Models for Adaptive Structures Using Shape Memory Actuators

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Abstract

Two models are presented for driving the shape of an elastic membrane using thin rods of shape memory alloys attached to the membrane. Existence and uniqueness of solutions is established for both models.

1 Introduction

This paper initiates a comprehensive study of adaptive structures with shape memory actuators. In particular, we study a model where one or more thin linear rods of shape memory alloy are attached to an elastic membrane. Heating and cooling these rods can be used to change their shape and in turn the deformation of the membrane. We assume that the membrane is over a bounded domain $\Omega \subset \mathbb{R}^2$. We will also assume that the motions of the rod is slow compared to the vibrations of the membrane. In this case it is sufficient to look at a quasi-static model, i. e. we will model the membrane statically and the rod dynamically. However, the membrane itself will act on the actuator via the elastic force. This feed-back of the structure on the actuator will change the dynamics of the entire system.

Several authors have studied related questions. A one-dimensional model of an adaptive structure was introduced in [3]. Two-dimensional models are found in the works of K.-H. Hoffmann and A. Zochowski [4, 9]. However, their model differs from ours in that they use visco-elastic shape memory materials, and that they do not consider the feed-back of the structure (in our case the membrane) on the actuator.

In the following section we will formulate a model which has an actuator on the boundary. This case is the less complicated one. We will establish a global existence and uniqueness result for the state equations. This system will admit classical smooth solutions.

In section 3 we look at the case when the actuator is in the interior of the domain. This situation is somewhat more complicated, since it involves the solution of an elliptic boundary value problem in a domain containing a cut. We will prove a local existence and uniqueness theorem for weak solutions. The cut in Ω reduces the regularity of the solutions, and it is necessary to consider a weak formulation of the problem. In both cases we will assume that the membrane itself is not conducting heat, and that the rod is thermally insulated.

In section 4 we will make some remarks about the possible extensions of the two models. In particular, we will point out the necessary changes in the case when the membrane is a heat conductor. Furthermore, we will mention the differences necessary to describe a fully dynamic system. Finally, the existing quasi-static system could be improved by using more realistic boundary conditions.

Throughout the article we will use the same model as in [7]. We refer the reader to this paper and the works cited therein for a derivation of this model and its properties.

2 Model with an actuator on the boundary

For this model we assume that $\Omega \in \mathbf{R}^2$ is a simply connected domain with a C^1 boundary $\partial\Omega = \Gamma \cup Q$. Here Q is a line segment. Without loss of generality we chose a coordinate system $\mathbf{x} = (x, y)$ such that $Q = \{(x, y) | \, x \in [0, 1], y = 0\}$. Furthermore, let $u(\mathbf{x}, t)$ be the vertical displacement of the membrane at time t, $(0 \leq t \leq T)$ and place $\mathbf{x} \in \Omega$. In a quasi-static approach u satisfies

$$\begin{aligned} -\Delta u(\mathbf{x}, t) &= f(\mathbf{x}, t) \quad \text{for} \quad \mathbf{x} \in \Omega, &(1)\\ u(\mathbf{x}, t)|_Q &= v(x, t), &(2)\\ u(\mathbf{x}, t)|_\Gamma &= w(\mathbf{x}, t), &(3)\end{aligned}$$

for any fixed $t \in [0, T]$. Observe, that we allow both f and w to explicitly depend on time. However, we assume that the changes are slow compared with the oscillations of the membrane. We will also assume that $f \in C^1([0, T]; H^1(\Omega))$ and $w \in C^1([0, T]; H^{\frac{5}{2}}(\Gamma))$. Furthermore, we will assume that

$$w(0, 0) = v(0, t) = v(1, t) = w(1, 0) = 0 \tag{4}$$

for all $t \in [0, T]$. This condition seems overly restrictive, however, we can always normalize the vertical displacement along the boundary by the line connecting $w(0, 0)$ and $w(1, 0)$, via $v(x, t) = w(0, 0)x + w(1, 0)(1 - x) + v'(x, t)$. The function $v(x, t)$ $(v'(x, t))$ is the vertical displacement of the actuator. It satisfies the nonlinear system for shape memory rods given in [7].

