A Simple Model for the Motion of Frog Sperm

Lucy Calvillo[1]
California State University
Northridge, CA 91330-8313

Werner Horn
California State University
Northridge, CA 91330-8313
werner.horn@csun.edu

Many processes in Biology show both random and deterministic effects. While the process gives "on average" the same outcome, when repeated many times, it will show distinctive differences in each individual experiment. The same initial conditions one will get significantly different outcomes. Despite these complications, such processes can often be simulated using elementary methods from calculus and differential equations, coupled with some methods from probability theory. Due to the elementary nature of the mathematical tools involved, such models are well suited for undergraduate research projects in applied mathematics.

The content of this paper started out as such an undergraduate research project in the NASA/JPL/CSUN PAIR program. During experiments involving *in vitro* fertilization of eggs of *Lepidobatrachus laevis* (a species of frog from South America, see [1, 2]), the individual paths of frog sperm on its way to an egg were video-taped. It is known, that the sperm can detect a chemical signature of the egg. After detection, of this signature, the sperm will start swimming toward the egg.

In this paper we will give a simple deterministic model for this situation, based on the known forces acting on the sperm (such as drag) and a simplified model of the sperm-egg interaction. This deterministic model is described in detail in the following section. We will then investigate some interesting analytical properties of this model and give some numerical simulations. In

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the final section of this article, we will introduce random effects to this model and simulate this new model numerically.

The interaction between different sperms has been investigated using game theory (see, for example, [4]). However, in this paper we will only try to derive a model for the physical interaction between the egg and a sperm. As in the above mentioned laboratory experiments, we will assume that the egg rests in shallow water and that the sperm is released near the egg. This is very similar to the actual situation in which frog eggs are in shallow ponds, however, we will exclude environmental effects especially surface waves and currents. We will restrict ourselves to a two dimensional geometry, however, the model can easily be extended to three dimensions.

The deterministic model

We will assume that the egg rests at the origin in $\mathbb{R}^2$. We will model the egg as a disk of radius r_0. The sperm is modeled by a point located at $\mathbf{x}(t) \in \mathbb{R}^2$, at time $t \geq 0$. For simplicity we also assume that the sperm has unit mass. The velocity of the sperm is given by

$$\mathbf{v}(t) = \mathbf{x}'(t).$$

In order to apply Newton's law to the motion of the sperm we need to model the forces acting on the sperm. We will assume that the plane is perpendicular to gravity, such that gravity (or buoyancy for that matter) will not have an effect on the motion of the sperm in the plane. The critical part is to make a reasonable model for the attraction of the sperm to the egg. It is assumed that the egg releases a chemical signature, and that the sperm starts swimming toward the egg as soon as the sperm detects this signature. For simplicity, we will assume that the sperm can only propel itself at a fixed rate. If the egg is located at the origin let $r_1 > r_0$ be the radius at which the sperm can detect the chemical signature of the egg. The geometry of this situation is shown in FIGURE 1.

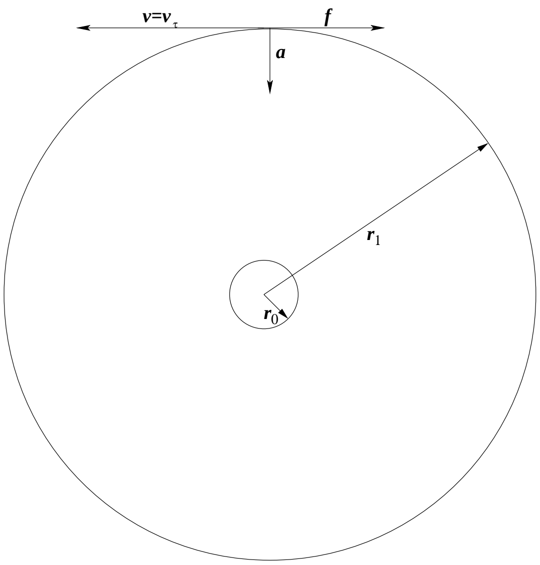

Figure 1: The geometry of the model

After the sperm detects the signature it will accelerate toward the origin at a fixed rate a. The attractive acceleration on the sperm can be written as:

