

## 8 LOW-DISTORTION EMBEDDINGS OF FINITE METRIC SPACES

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### INTRODUCTION

An  $n$ -point metric space  $(X, D)$  can be represented by an  $n \times n$  table specifying the distances. Such tables arise in many diverse areas. For example, consider the following scenario in microbiology:  $X$  is a collection of bacterial strains, and for every two strains, one is given their *dissimilarity* (computed, say, by comparing their DNA). It is difficult to see any structure in a large table of numbers, and so we would like to represent a given metric space in a more comprehensible way.

For example, it would be very nice if we could assign to each  $x \in X$  a point  $f(x)$  in the plane in such a way that  $D(x, y)$  equals the Euclidean distance of  $f(x)$  and  $f(y)$ . Such a representation would allow us to see the structure of the metric space: tight clusters, isolated points, and so on. Another advantage would be that the metric would now be represented by only  $2n$  real numbers, the coordinates of the  $n$  points in the plane, instead of  $\binom{n}{2}$  numbers as before. Moreover, many quantities concerning a point set in the plane can be computed by efficient geometric algorithms, which are not available for an arbitrary metric space.

This sounds too good to be generally true: indeed, there are even finite metric spaces that cannot be exactly represented either in the plane or in *any* Euclidean space; for instance, the four vertices of the graph  $K_{1,3}$  (a star with 3 leaves) with the shortest-path metric (see Figure 8.0.1). However, it *is* possible to embed the latter metric in a Euclidean space, if we allow the distances to be distorted somewhat. For example, if we place the center of the star at the origin in  $\mathbb{R}^3$  and the leaves at  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ , then all distances are preserved *approximately*, up to a factor of  $\sqrt{2}$  (Figure 8.0.1b).

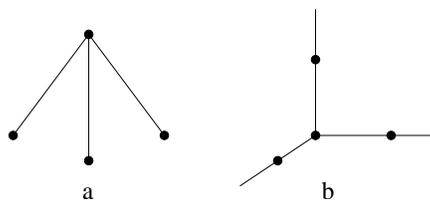


FIGURE 8.0.1  
A nonembeddable metric space.

Approximate embeddings have proven extremely helpful for approximate solutions of problems dealing with distances. For many important algorithmic problems, they yield the only known good approximation algorithms.

The normed spaces usually considered for embeddings of finite metrics are the spaces  $\ell_p^d$ ,  $1 \leq p \leq \infty$ , and the cases  $p = 1, 2, \infty$  play the most prominent roles.

**GLOSSARY**

**Metric space:** A pair  $(X, D)$ , where  $X$  is a set of *points* and  $D: X \times X \rightarrow [0, \infty)$  is a *distance function* satisfying the following conditions for all  $x, y, z \in X$ :

- (i)  $D(x, y) = 0$  if and only if  $x = y$ ,
- (ii)  $D(x, y) = D(y, x)$  (symmetry), and
- (iii)  $D(x, y) + D(y, z) \geq D(x, z)$  (triangle inequality).

**Separable metric space:** A metric space  $(X, D)$  containing a countable dense set; that is, a countable set  $Y$  such that for every  $x \in X$  and every  $\varepsilon > 0$  there exists  $y \in Y$  with  $D(x, y) < \varepsilon$ .

**Pseudometric:** Like metric except that (i) is not required.

**Isometry:** A mapping  $f: X \rightarrow X'$ , where  $(X, D)$  and  $(X', D')$  are metric spaces, with  $D'(f(x), f(y)) = D(x, y)$  for all  $x, y$ .

**(Real) normed space:** A real vector space  $Z$  with a mapping  $\|\cdot\|_Z: Z \rightarrow [0, \infty]$ , the *norm*, satisfying  $\|x\|_Z = 0$  iff  $x = 0$ ,  $\|\alpha x\|_Z = |\alpha| \cdot \|x\|_Z$  ( $\alpha \in \mathbb{R}$ ), and  $\|x + y\|_Z \leq \|x\|_Z + \|y\|_Z$ . The metric on  $Z$  is given by  $(x, y) \mapsto \|x - y\|_Z$ .

