

## Math 350 — Assignment 7, Solutions

1. (a) Suppose that  $|f(x+h) - f(x)| < K|h|^\alpha$  for some  $K > 0$  and  $\alpha > 0$ . Show that  $f$  is continuous at  $x$ .

**Solution:** Let  $\epsilon > 0$  and  $\delta = \frac{\epsilon^{\frac{1}{\alpha}}}{K}$  then  $|f(y) - f(x)| < \epsilon$  for all  $y$  with  $|y - x| < \delta = \frac{\epsilon^{\frac{1}{\alpha}}}{K}$ .

- (b) Suppose that  $|f(x+h) - f(x)| < K|h|^\alpha$  for some  $K > 0$  and  $\alpha > 1$ . Show that  $f$  is differentiable at  $x$ .

**Solution:** Actually,  $f'(x) = 0$ . Dividing the above inequality by  $|h|$  we get

$$\left| \frac{f(x+h) - f(x)}{h} \right| \leq K|h|^{\alpha-1}.$$

Since  $\alpha > 1$  the expression on the right will converge to 0 as  $h \rightarrow 0$ . Hence  $|f'(x)| = 0$  and  $f'(x) = 0$  and exists.

- (c) Suppose that  $|f(x+h) - f(x)| < K|h|$  for some  $K > 0$ . Is  $f$  necessarily differentiable? *Hint:* Consider

$$f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

at  $x = 0$ .

**Solution:** The function given in the hint satisfies the inequality, but

$$\lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \sin \frac{1}{h}$$

doesn't exist. So this function is not differentiable at 0.

2. Suppose  $f$  is differentiable on  $(a, b)$  with  $|f'(x)| \leq M$  for some  $M$  and all  $x \in (a, b)$ . Show that  $f$  is uniformly continuous on  $(a, b)$ .

**Solution:** Let  $x, y \in [a, b]$  with  $y > x$ . Then there exists a  $c \in (x, y)$  such that

$$|f(y) - f(x)| = |f'(c)|(y - x) \leq M|y - x|.$$

Therefore  $|f(x) - f(y)| < M|x - y|$  for all  $x, y \in [a, b]$  For  $\epsilon > 0$  one can now uniformly choose  $\delta = \frac{\epsilon}{M}$  to obtain uniform continuity.

3. Suppose that  $f$  is continuous on  $a, b$  and differentiable on  $(a, b) \setminus \{c\}$ , for some  $c \in (a, b)$ . Show that if there is a  $\delta > 0$  such that  $f'(x) > 0$  for  $x \in (c - \delta, c)$  and  $f'(x) < 0$  for  $x \in (c, c + \delta)$ , then  $f$  has a relative maximum at  $c$ .

**Solution:** Let  $x \in (c - \delta, c)$ .  $f$  satisfies the mean value theorem on  $[x, c]$  Therefore there exists a  $\xi \in (x, c)$  such that

$$f(c) - f(x) = f'(\xi)(c - x) > 0,$$

and thus  $f(c) > f(x)$  for all  $x \in (c - \delta, c)$ . Analogously one shows that  $f(c) > f(x)$  for all  $x \in (c, c + \delta)$ . Thus  $f$  has a local maximum at  $c$ .

4. Use Taylor's Theorem to find an approximation of  $\sqrt{8}$  which is accurate to 0.0001.

**Solution:**

$$\sqrt{8} = \sqrt{9 - 1} = \sqrt{9} + \frac{1}{2\sqrt{9}}(-1) - \frac{1}{4\sqrt{9}^3}(-1)^2 + \frac{3}{489^5}(-1)^3 - \frac{15}{384\sqrt{9}^7}(-1)^4 + \frac{105}{3840\sqrt{9}^9}(-1)^5.$$

for some  $c \in (8, 9)$ . Observe that  $\sqrt{c} > 2$  which implies that

$$\frac{105}{3840\sqrt{c}^9} < \frac{105}{3840 \cdot 512} < \frac{100}{1500000} = \frac{1}{15000} < 0.0001.$$

Thus

$$\sqrt{8} \approx 3 - \frac{1}{6} - \frac{1}{108} - \frac{1}{3888} - \frac{5}{128 \cdot 3^9}$$

5. Let  $f(x) = \frac{ax+b}{cx+d}$ .

- (a) Show that  $f$  is one-to-one if and only if  $ad \neq bc$ .

**Solution:**  $f$  is one-to-one if and only if  $f'(x) \neq 0$ . But  $f'(x) = \frac{a(cx+d) - (ax+b)c}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2} \neq 0$  if and only if  $ad - bc \neq 0$ .

- (b) Find  $f^{-1}$

**Solution:** Simple algebra gives:  $f^{-1}(x) = \frac{dx-b}{a-cx}$

- (c) For which values of  $a, b, c, d$  is  $f = f^{-1}$ .

**Solution:** Direct computation gives:  $a = -d$ , for any values of  $c, b$  or  $a = d$ , and  $c = b = 0$ .

Due the following problems for your own entertainment.

1. Let  $f$  and  $g$  be differentiable on  $(a, b)$  with  $f'(x) = g'(x)$ . Prove that  $f$  and  $g$  differ by a constant on  $(a, b)$ .

2. Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

You may assume that  $\lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0$  for all  $n \in \mathbb{N}$ . Prove that  $f$  is infinitely often differentiable. And that  $f^{(n)}(0) = 0$  for all  $n$ .

3. For the function in the previous problem write a Taylor expansion of order  $n$  around the origin. What is weird about this function?

4. Prove that  $\lim_{x \rightarrow 0} \frac{e^{-\frac{1}{x^2}}}{x^n} = 0$  for all  $n \in \mathbb{N}$ .

5. Let  $f$  be differentiable on  $(a, b)$  with a strictly monotonic derivative. Prove that  $f'(x)$  is continuous.