$$\mathbf{f}_0(\mathbf{x}(t)) = a\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}\mathbf{\Theta}(r_1 - |\mathbf{x}(t)|),$$

where $\mathbf{\Theta}$ denotes the usual Heaviside function:

$$\mathbf{\Theta}(\xi) = \begin{cases} 0 & \text{for} \quad \xi < 0 \\ 1 & \text{for} \quad \xi \geq 0 \end{cases}$$

We shall also assume that this force is turned off after the sperm reached the egg. This is easily done by multiplying the above expression with $\mathbf{\Theta}(|\mathbf{x}(t)| - r_0)$. Thus the final form of the attraction is

$$\mathbf{f}_1(\mathbf{x}(t)) = a\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}\mathbf{\Theta}(r_1 - |\mathbf{x}(t)|)\mathbf{\Theta}(|\mathbf{x}(t)| - r_0).$$

The only other force acting on the sperm is drag. If we assume that the body of the sperm is nearly spherical the drag is given by Stokes' formula:

$$\mathbf{f}_2 = -b(\mathbf{x})|\mathbf{v}|\mathbf{v},$$

where b is a positive function which depends on the size of the sperm and the viscosity of the medium in which the sperm moves. The viscosity may be dependent on the location. In particular, the entrance into the egg can be modeled by a rapid increase of the viscosity, i.e.

$$b(\mathbf{x}) = b_1 + b_2 \mathbf{\Theta}(r_0 - |\mathbf{x}(t)|),$$

where $b_2 > b_1 > 0$. We can now combine these forces with Newton's second law to get the following initial value problem:

$$\begin{aligned}
\mathbf{x}'(t) &= \mathbf{v}(t) && (1)\\
\mathbf{v}'(t) &= a\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|}\mathbf{\Theta}(r_1 - |\mathbf{x}(t)|)\mathbf{\Theta}(|\mathbf{x}(t)| - r_0) && (2)\\
&\qquad -(b_1 + b_2\mathbf{\Theta}(r_0 - |\mathbf{x}(t)|))|\mathbf{v}|\mathbf{v} &&\\
\mathbf{x}(0) &= \mathbf{x}_0 && (3)\\
\mathbf{v}(0) &= \mathbf{v}_0 && (4)
\end{aligned}$$

This is a nonlinear system of ordinary differential equations. Moreover, the right hand side of this system is not continuous.

Analytical Treatment of the Model

Despite the discontinuous right hand side, existence and uniqueness for this differential equation do not pose a problem. In each of the three regions $|\mathbf{x}| \leq r_0$, $r_0 < |\mathbf{x}| \leq r_1$ and $|\mathbf{x}| > r_1$, the right hand side of the equation satisfies Lipschitz conditions which guarantee unique solutions. In fact, in the one-dimensional case the explicit solutions, can be easily obtained by separating variables. However, problems may arise at the boundaries of these regions. If the trajectory of a sperm starts in the region $|\mathbf{x}| > r_1$ and reaches the circle $|\mathbf{x}|$ with a positive speed transversally, it is clear that this trajectory will enter the region $|\mathbf{x}| \leq r_1$, where it observes a different well-posed initial value problem. In the region $|\mathbf{x}| > r_1$, the trajectory will be a straight line, along which the sperm travels with speed v, that satisfies.

$$v' = -bv^2.$$

However, this equation has the unique solution

$$v(t) = \frac{v_0}{v_0 bt + 1}$$

which satisfies $v(t) > 0$ as long as the initial speed v_0 is positive. I.e. if the sperm reaches the region $|\mathbf{x}| \leq r_1$, it will do so with positive speed, and it will enter the region if it reaches it transversally.

