag{32}$$

$$u(\mathbf{x}, t)|_Q = v_t(x, t) \tag{33}$$

$$u(\mathbf{x}, t)|_\Gamma = 0 \tag{34}$$

We apply the same reasoning as above to this elliptic equation to get the second inequality in the lemma.

Proof of Proposition 2.1: It suffices to show that (4)–(5) admit a weak solution in the sense (24)–(25) for any right hand side $\mathcal{F}(v) \in C^1(0,t;(H^{\frac{1}{2}}(Q))')$. We start with two observations:
First for σ given by (6) and $(\hat{v},\hat{\theta}) \in X_1(t) \times X_2(t)$ we have $\sigma \in L^2(0,t;H^2(Q)) \cap C(0,t;H^1(Q))$.
Secondly if $f_1 \in C^1(0,t;(H^{\frac{1}{2}}(Q))')$ and $g \in C(0,t;H^1(Q))$ we can define the linear form $\mathcal{L}(g)$ as follows:

$$\begin{aligned} \mathcal{L}(g) &= \langle f_1(t), g(t)\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)} - \langle f_1(0), g(0)\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)} \\ &\quad - \int_0^t \langle f_{1t} g\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)}\, ds \qquad (35) \\ &= \int_0^t \langle f_1, g_t\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)}\, ds. \qquad (36) \end{aligned}$$

Observe, that the definition of $\mathcal{L}(g)$ only involves f_1 and f_{1t} acting on g, but not g_t. Formally, this is equivalent to integration by parts in the variable t.
To prove the proposition we consider the following linear problem:

$$v_{tt} + Rv_{xxxx} = f_1 + (\sigma(\hat{\theta}, \hat{v}_x))_x, \qquad (37)$$

$$\theta_t - \kappa\theta_{xx} = \hat{\theta}(\sigma(\hat{\theta}, \hat{v}_x))_{\hat{\theta}}\hat{v}_{xt} - g, \qquad (38)$$

where

$$f_1 = \mathcal{F}(\hat{v}). \qquad (39)$$

These equations are augmented by the initial and boundary conditions. Since the right-hand-side of (37) contains $f_1 \in C^1(0,t;(H^{\frac{1}{2}}(Q))')$, the solutions to this equations are not classical solutions, in we must interpret this equation in the same weak sense as (24).
The linear system (37)–(39) defines a map

$$\mathcal{G} : (\hat{v},\hat{\theta}) \mapsto (v,\theta). \qquad (40)$$

To continue define for positive constants M_0 and M_1 the following subset of $X_1(h) \times X_2(h)$:

$$\mathcal{B} = \left\{ \|v\|_{X_1(h)} \le M_0, \|\theta\|_{X_2(h)} \le M_1, \theta > 0 \right\} \qquad (41)$$

We will show that for sufficiently small h the map $\mathcal{G}$ is a contraction

$$\mathcal{G} : \mathcal{B} \to \mathcal{B} \tag{42}$$

We do this in several steps.

Step 1: We multiply (37) by v_t, integrate over $Q \times (0,t)$ and obtain after integration by parts:

$$\begin{aligned} &\frac{1}{2}\left(\|v_t(t)\|^2 + R\,\|v_{xx}(t)\|^2\right) \\ &\qquad \leq \frac{1}{2}\left(\|v_1\|^2 + R\,\|v_{0xx}\|^2\right) \\ &\qquad\quad + \int_0^t \left\|\sigma(\hat{\theta},\hat{v})_x\right\| \|v_t\| \; ds + \int_0^t \langle f_1, v_t\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)} \, ds. \end{aligned} \tag{43}$$