$\ell_p^d$ : The space  $\mathbb{R}^d$  with the  $\ell_p$ -norm  $\|x\|_p = (\sum_{i=1}^d |x_i|^p)^{1/p}$ ,  $1 \leq p \leq \infty$  (where  $\|x\|_\infty = \max_i |x_i|$ ).

**Finite  $\ell_p$  metric:** A finite metric space isometric to a subspace of  $\ell_p^d$  for some  $d$ .

$\ell_p$ : For a sequence  $(x_1, x_2, \dots)$  of real numbers we set  $\|x\|_p = (\sum_{i=1}^\infty |x_i|^p)^{1/p}$ . Then  $\ell_p$  is the space consisting of all  $x$  with  $\|x\|_p < \infty$ , equipped with the norm  $\|\cdot\|_p$ . It contains every finite  $\ell_p$  metric as a (metric) subspace.

**Distortion:** A mapping  $f: X \rightarrow X'$ , where  $(X, D)$  and  $(X', D')$  are metric spaces, is said to have distortion at most  $c$ , or to be a *c-embedding*, where  $c \geq 1$ , if there is an  $r \in (0, \infty)$  such that for all  $x, y \in X$ ,

$$r \cdot D(x, y) \leq D'(f(x), f(y)) \leq cr \cdot D(x, y).$$

If  $X'$  is a normed space, we usually require  $r = \frac{1}{c}$  or  $r = 1$ .

**Order of congruence:** A metric space  $(X, D)$  has order of congruence at most  $m$  if every finite metric space that is not isometrically embeddable in  $(X, D)$  has a subspace with at most  $m$  points that is not embeddable in  $(X, D)$ .

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## 8.1 THE SPACES $\ell_p$

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### 8.1.1 THE EUCLIDEAN SPACES $\ell_2^d$

Among normed spaces, the Euclidean spaces are the most familiar, the most symmetric, the simplest in many respects, and the most restricted. Every finite  $\ell_2$  metric embeds isometrically in  $\ell_p$  for all  $p$ . More generally, we have the following Ramsey-type result on the “universality” of  $\ell_2$ ; see, e.g., [MS86]:

**THEOREM 8.1.1** Dvoretzky's theorem (a finite quantitative version)

For every  $d$  and every  $\varepsilon > 0$  there exists  $n = n(d, \varepsilon) \leq 2^{O(d/\varepsilon^2)}$  such that  $\ell_2^d$  can be  $(1+\varepsilon)$ -embedded in every  $n$ -dimensional normed space.

Isometric embeddability in  $\ell_2$  has been well understood since the classical works of Menger, von Neumann, Schoenberg, and others (see, e.g., [Sch38]). Here is a brief summary:

**THEOREM 8.1.2**

- (i) (*Compactness*) A separable metric space  $(X, D)$  is isometrically embeddable in  $\ell_2$  iff each finite subspace is so embeddable.
- (ii) (*Order of congruence*) A finite (or separable) metric space embeds isometrically in  $\ell_2^d$  iff every subspace of at most  $d+3$  points so embeds.
- (iii) For a finite  $X = \{x_0, x_1, \dots, x_n\}$ ,  $(X, D)$  embeds in  $\ell_2$  iff the  $n \times n$  matrix  $(D(x_0, x_i)^2 + D(x_0, x_j)^2 - D(x_i, x_j)^2)_{i,j=1}^n$  is positive semidefinite; moreover, its rank is the smallest dimension for such an embedding.
- (iv) (*Schoenberg's criterion*) A separable  $(X, D)$  isometrically embeds in  $\ell_2$  iff the matrix  $(e^{-\lambda D(x_i, x_j)^2})_{i,j=1}^n$  is positive semidefinite for all  $n \geq 1$ , for any points  $x_1, x_2, \dots, x_n \in X$ , and for any  $\lambda > 0$ . (This is expressed by saying that the functions  $x \mapsto e^{-\lambda x^2}$ , for all  $\lambda > 0$ , are positive definite on  $\ell_2$ .)