If the sperm reaches the circle $|\mathbf{x}| = r_1$ tangentially, the situation is somewhat more complicated. To understand this case, let us assume that the sperm reaches the circle $|\mathbf{x}| = r_1$ at time $t = t_0$ in a tangential direction at a point $\mathbf{x}_1$. Without loss of generality we set $t_0 = 0$.

We then have to solve the initial value problem

$$\begin{aligned}
\mathbf{x}'(t) &= \mathbf{v}(t) \\
\mathbf{v}'(t) &= -a\frac{\mathbf{x}(t)}{|\mathbf{x}(t)|} - b_1|\mathbf{v}|\mathbf{v} \\
\mathbf{x}(0) &= \mathbf{x}_1 \\
\mathbf{v}(0) &= \mathbf{v}_1,
\end{aligned}$$

where $\mathbf{v}_1$ is tangential to the circle $|\mathbf{x}| = r_1$. In order to simplify the situation we decompose the velocity into a normal component v_r and a tangential component v_ϕ. Let $\mathbf{e}_r$ and $\mathbf{e}_\phi$ be the unit normal and tangent vectors. Then

$$\mathbf{v} = v_r\mathbf{e}_r + v_\phi\mathbf{e}_\phi,$$

and

$$\mathbf{v}' = v_r'\mathbf{e}_r + v_r\mathbf{e}_r' + v_\phi'\mathbf{e}_\phi + v_\phi\mathbf{e}_\phi'. \tag{5}$$

From the geometry and calculus we know that

$$\mathbf{e}_r' = \mathbf{e}_\phi\phi',$$

and that

$$d\phi = \frac{v_\phi dt}{r_1}.$$

It follows that

$$\phi' = r_1 v_\phi.$$

Similarly, we get that

$$\mathbf{e}'_\phi = -r_1 \mathbf{e}_r v_r.$$

Hence, (5) becomes

$$\mathbf{v} = \left(v'_r - r_1 v_\phi^2\right) \mathbf{e}_r + \left(v'_\phi + r_1 v_r^2\right) \mathbf{e}_\phi.$$

Locally, near the circle $|\mathbf{x}| = r_1$, the right hand side of the differential equation for $\mathbf{v}$ becomes:

$$-\mathbf{e}_r a - b_1 v v_\phi \mathbf{e}_\phi - b_1 v v_r \mathbf{e}_r,$$

where v denotes $|\mathbf{v}| = \sqrt{v_r^2 + v_\phi^2}$. We now obtain the following system for the components:

$$\begin{aligned} v'_r &= -a - b_1 v v_r + r_1 v_\phi^2 && (6) \\ v'_\phi &= -b_1 v v_\phi - r_1 v_r^2 && (7) \end{aligned}$$

At the time $t = 0$ when the trajectory is tangential to the circle we have $v_r = 0$. The right hand side of (6 becomes

$$-a + r_1 v_\phi^2 = -a + r_1 v^2.$$

The sperm will only be able to enter the circle if v'_r is negative at this time, i.e. if

$$a > r_1 v^2. \tag{8}$$

This is clearly a sufficient condition. If $a = r_1 v^2$, we are in the situation of uniform circular motion known from celestial mechanics, where the centrifugal acceleration $r_1 v^2$ balances the acceleration due to the central force. If drag where absent the sperm would start a uniform circular motion about the egg, (which is not observed in nature). However, due to drag, the tangential velocity will decrease and (8) will be satisfied instantaneously, and the sperm will enter the circle. Only if $a < r_1 v^2$ at the point of tangency will the sperm fail to enter the circle and continue along a straight line in the region $|\mathbf{x}| > r_1$. If the trajectory meets this boundary from the inside of the circle tangentially the same inequality applies. We thus have shown

PROPOSITION If the trajectory of a sperm meets the boundary of the region $|\mathbf{x}| < r_1$ tangentially. It will enter this region or stay in this region if only if

$$a \leq r_1 v^2 \tag{9}$$

holds.

