

## Course Information for Math 360: Abstract Algebra I (formerly 364)

- **Pre-requisites:** A grade of C or better in Math 262 and Math 320.
- **Course Description:** This is a first course in abstract algebra, to be taken after Math 320. While some mathematical maturity may be assumed because of the 320 prerequisite, it must be kept in mind that most of the structures of abstract algebra are essentially new to the student at this time. Thus, this course will serve as a first level introduction to the primary structures of algebra: rings, fields, vector spaces, and groups. The focus should be on developing competence with a wide assortment of examples including the ability to compute various features in these examples (such as being able to compute indexes of various subgroups and orders of elements in groups, to compute degrees of field extensions, to check a set of vectors for being a basis, etc.), and on proving very basic theorems about these structures (such as Lagranges theorem for subgroups of finite groups, unique factorization for polynomials, relations between ring homomorphisms, kernels, ideals, images, etc.)

The course starts with rings, as the student is already familiar with rings as number-systems (but has simply not seen the ring structure before). The sum of the minimum times suggested below add up to 33 hours, which leaves lots of room in a 15 week 3 hours per week semester. The sum of the maximum times below is 48, which clearly exceeds the time limit for a semester. The material on constructibility (3 hours, not included in the count above) is optional, but is highly recommended, so that students can see some fruits of the material developed in topics I, II, and III.

It is necessary to provide historical perspectives in this course since this is a core course for our Teaching Option majors. Examples of such perspectives are listed after the syllabus.

### 1. Rings (1521 hours)

- (a) Definition and Examples of rings:  $\mathbf{Z}$ ,  $\mathbf{Q}$ ,  $\mathbf{Q}[i]$ ,  $\mathbf{Q}[\sqrt{p}]$ ,  $\mathbf{Q}[\omega_n]$ ,  $\mathbf{R}$ ,  $\mathbf{C}$ ,  $\mathbf{Z}/n\mathbf{Z}$ , Matrix rings, Polynomial rings, Rational functions, Quaternions. Extensive computations with these rings, to get a feel for these objects and how the ring axioms are exemplified in them. Definitions and discussions of commutative rings, integral domains, fields (commutative and noncommutative), and classification of examples above into these categories. (68 hours).
- (b) Subrings, Homomorphisms, Ideals: Definitions, Examples, Extensive computations. (35 hours).
- (c) Polynomial Rings (with emphasis on similarity between  $\mathbf{Z}$  and  $\mathbf{F}[x]$ ): Division algorithm, GCD, Irreducible polynomials, Unique factorization in  $\mathbf{F}[x]$ , Roots of polynomials in fields, Rational roots test, Relation between roots and coefficients, Cyclotomic polynomials and roots of unity, Extensive computations. (68 hours).

2. Vector Spaces (Review; 57 hours)
  - (a) Definition and Examples:  $\mathbf{R}^n$ ,  $\mathbf{Q}[\sqrt{2}]$  as  $\mathbf{Q}$ -space,  $\mathbf{Q}[\text{cube-root } 2]$  as  $\mathbf{Q}$ -space,  $\mathbf{F}[x]$  as  $\mathbf{F}$ -space, Subspaces, Definition and examples of subspaces. (2 hours)
  - (b) Spanning Sets, Linear Independence, Bases, Dimension: Definitions and Examples. (35 hours)
3. Fields and Field Extensions (68 hours)
  - (a) Degree of Field Extensions, multiplicativity of degree. (2 hours)
  - (b) Algebraic and algebraically independent elements in a field extension. Definitions and Examples. (1 hour)
  - (c) Subfield  $\mathbf{F}(a)$  generated by an element in an extension  $K/\mathbf{F}$ . Difference between subring  $\mathbf{F}[a]$  generated by  $a$  and  $\mathbf{F}(a)$ , Minimal Polynomial,  $\mathbf{F}(a)$  as homomorphic image of  $\mathbf{F}[x]$ , Relation between degree of minimal polynomial and degree of  $K/\mathbf{F}$ . Solving polynomial equations as construction of field extensions. (35 hours).
4. Groups (812 hours)
  - (a) Definition and Examples. Much focus on examples, as groups are very new objects to most students. Dihedral groups, Permutation groups, 2X2 Matrix groups, Cyclic groups (concretely as finite subgroups of the unit circle and abstractly), Group of units of  $\mathbf{Z}/n\mathbf{Z}$ . Extensive computations. (34 hours).
  - (b) Subgroups. Definition and examples from the list above. Extensive computations. (12 hours).
  - (c) Cosets, Examples of cosets, Lagranges theorem. (23 hours).
  - (d) Cyclic Groups: Order of Elements and of Subgroups. Fermats little theorem. (23 hours).
5. V. Constructibility (Optional; 3 hours)
  - (a) Constructibility as a process of solving linear and quadratic equations ( 1 hour).
  - (b) Constructible numbers live inside tower of quadratic extensions, Constructible numbers have minimal polynomial of degree a power of 2. (1 hour)
  - (c) Impossibility of constructing 60 degree, doubling the cube. (1 hour).

Examples of historical perspectives to be provided: Relevance of studying general rings arising from early incomplete proofs of Fermats theorem, Hamiltons attempts to create more general rings than complex numbers in Physics, Galois and his use of symmetries to shed light on solutions of polynomials by radicals, extensive use of groups in Physics to describe symmetry. History of matrices. Emmy Noether, credited with systematic study of rings, class of Noetherian rings named in her honor.

- **Books used:** This material can be culled from any standard textbook, but it has to be done with care, since standard texts are often not written from a one-semester perspective. We also suggest that the department write its own textbook and put it out through Quick Copies.