

Math 350 — Assignment 8, due 11/29

Turn in the answers to the following five problems

1. Let f be a continuous function on $[0, 1]$. Prove that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f\left(\frac{k}{n}\right) = \int_0^1 f(x) dx.$$

2. Prove that a monotonic function on $[a, b]$ is Riemann integrable.
3. Let f, g be Riemann integrable on $[a, b]$ and $\alpha, \beta \in \mathbb{R}$. Prove that $\alpha f + \beta g$ is Riemann integrable and that

$$\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx.$$

4. Prove that f is Riemann integrable on $[a, b]$ if and only if for every $c \in (a, b)$ f is Riemann integrable on $[a, c]$ and $[c, b]$ and that

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

5. Suppose that f and g are continuous on $[a, b]$ and that $g(x) \geq 0$ on $[a, b]$. Show that there exists a $c \in [a, b]$ such that

$$\int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx.$$

Do the following problems to entertain your brain and prepare for big events in December.

1. Suppose that f and g are continuous on $[a, b]$ and that $g(x) \geq 0$ on $[a, b]$ and let

$$M = \sup\{f(x) : x \in [a, b]\}, \quad m = \inf\{f(x) : x \in [a, b]\}.$$

Show that there exists a $c \in [a, b]$ such that

$$\int_a^b f(x)g(x) dx = m \int_a^c g(x) dx + M \int_c^b g(x) dx.$$

Hint: Think Mean Value Theorem, actually always think of that first.

2. Some important functions are defined as improper integrals. One them is the Gamma function:

$$\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$$

- (a) Show that this integral converges for all $x > 0$.
 - (b) Prove that $\Gamma(x + 1) = x\Gamma(x)$ for $x > 0$.
 - (c) Show that $\Gamma(1) = 1$.
 - (d) Show that $\Gamma(n) = (n - 1)!$ for all positive integers n .
3. Suppose that $f(x) \geq 0$ and continuous on $a, b]$, and that $\int_a^b f(x) dx = 0$. Then $f(x) = 0$.
4. Suppose that $f(x) \geq 0$ on $[a, b]$ and Riemann integrable on $[a, b]$. If $\int_a^b f(x) dx = 0$ then $f(x) = 0$ almost everywhere.