$$\begin{aligned} v_{tt} - (\sigma(\theta, v_x))_x + R v_{xxxx} = f_1, &\quad (5)\\ \theta_t - \kappa\theta_{xx} - \theta\,(\sigma(\theta, v_x))_\theta\, v_{xt} = g. &\quad (6)\end{aligned}$$

Here θ is the absolute temperature of the rod and the function σ is given by

$$\sigma(\theta, v_x) = \gamma\,(\theta_1 - \theta)\, v_x - \beta v_x^3 + \alpha v_x^5. \tag{7}$$

θ_1, α, β, γ, κ and R are positive constants. We refer the reader to [7] for a detailed investigation of this model. v can be interpreted as either the tangential (as in [7]) or the normal displacement (as in [8]). We will interpret it as the normal displacement here. The function g represents external heating or cooling of the rod. f_1 represents an external force on the rod. In our case we assume that the external force is only the elastic force acting on the rod from the deformation of the attached membrane. Following Hooke's law the elastic force is proportional to the normal derivative of the displacement u on Q, i. e.

$$f_1(x, t) = c\,\frac{\partial u}{\partial n}\bigg|_Q = \tilde{c}\frac{\partial u}{\partial y}(x, 0, t). \tag{8}$$

We will assume that there are no other external forces, however, the presence of such would not create any additional difficulties in the model.

The system of equations (5)–(6) is augmented by the following set of initial and boundary conditions.

$$\theta_x(0,t) = \theta_x(1,t) = 0, \tag{9}$$
$$v(0,t) = v(1,t) = v_{xx}(0,t) = v_{xx}(1,t) = 0, \tag{10}$$
$$\theta(x,0) = \theta_0(x), \tag{11}$$
$$v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x). \tag{12}$$

In order to study this system we will need the following families of Banach spaces:

$$\begin{aligned} X_1(t) &= C(0,t;H^5(Q)) \\ &\quad \cap C^1(0,t;H^3(Q)) \\ &\quad\quad \cap C^2(0,t;H^1(Q)), \end{aligned} \tag{13}$$
$$\begin{aligned} X_2(t) &= C(0,t;H^3(Q)) \\ &\quad \cap C^1(0,t;H^1(Q)) \\ &\quad\quad \cap L^2(0,t;H^4(Q)), \end{aligned} \tag{14}$$
$$Y_1(t) = C^1(0,t;H^3(\Omega)). \tag{15}$$

Furthermore, we will make the following assumptions:

$$f \in C^1(0,T;H^1(\Omega)); w \in C^1(0,T;H^{\frac{5}{2}}(\Gamma_2)), \tag{16}$$
$$g \in L^2(0,T;H^2(Q)) \cap H^1(0,T;H^1(Q)), \tag{17}$$
$$g(x,t) \geq 0 \quad \text{on} \quad \overline{Q} \times [0,\infty), \tag{18}$$
$$v_0 \in H^5(Q), \tag{19}$$
$$v_0(0) = v_0(1) = v_{0xx}(0) = v_{0xx}(1) = 0, \tag{20}$$
$$v_1 \in H^3(Q), \tag{21}$$
$$v_1(0) = v_1(1) = v_{1xx}(0) = v_{1xx}(1) = 0, \tag{22}$$
$$\theta_0 \in H^3(Q), \tag{23}$$
$$\theta_{0x}(0) = \theta_{0x}(1) = 0. \tag{24}$$

The main result of this section is an existence and uniqueness theorem for this situation.

Proposition 2.1 *Let g, θ_0, v_0 and v_1 satisfy the conditions (16)–(24). Then there exists a unique triple $(u,v,\theta) \in Y_1(T) \times X_1(T) \times X_2(T)$ which satisfy (1)–(12) in the following sense:*

1. *For every fixed $t \in [0,T]$, $u(\mathbf{x},t)$ satisfies (1)–(3).*
2. *(v,θ) satisfy (5)–(12).*

Sketch of the Proof: The proof of this theorem uses the previous results of [7]. The details can be found in [1, 7, 8]. To begin let

$$\mathcal{F} : v \mapsto \left. \frac{\partial u}{\partial n} \right|_Q \tag{25}$$

be the composition of the solution operator to the elliptic problem (1)–(3), and the von Neumann boundary operator on Q.