At the circle $|\mathbf{x}| = r_0$, the situation is somewhat different, since there is no attraction inside this circle, and the trajectory will not enter the circle tangentially, independent of the speed at the time tangency. This, however is an artifact of the model itself, and can be changed, by allowing the attractive force to continue inside this circle. This circle after all represents the egg and we are only interested in modeling the path of the sperm toward the egg.

The above proposition illustrates the nonlinear behavior of the trajectories well. Only a small change in the initial velocity will make the difference on whether the trajectory reaches the egg or not. In other words, at any fixed t_0, the location $\mathbf{X}(t_0)$ does not depend continuously on the initial values.

Before going on to numerical simulations, we will establish an interesting result about the kinetic energy of the sperm. For this we rewrite equation (2 in the following way:

$$\mathbf{v}' = \mathbf{a}(|\mathbf{x}|)\frac{\mathbf{x}}{|\mathbf{x}|} - b(|\mathbf{x}|)|\mathbf{v}|\mathbf{v}, \tag{10}$$

where $\mathbf{a}$ and $\mathbf{b}$ are piecewise constant functions, which only depend on $|\mathbf{x}|$. If we take the scalar product of (10) with $\mathbf{v}$ we obtain

$$\mathbf{v}\mathbf{v}' = -\mathbf{a}\frac{\mathbf{x}}{|\mathbf{x}|}\mathbf{v} - b|\mathbf{v}|^3. \tag{11}$$

Next, since $\mathbf{v} = \mathbf{x}'$ we can write

$$\frac{\mathbf{x}}{|\mathbf{x}|}\mathbf{v} = \frac{\mathbf{x}}{|\mathbf{x}|}\mathbf{x}' = \frac{1}{2}\frac{\frac{d|\mathbf{x}|^2}{dt}}{|\mathbf{x}|} = \frac{1}{|\mathbf{x}|}\frac{d|\mathbf{x}|}{dt}.$$

Using this we get

$$-\mathbf{a}\frac{\mathbf{x}}{|\mathbf{x}|}\mathbf{v} = -\frac{d}{dt}\mathbf{a}\ln|\mathbf{x}|,$$

at all points $\mathbf{x}(t)$ which lie not on one of the circles with radius r_0 and r_1. This is true since $\mathbf{a}$ is constant in every region, except on these circles. Thus our equation becomes

$$\frac{1}{2}\frac{d|\mathbf{v}|^2}{dt} = -\mathbf{a}\frac{1}{|\mathbf{x}|}\frac{d|\mathbf{x}|}{dt} - \mathbf{b}|\mathbf{v}|^3.$$

We can integrate this equation from 0 to t to get.

$$\frac{1}{2}|\mathbf{v}(t)|^2 - \frac{1}{2}|\mathbf{v}(0)|^2 = -\int_0^t \frac{d}{dt}\mathbf{a}\ln|\mathbf{x}|\,ds - \int_0^t \mathbf{b}|\mathbf{v}|^3\,ds.. \tag{12}$$

The left-hand-side of this equation is the difference of kinetic energy along the trajectory starting at $t = 0$. The last term on the right is the loss of energy due to friction, which is always negative. The first term on the right is the work done by the sperm. This term need some further investigation. Let us assume that $|\mathbf{x}(0)| > r_1$ and that $r_0 < |\mathbf{x}(t)| < r_1$, and that the trajectory entered the circle $|\mathbf{x}| < r_1$ at time t_0. Then

$$-\int_0^t \frac{d}{dt}\mathbf{a}\ln|\mathbf{x}|\,ds = -\int_{t_0}^t \frac{d}{dt}\mathbf{a}\ln|\mathbf{x}|\,ds = -a\ln\frac{|\mathbf{x}(t)|}{|\mathbf{x}(t_0)|} = -a\ln\frac{|\mathbf{x}(t)|}{r_1}.$$

If the sperm enters the egg, the propulsion term is turned off, and the energy balance (12) becomes

$$\frac{1}{2}|\mathbf{v}(t)|^2 - \frac{1}{2}|\mathbf{v}(0)|^2 = -a\ln\frac{r_0}{r_1} - \int_0^t \mathbf{b}|\mathbf{v}|^3\,ds. \tag{13}$$

