Math 350, Assignment 4, Solutions

Turn in the following five problems.

1. Give an example of an infinite number of closed sets whose union is not closed.

   Solution: For $n \geq 2$ let $F_n = [\frac{1}{n}, 1 - \frac{1}{n}]$. Then each of the f_n is closed but

   $$\bigcup_{n=2}^{\infty} F_n = (0, 1),$$

   which is open.

2. (a) Let A be a bounded set of real numbers. Show that $\sup A$ and $\inf A$ are points of $\overline{A}$.

   Solution: Let $\alpha = \sup A$ and $\alpha \notin A$. Then for every $\epsilon > 0$ there is an $x \in A$ such that $\alpha - \epsilon < x < \alpha$, i.e. $x \in (\alpha - \epsilon, \alpha + \epsilon)$ and $x \neq \alpha$. Thus α is a limit point, and $\alpha \in \overline{A}$. If $\alpha \in A$ then $\alpha \in \overline{A}$. The analogous argument works for the infimum.

   (b) Is $\inf A$ necessarily a limit point of A.

   Solution: No. For example let $A = \{1\}$, which has no limit points, but it has a least upper bound.

3. Show that $A \subset B$ implies $\overline{A} \subset \overline{B}$

   Solution: It suffices to show that a limit point of A is also a limit point of B. To do this let a be a limit point of A and $\delta > 0$ be arbitrary. Then there is an $x \in (a - \delta, a + \delta) \cap A$ with $x \neq a$. But then $x \in (a - \delta, a + \delta) \cap B$ and $x \neq a$, and thus a is a limit point of B. Thiholds for all limit points of A, and therefore $\overline{A} \subset \overline{B}$.

4. (a) Show that the interior of *int* A is an open set.

   Solution: Let x be in the interior of A. Then there is a $\delta > 0$ such that $(x - \delta, x + \delta) \subset A$. Now let $y \in (x - \delta, x + \delta)$, since this set is open there is a $\delta' > 0$ such that $(y - \delta', y + \delta') \subset (x - \delta, x + \delta) \subset A$, and so y is in the interior of A. It follows that $(x - \delta, x + \delta)$ lies entirely in the interior of A, thus the interior is open.

(b) Show that if B is open and $B \subset A$, the B lies in the interior of A, and the interior is the largest open set contained in A.

Solution: Let $x \in B$ since B is open there is a $\delta > 0$ such that $(x - \delta, x + \delta) \subset B \subset A$, and there fore x is in the interior of A.

(c) Show that $int(A) \cap int(B) = int(A \cap B)$.

Solution: Let $x \in int(A) \cap int(B)$ then there are $\delta_1, \delta_2 > 0$ such that $(x - \delta_1, x + \delta_1) \subset A$ and $(x - \delta_2, x + \delta_2) \subset B$. Let $\delta = \min |[\delta_1, \delta_2\}$, then $(x - \delta, x + \delta) \subset A$ and $(x - \delta, x + \delta) \subset B$, and thus $(x - \delta, x + \delta) \subset A \cap B$ and $x \in int(A \cap B)$. Conversely, if $x \in int(A \cap B)$ there exists a $\delta.0$ such that $(x - \delta, x + \delta) \subset A \cap B$. Therefore, $(x - \delta, x + \delta) \subset A$ and $(x - \delta, x + \delta) \subset B$, and $x \in int(A) \cap int(B)$.

(d) Give an example where $int(A \cup B) \neq int(A) \cup int(B)$.

Solution: Let $A = [-1, 0]$ and $B = [0, 1]$. Then $int(A \cup B) = (-1, 1) \neq (-1, 0) \cup (0, 1) = int(A) \cup int(B)$.

5. This exercise shows that if $\{A_\alpha\}_{\alpha \in \mathcal{A}}$is a collection of open sets, thenthere is a countable subcollection $\{A_n\}$ of $\{A_\alpha\}$ such that

$$\bigcup_{\alpha \in \mathcal{A}} A_\alpha = \bigcup_{j=1}^{\infty} A_j$$

Solution: Let $B = \bigcup_{\alpha \in \mathcal{A}} A_\alpha$ and $x \in B$. Let A_x be a set in the collection such that $x \in A$. Then there exists a $\delta > 0$ such that $(x - \delta, x + \delta) \subset A_x$. There exists a $p \in \mathbb{Q}$ such that $x - \delta < p < x$ and a $q \in \mathbb{Q}$ such that $x < q < x + \delta$. Therefore,

$$I_x = (p, q) \subset (x - \delta, x + \delta) \subset A_x.$$

To continue we prove that there only countably many different I_x. To to this observe that we have a one-to-one map from the set of all I_x to $\mathbb{Q} \times \mathbb{Q}$by $I_x \mapsto (p, q)$. Now observe that

$$\mathbb{Q} \times \mathbb{Q} = \bigcup_{p \in \mathbb{Q}} \{(p, q) : q \in \mathbb{Q}\},$$

and that for a fixed p the set $\{(p, q) : q \in \mathbb{Q}\}$ is countable since the map $(p, q) \mapsto q$ is a one-to-one and onto map from this set to $\mathbb{Q}$. This makes$\mathbb{Q} \times \mathbb{Q}$

the countable union of countable sets, and therefore countable. Hence, there are only countably many different I_x, label them I_n for $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ let A_n be the set in our original cover such that $I_n \subset A_n$. This yields a countable collection $\{A_n\}_{n \in \mathbb{N}}$. Let $y \in B$then there is an $n \in \mathbb{N}$ such that $y \in I_n \subset A_n$, and hence $y \in \bigcup_{n=1}^{\infty} A_n$. Thus $B \subset \bigcup_{n=1}^{\infty} A_n$, and since the converse clearly holds we have

$$\bigcup_{\alpha \in \mathcal{A}} A_\alpha = \bigcup_{j=1}^{\infty} A_j.$$