

Math 350, Assignment 5, due Thursday Oct 18

Turn in the following five problems.

1. Let f and g be functions such that $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = L$. Suppose that h is a function for which $f(x) \leq h(x) \leq g(x)$ if $x \in (a - \delta, a + \delta) \setminus \{a\}$ for some $\delta > 0$. Prove that $\lim_{x \rightarrow a} h(x) = L$.

Solution: Observe that for $x \in (a - \delta, a + \delta)$ we have

$$\begin{aligned} h(x) - L &\leq g(x) - L \leq |g(x) - L| \\ L - h(x) &\leq L - f(x) \leq |f(x) - L| \end{aligned}$$

Combining these inequalities we get

$$|h(x) - L| \leq \max\{|f(x) - L|, |g(x) - L|\}.$$

Now let $\epsilon > 0$, then there exists a $\delta_1 > 0$ such that $|f(x) - L| < \epsilon$ for all $x \in (a - \delta_1, a + \delta_1) \setminus \{a\}$ and a $\delta_2 > 0$ such that $|g(x) - L| < \epsilon$ for all $x \in (a - \delta_2, a + \delta_2) \setminus \{a\}$. Therefore,

$$\max\{|f(x) - L|, |g(x) - L|\} < \epsilon$$

for all $x \in (a - \delta', a + \delta') \setminus \{a\}$, where $\delta' = \min\{\delta, \delta_1, \delta_2\}$. This immediately implies the result.

2. Use the definition of limit to show that $\lim_{x \rightarrow 1} \frac{1}{1+x} = \frac{1}{2}$.

Solution: For $\frac{1}{2} \leq x \leq \frac{3}{2}$ we have $\frac{3}{2} \leq x + 1$. Let $\epsilon > 0$ and choose $\delta = \min\{\frac{1}{2}, \frac{3}{4}\epsilon\}$. Then

$$\left| \frac{1}{x+1} - \frac{1}{2} \right| = \frac{2|x-1|}{x+1} \leq \frac{4}{3}|x-1| < \frac{3}{4}\delta \leq \epsilon,$$

for all $x \in (\frac{1}{2} - \delta, \frac{1}{2} + \delta)$

3. Prove that $f(x) = |x|$ is continuous on its domain. Use the $\epsilon - \delta$ definition of continuity.

Solution: Let $\epsilon > 0$ and choose $\delta = \epsilon$. We have

$$||x| - |x_0|| < |x - x_0| < \delta = \epsilon,$$

for all $x \in (x_0 - \delta, x_0 + \delta)$.

4. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function, and C be a closed subset of \mathbb{R} . Prove that $f^{-1}(C)$ is closed.

Solution: Let C be closed then C^C is open. Since f is continuous $f^{-1}(C^C)$ is open. Claim: $f^{-1}(C^C) = (f^{-1}(C))^C$. To prove this claim let $x \in f^{-1}(C^C)$, then $f(x) \in C^C$, and $f(x) \notin C$. Therefore, $x \notin f^{-1}(C)$, and $x \in (f^{-1}(C))^C$. Conversely, if $x \in (f^{-1}(C))^C$, then $x \notin f^{-1}(C)$, and $f(x) \notin C$. Therefore, $x \in f^{-1}(C^C)$. This proves the claim. Now since $f^{-1}(C^C)$ is open, $(f^{-1}(C))^C$ is open and $f^{-1}(C)$ is closed.

5. Prove that if f is continuous then so is $|f|$, but the converse is not true.

Solution: In a previous problem we showed that $g(x) = |x|$ is a continuous function. This makes the function $|f(x)| = g \circ f(x)$ the composition of two continuous functions, which is continuous by a theorem covered in class. For the converse let

$$f(x) = \begin{cases} -1, & x \leq 0 \\ 1, & x > 0 \end{cases}$$

Then f is clearly not continuous, but $|f(x)| = 1$ is continuous.

For your own practice do the following, but don't turn them in.

1. Let $f : [0, 1] \rightarrow [0, 1]$ be a continuous function. Prove that there exists a $\xi \in [0, 1]$ such that $f(\xi) = \xi$. *Hint: Think about the intermediate value theorem*

Solution: Define $F(x) = f(x) - x$ If $f(0) = 0$ or $f(1) = 1$ the desired ξ is either 0 or 1. If not $F(0) = f(0) > 0$ and $F(1) = f(1) - 1 < 0$. F is continuous and the intermediate value theorem implies that there is a $\xi \in (0, 1)$ such that $F(\xi) = 0$. But then $f(\xi) = \xi$.

2. Let

$$f(x) = \begin{cases} x^2, & x \leq 1 \\ x, & x > 1 \end{cases}$$

Prove that f is continuous.

Solution: Let $\epsilon > 0$. Choose $\delta = \epsilon/2$. Then for $x \leq 1$ we have:

$$|f(x) - f(1)| = |x - 1||x + 1| \leq 2|x - 1| < 2\delta = \epsilon,$$

for all $x \in (1 - \delta, 1)$. Furthermore we have:

$$|f(x) - f(1)| = |x - 1| < \delta = \epsilon/2 < \epsilon,$$

for all $x \in (1, 1 + \delta)$. Combining these two statements gives the continuity of f at 1. At other points f is also continuous.

3. Show that

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

is a continuous function.

Solution: f is continuous if $x \neq 0$. Next observe that $|\sin x| \leq |x|$, and therefore

$$\frac{\sin x}{x} \leq 1$$

near $x = 0$. Similarly, $|\tan x| \geq |x|$, and so

$$\left| \frac{\sin x}{x} \right| \geq |\cos x|$$

near $x = 0$. Since $\lim_{x \rightarrow 0} \cos x = \lim_{x \rightarrow 0} 1 = 1$, the result follows by a theorem in the book.

4. Let f be a function with the property, that there exists an $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|$$

for all x and y in the domain of f . Such a function is called Lipschitz continuous. Prove that a Lipschitz continuous function is continuous.

Solution: Let $\epsilon > 0$ and $\delta = \epsilon/L$. Then

$$|f(x) - f(y)| \leq L|x - y| < L\delta = \epsilon,$$

for all $x, y \in \mathcal{D}(f)$ with $|x - y| < \delta$. Hence, f is *uniformly* continuous.

5. Give an example of a function on $[0, 1]$ that does not assume its supremum.

Solution: Let

$$f(x) = \begin{cases} x, & x \in [0, 1) \\ 0, & x = 1 \end{cases}$$

Then $\sup f = 1$, but $f(x) < 1$ for all $x \in [0, 1]$.