We apply (35), Hölder's and Young's inequalities to the last term on the right to get

$$\begin{aligned} \frac{1}{2}\left(\|v_t(t)\|^2 + R\,\|v_{xx}(t)\|^2\right) &\leq \frac{1}{2}\left(\|v_1\|^2 + R\,\|v_{0xx}\|^2\right) \\ &+ C_1 t + \frac{1}{2}\int_0^t \|v_t\|^2 \; ds + \frac{1}{2}\int_0^t \|\mathcal{F}(\hat{v}_t)\|^2_{H^{-1}(Q)} \; ds \\ &+ \frac{1}{2}\int_0^t \|v\|^2_{H^1(Q)} \; ds \\ &+ \langle \mathcal{F}(\hat{v}(t)), v(t)\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)} - \langle \mathcal{F}(\hat{v}_0), v_0\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)}, \end{aligned} \tag{44}$$

for a suitable constant C_1. The second to last term can be treated via Hölder's and young's inequalities again. The last term is bounded by $C_2\,\|v_0\|^2_{H^1(Q)}$. Since $v(0,s) = v(1,s) = 0$, there is a $\xi \in Q$ for each $s \in [0,t]$ such that $v_x(\xi, s) = 0$. We can therefore apply Poincaré's inequality to both v and v_x to obtain:

$$\|v\|_{H^1(Q)} \leq C_3\,\|v_{xx}\|\,, \tag{45}$$

for a suitable C_3 (see, for example, [2] for details). Combining these results we get

$$\begin{aligned} \frac{1}{2}\left(\|v_t(t)\|^2 + \hat{R}\,\|v_{xx}(t)\|^2\right) \;\; &\leq \;\; \frac{1}{2}\left(\|v_1\|^2 + R\,\|v_{0xx}\|^2\right) \\ &\qquad + C_4 t + \int_0^t \left(\|v_t\|^2 + \|v_{xx}\|^2\right) \, ds, \end{aligned}$$

for a suitable positive constant C_4 which only depends on the initial data, and $(\hat{\theta}, \hat{v}) \in \mathcal{B}$. Applying Gronwall's inequality we get:

$$\begin{aligned} &\|v_t(t)\|^2 + \hat{R} \|v_{xx}(t)\|^2 \\ &\quad \leq e^t \left(\|v_1\|^2 + R \|v_{0xx}\|^2 + C_5 t \right), \end{aligned} \tag{46}$$

where C_5 only depends on M_0 and M_1.

Step 2: We multiply (38) by θ to get after integration by parts:

$$\frac{1}{2} \|\theta(t)\|^2 + \int_0^t \|\theta_x\|^2 \, ds \leq \frac{1}{2} \|\theta_0\| + C_6 + \int_0^t \|\theta\|^2 \, ds, \tag{47}$$

for a suitable constant C_6. Next we multiply (38) by θ_t and obtain integration by parts and applying Young's inequality:

$$\frac{1}{2} \|\theta_x(t)\|^2 + \int_0^t \|\theta_t\|^2 \, ds \leq \frac{1}{2} \|\theta_{0x}\| + C_7, \tag{48}$$

for a suitable constant C_7. We combine these last two results and apply Gronwall's lemma to get:

$$\|\theta(t)\|^2_{H^1(Q)} \leq e^t \left(\|\theta_0\|^2_{H^1(Q)} + tC_3 \right), \tag{49}$$

where C_3 again only depends on M_0 and M_1.

Step 3: In this step we multiply (37) by $-v_{xxt}$, integrate over $Q \times (0,t)$ and integrate by parts. The right hand side of the resulting equation contains the term

$$\int_0^t \langle f_1, v_{xxt} \rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)},$$

which is again treated using (35). After a similar argument as in Step 1 we get:

$$\|v_{xt}(t)\|^2 + \tilde{R} \|v_{xxx}\|^2 \leq e^t \left(\|v_{1x}(t)\|^2 + \tilde{R} \|v_{0xxx}\|^2 + tC_8 \right), \tag{50}$$

where C_8 only depends on M_0 and M_1.

Step 4: We combine the first three steps to get the following inequalities:

$$\|v\|^2_{X_1(t)} \leq e^t \left(\|v_0\|^2_{H^3(Q)} + \|v_1\|^2_{H^1(Q)} + tK_1 \right), \tag{51}$$

$$\|\theta\|^2_{X_2(t)} \leq e^t \left(\|\theta_0\|^2_{H^1(Q)} + tK_2 \right), \tag{52}$$

where K_1, and K_2 depend on M_0 and M_1. Furthermore, since (38) satisfies a maximum principle we have that $\theta > 0$. We can now pick M_0, M_1 and h such that the map $\mathcal{G}$ satisfies

