Math 350, Assignment 6, Solutions

Turn in the following five problems.

1. Show that the sum and product of two uniformly contiuous functions on $[a, b]$ are uniformly continuous.

   Solution: Denote the functions by f and g For the sum, let $\epsilon > 0$ then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that.

$$|f(x) - f(y)| < \frac{\epsilon}{2}, \qquad \forall x, y \in [a, b], \quad |x - y| < \delta_1,$$

and

$$|g(x) - g(y)| < \frac{\epsilon}{2}, \qquad \forall x, y \in [a, b], \quad |x - y| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ then

$$|(f(x) + g(x)) - (f(y) + g(y))| < \epsilon$$

for all $x, y \in [a, b]$ with $|x - y| < \delta$. For the product observe that both f and g are bounded on $[a, b]$. Then there exists M such that $|g(x)| < M$ aand $|f(x)| < M$ for all $x \in [a, b]$. Let $\epsilon > 0$ then there exist $\delta_1 > 0$ and $\delta_2 > 0$ such that.

$$|f(x) - f(y)| < \frac{\epsilon}{2M}, \qquad \forall x, y \in [a, b], \quad |x - y| < \delta_1,$$

and

$$|g(x) - g(y)| < \frac{\epsilon}{2M}, \qquad \forall x, y \in [a, b], \quad |x - y| < \delta_2.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ then

$$|(f(x)g(x)) - (f(y)g(y))| \leq |g(x)||f(x) - f(y)| + |f(y)||g(x) - g(y)| \leq M|f(x) - f(y)| + M|g(x) - g($$

for all $x, y \in [a, b]$ with $|x - y| < \delta$.

2. Let f be uniformly continuous on $[a, b]$. Show that $f([a, b])$ is either a closed and bounded interval or a single point.

   Solution: Since $[a, b]$ is compact $f([a, b])$ is closed and bounded. Suppose it is not a single point and there exist $y_1, y_2 \in f([a, b])$ with $y_1 \neq y_2$. WLOG $y_1 < y_2$. Let $x_1, x_2 \in [a, b]$ be such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Let y any number between y_1 and y_2. By the intermediate value theorem, there exists an x between x_1 and x_2 such that $y = f(x)$. Therefore, $y \in f([a, b])$ and this set is an interval. (Recall the definition of interval).

3. Let f be a uniformly continuous function on (a, b) and $\{x_n\}$ be a Cauchy sequence in (a, b). Show that $\{f(x_n)\}$ is a Cauchy sequence.

   Solution: f is uniformly continuous. For every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in (a, b)$ such that $|x - y| < \delta$. Let x_n be a Cauchy sequence and $\epsilon > 0$ the same as above. Then for the $\delta > 0$ there exists a N such that $|x_n - x_m| < \delta$ for all $n, m > N$. Combining these two statements we get:

   For every $\epsilon > 0$ there exists N such that

   $$|f(x_n) - f(x_m)|, \epsilon$$

   for all $n, m > N$.

4. Find a function f that is continuous but not uniformly continuous on $(0, 1)$ and a Cauchy sequence $\{x_n\}$ in this interval such that $\{f(x_n)\}$ is not a Cauchy sequence.

   Solution: Consider the function $f(x) = \frac{1}{x}$ on $(0, 1)$ and the sequence $x_n = \frac{1}{n}$. This sequence converges and it is therefore a Cauchy sequence. But $f(x_n) = n$ divergeges aan can therefore not be a Cauchy sequence.

5. A function f is **convex** on $[a, b]$ if for $x_1, x_2 \in [a, b]$ and for any $\lambda \in [0, 1]$ we have

   $$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2).$$

   Show that a convex function is continuous.

   Solution: Let $x \leq x' < y'$ then we may write

   $$x' = \lambda x + (1 - \lambda)y'$$

   with

   $$\lambda = \frac{x' - y'}{x - y'}.$$

The convexity of the function gives

$$f(x') \le \lambda(f(x) - f(y')) + f(y') = \frac{x' - y'}{x - y'}(f(x) - f(y')) + f(y')$$

Simple algebra and the fact that $x' < y'$ gives

$$\frac{f(x') - f(y')}{x' - y'} \ge \frac{f(x) - f(y')}{x - y'}. \tag{1}$$

Next let $x < y \le y'$. Again we can write y=$\lambda y' + (1-\lambda)x with \lambda = \frac{y'-x}{y-x} The same algebra as above gives \frac{f(\ldots}{}$
$\frac{f(y') - f(x)}{y' - x}$.(2)Combining these two equations yield:

$$\frac{f(x) - f(y)}{x - y} \le \frac{f(x') - f(y')}{x' - y'}. \tag{3}$$

. Let $x_0, x_1 \in (a, b)$ be fixed with $x_0 < x_1$. Let $x, y \ge x_0$. Applying equation (3) with $a = x$, $x_0 = b$, $x = x$, and $y = y'$ yields

$$\frac{f(x_0) - f(a)}{x_0 - a} \le \frac{f(x) - f(y)}{x - y}, \tag{4}$$

for all $x, y \ge x_0$. Next we apply equation (3) with $x = x$, $y = y$, $b = x'$, and $x_1 = y'$ to get:

$$\frac{f(x) - f(y)}{x - y} \le \frac{f(b) - f(x_1)}{b - x_1} \tag{5}$$

For fixed x_0, x_1 let

$$M(x_0, x_1) = \max\left\{ \left| \frac{f(x_0) - f(a)}{x_0 - a} \right|, \left| \frac{f(b) - f(x_1)}{b - x_1} \right| \right\}.$$

With this we have

$$\left| \frac{f(x) - f(y)}{x - y} \right| \le M(x_0, x_1)$$

for all $x, y \in [x_0, x_1]$. It follows that

$$|f(x) - f(y)| < M(x_0, x_1)|x - y|$$

for all $x, y \in x_0, x_1$. A previou homework assignment shows that f is uniformly continuous on $[x_0, x_1]$. Since f is continuous on $[x_0, x_1]$ for all $a < x_0 < x_1 < b$, f is continuous on $\bigcup_{b > x_0 > a} \bigcup_{x_0 < x_1 < b} [x_0, x_1] = (a, b)$.

For your own enlightenment do the following, but don't turn them in.

1. Show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

2. Let f be a monotone function on $[a, b]$. Show that f is one-to-one.

3. Give an example of a one-to-one function on $[-1, 1]$ thatt is not monotonic.

4. Suppose that f is continuous on (a, b) and that $\lim_{x\to a} f(x) = A$ and $\lim_{x\to b} f(x) = B$. Define
$$F(x) = \begin{cases} f(x), & x \in (a, b) \\ A, & x = a \\ B, & x = b \end{cases}$$
Show that F is **uniformly continuous** on $[a, b]$.

5. Give an example of a convex function.