The elliptic equation has well known regularity properties. In particular we have that this equation has for every $v \in H^{\frac{5}{2}}(Q)$ a unique solution $u \in H^3(\Omega)$ which satisfies:

$$\begin{aligned} \|u\|_{H^3(\Omega)} &\leq C_1 \|f\|_{H^1(\Omega)} \\ &\qquad + C_2 \|v\|_{H^{\frac{5}{2}}(Q)} + C_3 \|w\|_{H^{\frac{5}{2}}(\Gamma_2)} \end{aligned} \tag{26}$$

$$\leq C_4 + C_2 \|v\|_{H^{\frac{5}{2}}(Q)} \tag{27}$$

We combine this with the trace theorem and the definition of f_1 in (5) to get

$$\|\mathcal{F}(v)\| = \|f_1\|_{H^{\frac{3}{2}}}(Q) \leq C_4 + C_2 \|v\|_{H^{\frac{5}{2}}(Q)}, \tag{28}$$

which implies

$$\|f_1\|_{H^1(Q)} \leq C_4 + C_2 \|v\|_{H^{\frac{5}{2}}(Q)}. \tag{29}$$

Next observe that u is a function of t and that u_t satisfies

$$\Delta u_t = 0 \quad \text{in} \quad \Omega, \tag{30}$$

$$u_t = v_t \quad \text{on} \quad Q, \tag{31}$$

$$u_t = 0 \quad \text{on} \quad \Gamma, \tag{32}$$

which in turn implies

$$\|u_t\|_{H^3(\Omega)} \leq \hat{C}_2 \|v_t\|_{H^{\frac{5}{2}}(Q)}. \tag{33}$$

We can again combine this with the trace theorem to get

$$\|(f_1)_t\|_{H^{\frac{3}{2}}(Q)} \leq \hat{C}_2 \|v_t\|_{H^{\frac{5}{2}}(Q)}. \tag{34}$$

We can now continue exactly as in [7, 8]. Observe that f_1 in (5) has more regularity than was required in the cited papers.

We start by sketching the proof of local existence. To do this we define the following mapping

$$\mathcal{G} : X_1(t) \times X_2(t) \rightarrow X_1(t) \times X_2(t), \tag{35}$$

$$(\hat{v}, \hat{\theta}) \mapsto (v, \theta), \tag{36}$$

given by the following linear problem:

$$v_{tt} + Rv_{xxxx} = f_1 + (\sigma(\hat{\theta}, \hat{v}_x))_x, \tag{37}$$

$$\theta_t - \kappa\theta_{xx} - \theta(\sigma(\hat{\theta}, \hat{v}_x))_{\hat{\theta}} \hat{v}_{xt} = g, \tag{38}$$

together with the initial and boundary conditions of the original problem. Here f_1 depends on $\hat{v}$ via (1) and satisfies (28).

Using energy estimates (exactly as in Lemma 4.2.1 of [8]) one can show that for a sufficiently small value of h the map $\mathcal{G}$ is a contraction on the set $X_1^h(M_0) \times X_2^h(M_1)$, where $X_1^h(M_0)$ is the set of all $v \in X_1(h)$ with the following property:

$$\begin{aligned} \max_{0 \leq t \leq h} &\left(\|v(t)\|^2_{H^5(Q)} + \|v_t(t)\|^2_{H^3(Q)} \right) \\ &\qquad + \max_{0 \leq t \leq h} \|v_{tt}(t)\|^2_{H^1(Q)} \leq M_0. \end{aligned} \tag{39}$$

$X_2^h(M_1)$ is the set of all $\theta \in X_2(h)$ satisfying:

$$\max_{0\le t\le h}\left(\|\theta(t)\|^2_{H^3(Q)} + \|\theta_t(t)\|^2_{H^1(Q)}\right) + \int_0^h \|\theta(t)\|^2_{H^4(Q)}\,dt + \int_0^h \|\theta_t(t)\|^2_{H^2(Q)}\,dt + \int_0^h \|\theta_{tt}(t)\|^2\,dt \le M_1. \tag{40}$$

M_0 and M_1 are two constants which only depend on the initial conditions and the data for (1). It follows that this fixed point is a solution to our problem in $X_1(h) \times X_2(h)$.

To continue one can do exactly the same uniform a priori estimates as in [7, 8] to show that this solution is uniformly bounded in $X_1(T) \times X_2(T)$, which implies global existence of the solution.