If the sperm stays inside the egg, this is the final balance for the kinetic energy. However, our model leaves the possibility that the sperm leaves the egg again, and we will show below that this will actually happen. Then the new balance is

$$\frac{1}{2}|\mathbf{v}(t)|^2 - \frac{1}{2}|\mathbf{v}(0)|^2 = -a\ln\frac{r_0}{r_1} - a\ln\frac{|\mathbf{x}(t)|}{r_0} - \int_0^t \mathbf{b}|\mathbf{v}|^3\,ds.$$

This shows that the terms $\ln r_j$, $j = 0, 1$, cancel whenever the sperm exits the region $|\mathbf{x}| < r_j$, which it previously entered. As $t \to \infty$ the sperm cannot come to rest in the region $r_0 < |\mathbf{x}| < r_1$, since there is a constant central force there. Therefore, the sperm will be either eventually inside the egg or in the region $|\mathbf{x}| > r_1$. Hence, we have shown

PROPOSITION For $t\infty$ the kinetic energy $U(t) = \frac{1}{2}|\mathbf{v}(t)|^2$ satisfies

$$U(t) - U(0) = -a \ln \frac{r_0}{r_1} - \int_0^t b(2U(s))^{\frac{3}{2}}\, ds,$$

if the sperm is inside the egg at t, and

$$U(t) - U(0) = -\int_0^t b(2U(s))^{\frac{3}{2}}\, ds,$$

if the sperm is in the region $|\mathbf{x}| > r_1$.

Finally, we analyze the distance traveled by the sperm. Assume that after some finite time t_0 the sperm is in the region $|\mathbf{x}| < r_0$ or $|\mathbf{x}| > r_1$. Then after replacing $\frac{1}{2}|\mathbf{v}|^2$ by $U(t)$ equation (11) becomes

$$U'(t) = -cU^{\frac{3}{2}}(t),$$

we can integrate this equation to get

$$U^{\frac{1}{2}}(t) = \frac{2}{c(t - t_0) + U^{\frac{1}{2}}(t_0)}.$$

The right hand side is always positive, and therefore the kinetic energy will always be positive. Furthermore, since $U^{\frac{1}{2}} = \frac{|\mathbf{v}|}{\sqrt{2}} = \frac{v}{\sqrt{2}}$ we get the following equation for the speed $v(t)$.

$$v(t) = \frac{2\sqrt{2}}{c(t - t_0) + \frac{v(t_0)}{\sqrt{2}}}.$$

Clearly, the distance, traveled is

$$d(t) = \int_{t_0}^t \frac{2\sqrt{2}\, ds}{c(s - t_0) + \frac{v(t_0)}{\sqrt{2}}},$$

which is an integral that diverges as $t \to \infty$. Outside the annulus $r_0 < |\mathbf{x}|, r_1$, the trajectory will be straight lines. This implies that the trajectory will not stay inside the region $|\mathbf{x}| < r - 0$. It will leave the egg and possibly re-enter it again and so on. Hence we have

PROPOSITION The trajectory of the sperm does not stop in finite time. Furthermore the sperm will travel an infinite distance.

This last proposition clearly shows the limitations of our model. It is an inherent property of the Stokes drag term $-b|\mathbf{v}|\mathbf{v}$. Any particle moving with positive speed and subject to Stokes drag will not stop in either finite time or finite distance. In order to get a simple drag model which leads to stopping in finite time we need to look at an equation of the type

$$v' = -cv^{\alpha},$$

for some $0 < \alpha < 1$. A particle which is subject to a drag term of this type will stop in finite time and travel only a finite distance. If $\alpha = 1$, the particle will travel for an infinite time , but cover only a finite distance.

The model is only good for the first approach to the egg. We could of course fix this by, changing the original differential equation and have $\mathbf{v} = \mathbf{x}' = 0$ if $|\mathbf{x}| < r_0$. This model, however, would have less nice analytic properties.