$$\mathcal{G} : \mathcal{B} \to \mathcal{B} \tag{53}$$

Step 5: it remains to be shown that the map $\mathcal{G}$ is actually a contraction. To do this observe that $\mathcal{F}$ is linear and therefore we have:

$$\begin{aligned} \left\| \mathcal{F}(v^1 - v^2) \right\|_{(H^{\frac{1}{2}}(Q))'} &\leq C_1 \left\| v^1 - v^2 \right\|_{H^1(Q)}, & (54) \\ \left\| (\mathcal{F}(v^1 - v^2))_t \right\|_{(H^{\frac{1}{2}}(Q))'} &\leq C_3 \left\| v^1_t - v^2_t \right\|_{H^1(Q)}. & (55) \end{aligned}$$

for any two functions v^1 and v^2 in $C^1(0, t; H^1(Q))$. To prove that $\mathcal{G}$ is a contraction we will use similar apriori estimates for $\mathcal{G}(v^1, \theta^1) - \mathcal{G}(v^2, \theta^2)$ as in the previous steps. The feed-back term can be treated using (54)–(55) combined with the techniques of the previous steps. If necessary we may use smaller values for h, M_0, and M_1 in order to show that $\mathcal{G}$ is a contraction.
We can now apply the Banach Fixed Point Theorem to obtain the existence of a unique pair $(v, \theta) \in X_1(t) \times X_2(t)$ which solves (24)–(25). To get Proposition 2.1, we solve (16) for the given v.

4 Uniform a priori estimates

In this section we will prove some uniform a priori estimates which will then imply Proposition 2.2. These estimates follow in general the same lines as the estimates in [15], however, because of the lower regularity of f_1 (than in the cited paper) we will need the following preliminary Lemma. We will, however, only use the third assertion of this Lemma, but we will state and proof the others for the sake of completeness.

Lemma 4.1 *Let u satisfy (16)–(18). Define a bilinear form*

$$\mathcal{B} : H^{\frac{1}{2}}(Q) \times H^{\frac{1}{2}}(Q) \to \mathbb{R}$$

as follows:

$$\mathcal{B}(\phi, \psi) = \langle \mathcal{F}(\phi), \psi \rangle_{(H^{\frac{1}{2}}(Q))' \times H^{\frac{1}{2}}(Q)}$$

Then the following estimates hold:

$$\begin{aligned}
\mathcal{B}(v,v) &\leq 0, && (56)\\
|\mathcal{B}(v,v)| &\leq C\|v\|^2_{H^1(Q)}, && (57)\\
\int_0^t \mathcal{B}(v,v_t)\,ds &\leq \frac{1}{2}\|u(\cdot,0)\|^2_{H^1(\Omega)} < \hat{C}\|v_0\|^2_{H^1(Q)}, && (58)\\
\left|\int_0^t \mathcal{B}(v,v_t)\,ds\right| &\leq \tilde{C}\max\left\{\|v(t)\|^2_{H^1(Q)}, \|v_0\|^2_{H^1(Q)}\right\}, && (59)
\end{aligned}$$

where the constants C, $\hat{C}$, and $\tilde{C}$ depend only on the data.

Proof: For the first two assertions, observe that by the definition of $\lfloor$ we have:

$$\mathcal{B}(v,v) = \langle \mathcal{F}(v), v\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)} = \int_\Omega |\nabla u|^2\,dx,$$

since u is an extension of v to $H^1_0(D)$. The result follows immediately.
For the third and fourth assertion we have

$$\mathcal{B}(v,v_t) = \langle \mathcal{F}(v), v_t\rangle_{(H^{\frac{1}{2}}(Q))'\times H^{\frac{1}{2}}(Q)} = \int_\Omega \nabla u \nabla u_t\,dx,$$

The result follows after integration over $(0,t)$.
We can now proceed analogously to [15]. We will state the estimates. However, we will not give the proofs unless there is a significant difference. The only differences are due to the terms involving $\mathcal{F}(v)$.
We begin with.