Q.E.D.

Remark: We required that $f \in H^1(\Omega)$ and $w \in H^{\frac{3}{2}}(\Gamma_2)$. These requirements are not overly restrictive. Existence and uniqueness can also be shown for $f \in L^2(\Omega)$ and $w \in H^{\frac{1}{2}}(\Gamma_1)$. In this case we only get $H^4(Q)$ as the maximal regularity of v. The a priori estimates need to be explicitly redone.

3 A model with an interior actuator.

In this section we will discuss the situation with an actuator in the interior of the domain. We again assume that $\Omega \in \mathbf{R}^2$ is a bounded simply connected domain with a C^1 boundary Γ. Let Q be a line segment which lies in the interior of Ω. We will again assume that $Q = \{\mathbf{x} \in \Omega : 0 \le x \le 1, y = 0\}$. Again the vertical displacement u of the membrane satisfies

$$-\Delta u = f(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in \Omega, \tag{41}$$
$$u(\mathbf{x},t)|_Q = v(x,t), \tag{42}$$
$$u(\mathbf{x},t)|_\Gamma = w(\mathbf{x}), \tag{43}$$

for all $0 \le t \le T$. Furthermore we assume $v(0,t) = v(1,t) = 0$. The vertical displacement v of the shape memory rod also satisfies (5)-(6) and the initial and boundary values for it. However, f_1 has now a slightly different definition, namely

$$f_1 = c[\frac{\partial u}{\partial y}], \tag{44}$$

where

$$[\frac{\partial u}{\partial y}] = \left(-\lim_{y\to 0^+}\frac{\partial u}{\partial y} + \lim_{y\to 0^-}\frac{\partial u}{\partial y}\right) \tag{45}$$

Again we let $\mathcal{F}$ be defined by

$$\mathcal{F} : v \mapsto f_1 = [\frac{\partial u}{\partial y}] \tag{46}$$

Formally, the situation is identical to the situation in the previous section. However, the fact that Q lies inside the domain makes (41)– (43) a very different problem than before. The pseudo-differential operator $\mathcal{F}$ has less regularity that in the previous case. In particular we have the following:

Lemma 3.1 *Let $f \in H^{-1}(\Omega)$ and $w \in H^{\frac{1}{2}}(\Gamma)$. Then $\mathcal{F}$ is a bounded operator from $C^1(0,t;H^1(Q))$ to $C^1(0,t;H^{-\frac{1}{2}}(Q))$, i.e. there are constants C_1, C_2, C_3 which only depend on f, w and Ω such that:*

$$\|\mathcal{F}(v)\|_{H^{-\frac{1}{2}}(Q)} \leq C_1 + C_2 \|v\|_{H^1(Q)}, \tag{47}$$

$$\|(\mathcal{F}(v))_t\|_{H^{-\frac{1}{2}}(Q)} \leq C_3 \|v_t\|_{H^1(Q)}. \tag{48}$$

Proof: Elliptic regularity theory implies:

$$\|u\|_{H^1(\Omega)} \leq C\left(\|f\|_{H^{-1}(\Omega)} + \|w\|_{H^{\frac{1}{2}}(\Gamma)} + \|v\|_{H^{\frac{1}{2}}(Q)}\right). \tag{49}$$

(see, for example, [5, 6]). We combine this estimate with the trace theorem and the trivial imbedding $H^1(Q) \to H^{\frac{1}{2}}(Q)$ and get the first estimate of the lemma.

Next observe that u_t satisfies

$$-\Delta u_t = 0 \quad \text{on} \quad \Omega \tag{50}$$

$$u(\mathbf{x},t)|_Q = v_t(x,t) \tag{51}$$

$$u(\mathbf{x},t)|_\Gamma = 0 \tag{52}$$

We apply the same reasoning as above to this elliptic equation to get the second inequality in the lemma.

Q.E.D.

Remark: We only require that $f \in H^{-1}(\Omega)$. However, demanding more regularity on f will have no impact on the regularity of u. The fact that $u \in H^1(\Omega)$ depends on the geometry of Ω and can not be improved (see, for example, [5, 6]).