Numerical Simulations

The initial value problem can be solved numerically. This can be done using one of the standard solvers which are available. The trajectories in the simulations below were obtained by using the MATLAB function "ode45", which is based on an explicit Runge-Kutta (4,5) method (see [6]. In all simulations we used $r_0 = 0.2$, $r_1 = 1$, $a = 1$, $b_1 = 0.5$ and $b_2 = 10000$.

For the ten trajectories in FIGURE 2 we used the following initial data:

$$\mathbf{v}_0 = (-3, 0); \qquad \mathbf{x}_0 = (1.5, x_2),$$

where x_2 ranged from 0.45 to 1.35 in increments of 0.1.

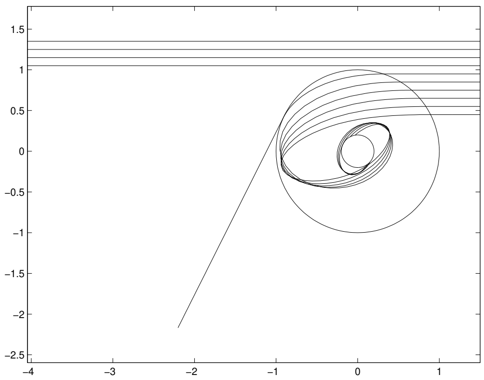

Figure 2: Ten trajectories with different initial data

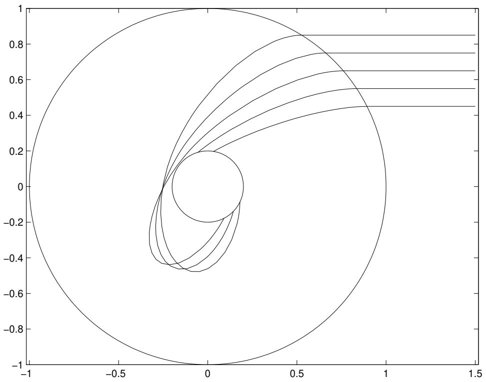

Figure 3: Five trajectories with different initial data

We see that there is a several different possibilities for the trajectories. If they reach the circle $|\mathbf{x}| = r_1$, they may exit this region again and never return, or they may eventually end up in the egg. For a second simulation we choose the initial data

$$\mathbf{v}_0 = (-0.8, 0); \qquad \mathbf{x}_0 = (1.5, x_2),$$

where x_2 ranged from 0.45 to 0.85 in increments of 0.1. The resulting five trajectories are shown in FIGURE 3.

Noise and stochastic differential equations

While the deterministic model shows the expected behavior of the trajectories of sperm, the actual observed path of sperm are somewhat more irregular. One possible reason for this is Brownian motion of the water molecules. The sperm exchanges momentum with water molecules around it. Due to the high polarity of water molecules and strong hydrogen bonds between them, liquid water does not consist of individual molecules, but of larger collections of many molecules which are bound together and move as single units. Therefore the momentum exchange with these units is not negligible (as it would be with single molecules). The motion of the sperm is now described by the two random variables $\mathbf{x}_t$ and $\mathbf{v}_t$, which satisfy the coupled system of integral equations.

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t \mathbf{v}_s\, ds + c_1 \int_0^t d\mathbf{M}_s \tag{14}$$

$$\mathbf{v}_t = \mathbf{V}_0 + \int_0^t \mathbf{f}(\mathbf{x}_s, \mathbf{v}_s)\, ds + c_2 \int_0^t d\mathbf{N}_s, \tag{15}$$

where, for each $t > 0$, $\mathbf{M}_t$ and $\mathbf{N_t}$ are random variables. The function $\mathbf{f}$ is the right-hand side of (2), i.e.