Lemma 4.2 *There exists a constant C which only depends on the initial data and g such that:*

$$\sup_{0<t<T}\left(\|v_t(t)\|^2 + \|v(t)\|^2_{H^2(Q)} + \|v_x(t)\|^6_{L^6(Q)} + \|v_x(t)\|^2_{L^\infty(Q)} + \|\theta\|_{L^1(Q)}\right) \leq C. \quad (60)$$

Proof: We start by multiplying (4) by v_t and integrating the result over Q to get:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\|v_t\|^2 + \frac{c_1}{2}\|v_x\|^2 + \frac{\alpha}{6}\|v_x\|^6_{L^6(Q)} + \frac{R}{2}\|v_{xx}\|^2\right)\\
&\qquad = \mathcal{B}(v,v_t) + \frac{\beta}{4}\frac{d}{dt}\|v_x\|^4_{L^4(Q)} - \int_Q \gamma\theta v_x v_{xt}\,dx.
\end{aligned}$$

Next we take (5) and integrate it over Q to get

$$\frac{d}{dt}\|\theta\| = \int_Q g\,dx + \int_Q \gamma\theta v_x v_{xt}\,dx$$

Adding this equation to the above equation the coupling term

$$\int_Q \gamma\theta v_x v_{xt}\,dx$$

cancels. To continue we integrate over $(0,t)$ and apply (58) to estimate the term $\int_0^t \mathcal{B}(v,v_t)\,ds$. The L^4 term on the right can be estimated against the L^6 term on the left. The same argument as in Step 1 of the preceding section is used to estimate the H^2 norm with $\|v_{xx}\|$. The result then follows by taking the supremum over $(0,T)$ and applying the Sobolev Imbedding Theorem to v_x in order to get an estimate for $\|v_x\|_{L^\infty(Q)}$.
The next estimate is only concerned with the energy balance (5). Thus, its proof is identical to the proof in the previous papers (see, for example [15]). We state result for completeness.

Lemma 4.3 *There exists a constant C which only depends on the initial data and the inhomogeneity g such that:*

$$\sup_{0<t<T} \|\theta(t)\|^2 + \int_0^T \left(\|\theta_x(s)\|^2 + \|\theta(s)\|^2_{L^\infty(Q)}\right) ds \le C. \tag{61}$$

We continue as in the proof of Lemma 2.6 of [15] by multiplying (4) by $-v_{xxt}$ and (5) by θ_t. Only the term

$$-\int_0^t \mathcal{B}(v, v_{xxt})\,ds$$

is different from the treatment in [15]. For this term we use (35) to get

$$\begin{aligned} -\int_0^t \mathcal{B}(v, v_{xxt})\,ds = &\int_0^t \mathcal{B}(v_t, v_{xx})\,ds \\ &-\mathcal{B}(v(t), v_{xx}(t)) + \mathcal{B}(v_0, v_{0xx}). \end{aligned}$$

The last two terms on the right are bounded by virtue of Lemma 4.2. For the first term on the right observe that

$$\left|\int_0^t \mathcal{B}(v_t, v_{xx})\,ds\right| \le \frac{1}{2}\int_0^t \left(\|v_t(s)\|^2_{H^1(Q)} + \|v_{xx}(s)\|^2_{H^1(Q)}\right) ds.$$

This term will be estimated by an application of Gronwall's inequality using the terms

$$\frac{1}{2}\left(\|v_{xt}(t)\|^2 + \|v_{xxx}(t)\|^2\right),$$

which appear on the left. Carrying out the steps of the proof of Lemma 2.6 in [15] we arrive at

Lemma 4.4 *There exists a constant C which only depends on the initial data and g such that:*

$$\sup_{0<t<T}\left(\|v_{xt}(t)\|^2 + \|v_{xxx}(t)\|^2 + \|\theta_x(t)\|^2\right) + \int_0^T \left(\|\theta_t(s)\|^2 + \|\theta_{xx}(s)\|^2\right) ds \leq C. \quad (62)$$

Finally, we can combine all the previous estimates to deduce that

$$\sup_{0<t<T} \|v_{tt}(t)\|^2_{H^{-1}(Q)} \leq C, \quad (63)$$

for a constant C that only depends on the initial data and g.
After these estimates Proposition 2.2 follows immediately.

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