We expect that the solutions (v,θ) of (5)-(6) will also be less regular. In particular we will show the existence of weak solutions in this case. To do this let us introduce the following spaces:

$$\begin{aligned} X_1(t) &= C(0,t;H^3(Q)) \\ &\qquad \cap C^1(0,t;H^1(Q)) \end{aligned} \tag{53}$$

$$\begin{aligned} X_2(t) &= L^2(0,t;H^2(Q)) \\ &\qquad \cap C(0,t;H^1(Q)) \\ &\qquad\qquad \cap C^1(0,t;L^2(Q)). \end{aligned} \tag{54}$$

Let $\|\cdot\|_{X_i(t)}$ denote the norms in these spaces.

We say the pair $(v,\theta) \in X_1(t) \times X_2(t)$ is a weak solution to (5)-(6) together with the initial and boundary conditions, if (v,θ) satisfy the initial conditions and

$$\int_0^t \left(\langle v_t, \phi_t\rangle - \langle \sigma, \phi_x\rangle + R\langle v_{xxx}, \phi_x\rangle\right) ds \tag{55}$$

$$= \langle v_1, \phi\rangle - \int_0^t \langle f_1, \phi\rangle\, ds, \tag{56}$$

$$\int_0^t \left(\langle \theta_t, \psi\rangle + \kappa\langle \theta_x, \psi_x\rangle - \langle \sigma_\theta v_{xt}\theta, \psi\rangle\right) ds \tag{57}$$

$$= \int_0^t \langle g, \psi\rangle\, ds, \tag{58}$$

for every pair $(\phi,\psi) \in \hat{X}_1(t) \times \hat{X}_2(t)$. Here

$$\hat{X}_1(t) = \left\{\phi \in H^1(0,t;H^1(Q)) : \phi(x,0) = 0\right\} \tag{59}$$

$$\hat{X}_2(t) = L^2(0,t;H^1(Q)). \tag{60}$$

We can now state the main result of this section;

Proposition 3.1 *For sufficiently small $t > 0$ there exists a unique triple $(u, v, \theta) \in H^1(\Omega) \times X_1(t) \times X_2(t)$ such that u solves (41)–(43), and (v, θ) satisfy the initial conditions and (56)–(58).*

Proof: It suffices to show that (5)–(6) admit a weak solution in the sense (56)–(58) for any right hand side $\mathcal{F}(v) \in C^1(0, t; H^{-\frac{1}{2}}(Q))$. We start with two observations:

First for σ given by (7) and $(\hat{v}, \hat{\theta}) \in X_1(t) \times X_2(t)$ we have $\sigma \in L^2(0, t; H^2(Q)) \cap C(0, t; H^1(Q))$.

Secondly if $f_1 \in C^1(0, t; H^{-\frac{1}{2}})$ and $g \in C(0, t; H^1(Q))$ we can compute

$$\int_0^t \int_Q f_1 g_t \, dx \, ds \tag{61}$$

as follows:

$$\begin{aligned} \int_0^t \int_Q f_1 g_t \, dx \, ds \quad = \quad & \int_Q f_1(t) g(t) \, dx \\ & - \int_Q f_1(0) g(0) \, dx \\ & - \int_0^t \int_Q f_{1t} g \, dx \, ds \end{aligned} \tag{62}$$

All three integrals on the right-hand-side are well defined, and therefore the integral on the left makes sense. This process only depends on the regularity of g, f_1, and f_{1t} not on the regularity of g_t.

To prove the proposition we consider the following linear problem:

$$v_{tt} + R v_{xxxx} = f_1 + (\sigma(\hat{\theta}, \hat{v}_x))_x, \tag{63}$$

$$\theta_t - \kappa \theta_{xx} = \hat{\theta} (\sigma(\hat{\theta}, \hat{v}_x))_{\hat{\theta}} \hat{v}_{xt} - g, \tag{64}$$

where

$$f_1 = \mathcal{F}(\hat{v}). \tag{65}$$

These equations are augmented by the initial and boundary conditions. Since the right-hand-side of (63) contains $f_1 \in C^1(0, t; H^{-\frac{1}{2}}(Q))$, the solutions to this equations are not classical solutions, in we must interpret this equation in the same weak sense as (56).