$$\begin{aligned} \mathbf{f}(\mathbf{x}_t, \mathbf{v}_t) &= a\frac{\mathbf{x}_t}{|\mathbf{x}_t|}\mathbf{\Theta}(r_1 - |\mathbf{x}_t|)\mathbf{\Theta}(|\mathbf{x}_t| - r_0) \\ &\qquad -(b_1 + b_2\mathbf{\Theta}(r_0 - |\mathbf{x}_t|))|\mathbf{v}_t|\mathbf{v}_t \end{aligned}$$

The random variables $\mathbf{M}_t$ and $\mathbf{N}_t$ have the property that if $t_1 \neq t_2$ then $\mathbf{M}_{t_1}$ and $\mathbf{M}_{t_2}$ are independent and $\mathbf{N}_{t_1}$ and $\mathbf{N}_{t_2}$ are independent. To continue, we need to model these random variables. The momentum exchange will only effect the momentum equation, it is therefore reasonable to assume that the random variable $\mathbf{M}_t = 0$, for all $t \geq 0$. After all, we do not assume that the sperm can change its position at random. For $\mathbf{N}_t$ it is reasonable to assume that is a normally distributed random variable with values in $\mathbb{R}^2$ and mean 0. It is known that the velocities in Brownian motion satisfy a normal distribution. Using this we can rewrite (14) and (15) in differential form:

$$d\mathbf{x}_t = \mathbf{v}_t\, dt \tag{16}$$

$$d\mathbf{v}_t = \mathbf{f}(\mathbf{x}_t, \mathbf{v}_t)\, dt + c_2\, d\mathbf{N}_t, \tag{17}$$

with given initial values $\mathbf{x}_0$ and $\mathbf{v}_0$. For the theory of the solution of such a stochastic differential equation we refer the reader to [5]. It suffices to to say here that $\mathbf{f}$ satisfies a Lipschitz condition in each of the three regions of Figure 1. Then by Theorem 5.5 of [5], the initial value problem has a unique solution in this region. However, we cannot give a condition similar to Proposition 1, on what happens if the trajectory enters a new region tangentially, due to the random variable $\mathbf{N}_t$.

For the remainder of this section we will concentrate on the numerical simulation of the stochastic differential equation above. high order numerical solvers do not work well for stochastic differential equations. The most common method is a variation of the traditional Euler method, which is known as the Euler-Mayurama method. In (17) the function $\mathbf{N}_t$ represents a two dimensional Brownian process. Since in such a process, the random variable $\mathbf{N}_{t+\Delta t}$ is independent from $\mathbf{N}_t$, this process can be initially determined for all $t \in [0, T]$. The Euler-Mayurama method now works as follows. First the function $\mathbf{N}_t$ is determined for all $t \in [0, T]$, to do this let Δt be a given discretization of the time interval $[0, T]$. Then at time $t_k = k\Delta t$, we have

$$\Delta \mathbf{N}_{t_k} = \sqrt{\Delta t}\mathbf{Y},$$

where $\mathbf{Y}$ is a normal random variable with mean 0 and standard deviation 1. Using this we get:

$$\mathbf{N}_{t_k} = \sum_{j=1}^{k} \Delta \mathbf{N}_{t_y}.$$

Having established $\mathbf{N}_t$, we solve the initial value system using the regular Euler method with a time discretization $t_j = jh$, where h is a multiple of Δt.

FIGURE 4 below shows ten trajectories computed by the Euler-Mayurama method, each with the initial conditions $\mathbf{x}_0 = (1.5, 0.5)$ and $\mathbf{v}_0 = (-0.8, 0)$. FIGURE 5 contains five trajectories with initial data $\mathbf{x}_0 = (1.5, 0.8)$ and $\mathbf{v}_0 = (-1, 0)$. In both simulations we used the same constants $r_0 = 0.2$, $r_1 = 1$, $a = 1$ and $b_1 = 0.5$.However, due do the relatively bad behavior of the Euler method (as compared to the 4th order Runge-Kutta method) we needed to change b_2 to a smaller value $b_2 = 100$. This only effects the behavior of the trajectories inside the egg.