As in the previous section this linear system defines a map

$$\mathcal{G} : (\hat{v}, \hat{\theta}) \mapsto (v, \theta). \tag{66}$$

To continue define for positive constants M_0 and M_1 the following subset of $X_1(h) \times X_2(h)$:

$$\mathcal{B} = \left\{ \|v\|_{X_1(h)} \leq M_0, \|\theta\|_{X_2(h)} \leq M_1, \theta > 0 \right\} \tag{67}$$

We will show that for sufficiently small h the map $\mathcal{G}$ is a contraction

$$\mathcal{G} : \mathcal{B} \to \mathcal{B} \tag{68}$$

We do this in several steps.

Step 1: We multiply (63) by v_t and obtain after integration by parts:

$$\begin{aligned} & \frac{1}{2} \left(\|v_t(t)\|^2 + R \|v_{xx}(t)\|^2 \right) \\ & \quad \leq \frac{1}{2} \left(\|v_1\|^2 + R \|v_{0xx}\|^2 \right) \\ & \quad\quad + \int_0^t \left\| \sigma(\hat{\theta}, \hat{v})_x \right\| \|v_t\| \, ds \\ & \quad\quad + \int_0^t \int_0^x f_1 v_t \, dx ds. \end{aligned} \tag{69}$$

We apply (62) to the last term on the right to get

$$
\begin{aligned}
&\frac{1}{2}\left(\|v_t(t)\|^2 + R\,\|v_{xx}(t)\|^2\right)\\
&\quad\le \frac{1}{2}\left(\|v_1\|^2 + R\,\|v_{0xx}\|^2\right)\\
&\quad\quad +C_1 t + \frac{1}{2}\int_0^t \|v_t\|^2\, ds\\
&\quad\quad\quad +\frac{1}{2}\int_0^t \|f_{1t}\|_{H^{-1}(Q)}^2\, ds\\
&\quad\quad\quad\quad +\frac{1}{2}\int_0^t \|v\|_{H^1(Q)}^2\, ds\\
&\quad\quad\quad\quad\quad + \|f_1\|_{H^{-1}(Q)}\,\|v(t)\|_{H^1(Q)}\\
&\quad\quad\quad\quad\quad\quad + \|f_1(0)\|_{H^{-1}(Q)}\,\|v_0\|_{H^1(Q)}\,,
\end{aligned}
\tag{70}
$$

for a suitable constant C_1. Since $v(0,s) = v(1,s) = 0$, there is a $\xi \in Q$ for each $s \in [0,t]$ such that $v_x(\xi, s) = 0$. We can therefore apply Poincaré's inequality to both v and v_x to obtain:

$$
\|v\|_{H^1(Q)} \le C_2\,\|u_{xx}\|\,,
\tag{71}
$$

for a suitable C_2 (see, for example, [2] for details). Combining these results we get

$$
\begin{aligned}
&\frac{1}{2}\left(\|v_t(t)\|^2 + \hat{R}\,\|v_{xx}(t)\|^2\right)\\
&\quad\le \frac{1}{2}\left(\|v_1\|^2 + R\,\|v_{0xx}\|^2\right)\\
&\quad\quad C_3 t + \int_0^t \left(\|v_t\|^2 + \|v_x x\|^2\right)\, ds,
\end{aligned}
\tag{72}
$$

for a suitable positive constant C_3 which only depends on the initial data, and $(\hat{\theta}, \hat{v}) \in \mathcal{B}$. Applying Gronwall's inequality we get:

$$
\begin{aligned}
&\|v_t(t)\|^2 + \hat{R}\,\|v_{xx}(t)\|^2\\
&\quad\le e^t\left(\|v_1\|^2 + R\,\|v_{0xx}\|^2 + C_4 t\right),
\end{aligned}
\tag{73}
$$

where C_4 only depends on M_0 and M_1.

Step 2: We multiply (64) by θ to get after integration by parts:

$$
\begin{aligned}
&\frac{1}{2}\|\theta(t)\|^2 + \int_0^t \|\theta_x\|^2\, ds\\
&\quad\le \frac{1}{2}\|\theta_0\| + C_1 + \int_0^t \|\theta\|^2\, ds,
\end{aligned}
\tag{74}
$$

for a suitable constant C_1. Next we multiply (64) by θ_t and obtain integration by parts and applying Young's inequality:

$$
\begin{aligned}
&\frac{1}{2}\|\theta_x(t)\|^2 + \int_0^t \|\theta_t\|^2\, ds\\
&\quad\le \frac{1}{2}\|\theta_{0x}\| + C_2,
\end{aligned}
\tag{75}
$$

for a suitable constant C_2. We combine these last two results and apply Gronwall's lemma to get:

$$\|\theta(t)\|^2_{H^1(Q)} \leq e^t \left(\|\theta_0\|^2_{H^1(Q)} + tC_3\right), \tag{76}$$

where C_3 again only depends on M_0 and M_1.

Step 3: In this step we multiply (63) by $-v_{xxt}$, integrate by parts and use (62). After a similar argument as in Step 1 we get:

$$\begin{aligned} &\|v_{xt}(t)\|^2 + \tilde{R}\,\|v_{xxx}\|^2 \\ &\qquad \leq e^t \left(\|v_{1x}(t)\|^2 + \tilde{R}\,\|v_{0xxx}\|^2 + tC_5\right), \end{aligned} \tag{77}$$

where C_5 only depends on M_0 and M_1. v_{xxt} is only a $H^{-1}(Q)$ function. However, after the use of (62), the equation makes sense.

Step 4: We combine the first three steps to get the following inequalities:

$$\|v\|^2_{X_1(t)} \leq e^t \left(\|v_0\|^2_{H^3(Q)} + \|v_1\|^2_{H^1(Q)} + tK_1\right), \tag{78}$$

$$\|\theta\|^2_{X_2(t)} \leq e^t \left(\|\theta_0\|^2_{H^1(Q)} + tK_2\right), \tag{79}$$

where K_1, and K_2 depend on M_0 and M_1. Furthermore, since (64) satisfies a maximum principle we have that $\theta > 0$. We can now pick M_0, M_1 and h such that the map $\mathcal{G}$ satisfies

$$\mathcal{G} : \mathcal{B} \to \mathcal{B} \tag{80}$$

Finally, we make similar estimates as in Step 1 – Step 3, to show that the map is actually a contraction. We can use smaller values of h, M_0, and M_1 if necessary.The result follows immediately from the Banach Fixed Point Theorem.

Q.E.D.

4 Concluding Remarks

In this section we should remark on the following items:

- If v_{tt} is in $L^2(0, t; L^2(Q))$, and if f and w are twice differentiable in t, the regularity of f_{1tt} is $L^2(0, t; H^{-1}(Q))$. This can be seen by differentiating (41) twice with respect to t and applying elliptic regularity theorems and trace theorems to this new equation. One can then possibly improve the result. However, since the right-hand side of equation (5) has at best the regularity $H^{-\frac{1}{2}}(Q)$ in the spatial variable, we cannot expect more than $H^{\frac{7}{2}}(Q)$ for its solution v. It is therefore impossible to find classical solutions to this equation.

- In the second case (interior actuator) the boundary conditions $v(0, t) = v(1, t) = 0$ are very restrictive. A more reasonable model is the case when the actuator is "glued" to the membrane and moves with the membrane, even if it is not deformed. A way to model this situation is to let $u = u_1 + u_2$, where u_1 satisfies

$$-\Delta u_1 = f \quad \text{on} \quad \Omega \tag{81}$$

$$\left[\frac{\partial u_1}{\partial y}\right]\bigg|_Q = 0 \tag{82}$$

$$u_1|_\Gamma = w, \tag{83}$$

and u_2 satisfies

$$
\begin{aligned}
-\Delta u_2 &= 0 \quad \text{on} \quad \Omega && (84)\\
u_2|_Q &= v && (85)\\
u_2|_\Gamma &= 0. && (86)
\end{aligned}
$$

This can always be done since this an linear equation and be can linearly superpose solutions. Only the equation for u_2 involves v and the feed-back on v only comes from u_2. The results of section 3 immediately apply.

- In the fully dynamic model, the elliptic equation for u has to be replaced by a wave equation. The analysis of this model is left for a future paper.
- If the membrane is conducting heat, we need to couple the existing system with a linear heat equation on Ω. The coupling will be done via Neumann conditions on Q for the heat equation on Ω and a right hand side in (6). This will also be investigated in a later paper.
- Finally, we want to remark that the situation here will give rise to some natural control problems. In particular one would consider the use of g in (6) to control the displacement of the membrane u. This will also be investigated in a later article.

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