We see that the individual trajectories may vary significantly, they all follow a general trend. The theory of stochastic differential equations predicts that the means of samples of n trajectories will be distributed normally about the trajectory $\mathbf{x}(t)$ of the corresponding deterministic equation. This is tested in a final numerical experiment. FIGURE 6 shows 50 trajectories for

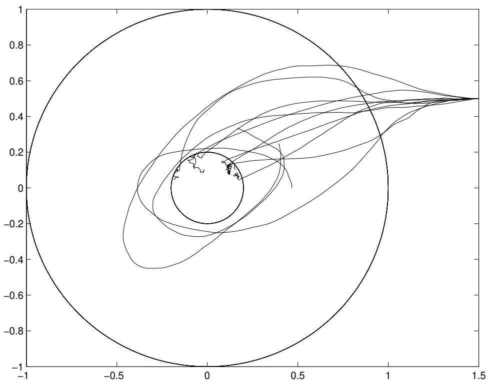

Figure 4: Ten trajectories for identical initial data

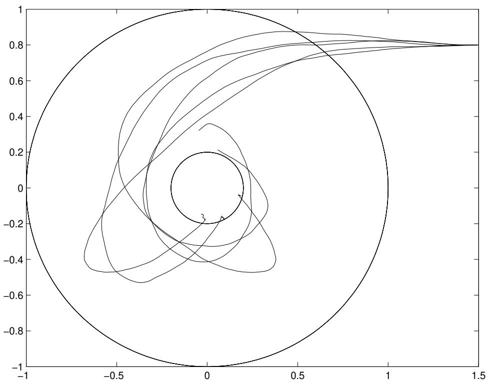

Figure 5: Five trajectories for identical initial data

the stochastic differential equation using the same initial data as in FIGURE 5.FIGURE 7 shows the mean of these 50 trajectories and the solution the deterministic equation. We see that there is relatively good agreement between the mean of the trajectories and the trajectory of the deterministic equation. Perfect agreement is of course not expected, specifically with a small sample of 50 trajectories. We also see that the deviation between the mean trajectory and the deterministic trajectory increases along the trajectories. This is to be expected, since we look at trajectories of the spatial variable $\mathbf{x}_t$, which is the integral of $\mathbf{v}_t$. Small variations in the velocity at time t_1 can have a large effect in the position at a later time.

Conclusion

The model developed is clearly accessible to undergraduate students with a good Calculus background and some backgroud in probability theory. Biology is full of processes which are approachable with a relatively modest background in mathematics. Such processes are ideal for undergraduate projects in mathematical modeling.

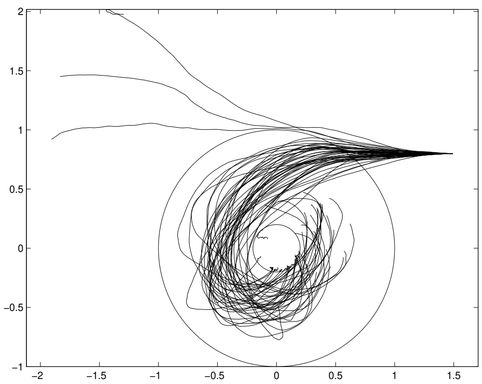

Figure 6: 50 trajectories for identical initial data.

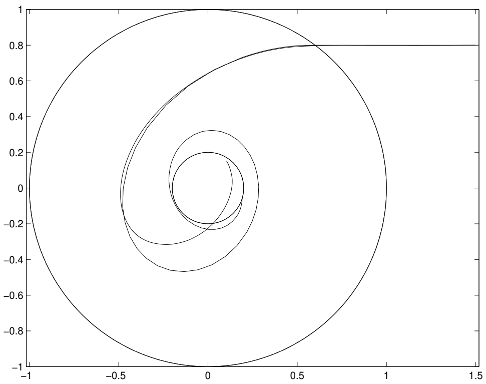

Figure 7: The mean of the 50 trajectories in Figure 6, and the deterministic trajectory for the same initial data

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