Using similar ideas, the problem of finding the smallest  $c$  such that a given finite  $(X, D)$  can be  $c$ -embedded in  $\ell_2$  can be formulated as a semidefinite programming problem and thus solved in polynomial time [LLR95]. For embedding into  $\ell_2^d$  no similar result is possible. It has been shown that for any constant  $d \geq 1$ , approximating even to within a polynomial factor the minimum distortion embedding into  $\ell_2^d$  is NP-hard [MS10]

**8.1.2 THE SPACES  $\ell_1^d$** **GLOSSARY**

**Cut metric:** A pseudometric  $D$  on a set  $X$  such that, for some partition  $X = A \dot{\cup} B$ , we have  $D(x, y) = 0$  if both  $x, y \in A$  or both  $x, y \in B$ , and  $D(x, y) = 1$  otherwise.

**Hypermetric inequality:** A metric space  $(X, D)$  satisfies the  $(2k+1)$ -point hypermetric inequality (also called the  $(2k+1)$ -gonal inequality) if for every multiset  $A$  of  $k$  points and every multiset  $B$  of  $k+1$  points in  $X$ ,  $\sum_{a, a' \in A} D(a, a') + \sum_{b, b' \in B} D(b, b') \leq \sum_{a \in A, b \in B} D(a, b)$ . (We get the triangle inequality for  $k=1$ .)

**Hypermetric space:** A space that satisfies the hypermetric inequality for all  $k$ .

**Cocktail-party graph:** The complement of a perfect matching in a complete graph  $K_{2m}$ ; also called a **hyperoctahedron graph**.

**Half-cube graph:** The vertex set consists of all vectors in  $\{0, 1\}^n$  with an even number of 0's, and edges connect vectors with Hamming distance 2.

**Cartesian product of graphs  $G$  and  $H$ :** The vertex set is  $V(G) \times V(H)$ , and the edge set is  $\{(u, v), (u, v')\} \mid u \in V(G), \{v, v'\} \in E(H\} \cup \{(u, v), (u', v)\} \mid \{u, u'\} \in E(G), v \in V(H)\}$ . The cubes are Cartesian powers of  $K_2$ .

**Girth of a graph:** The length of the shortest cycle.

The  $\ell_1$  spaces are important for many reasons, but considerably more complicated than Euclidean spaces; a general reference here is [DL97]. Many important and challenging open problems are related to embeddings in  $\ell_1$  or in  $\ell_1^d$ .

Unlike the situation in  $\ell_2^n$ , not every  $n$ -point  $\ell_1$ -metric lives in  $\ell_1^n$ ; dimension of order  $\Theta(n^2)$  is sometimes necessary and always sufficient to embed  $n$ -point  $\ell_1$ -metrics isometrically (similarly for the other  $\ell_p$ -metrics with  $p \neq 2$ ).

The  $\ell_1$  metrics on an  $n$ -point set  $X$  are precisely the elements of the *cut cone*; that is, linear combinations with nonnegative coefficients of cut metrics on  $X$ . Another characterization is this: A metric  $D$  on  $\{1, 2, \dots, n\}$  is an  $\ell_1$  metric iff there exist a measure space  $(\Omega, \Sigma, \mu)$  and sets  $A_1, \dots, A_n \in \Sigma$  such that  $D(i, j) = \mu(A_i \Delta A_j)$ .

Every  $\ell_1$  metric is a hypermetric space (since cut metrics satisfy the hypermetric inequalities), but for 7 or more points, this condition is not sufficient. Hypermetric spaces have an interesting characterization in terms of *Delaunay polytopes* of lattices; see [DL97].

## ISOMETRIC EMBEDDABILITY

Deciding isometric embeddability in  $\ell_1$  is NP-hard. On the other hand, the embeddability of *unweighted* graphs, both in  $\ell_1$  and in a Hamming cube, has been characterized and can be tested in polynomial time. In particular, we have:

### THEOREM 8.1.3

- (i) *An unweighted graph  $G$  embeds isometrically in some cube  $\{0, 1\}^m$  with the  $\ell_1$ -metric iff it is bipartite and satisfies the pentagonal inequality.*
- (ii) *An unweighted graph  $G$  embeds isometrically in  $\ell_1$  iff it is an isometric subgraph of a Cartesian product of half-cube graphs and cocktail-party graphs.*

A first characterization of cube-embeddable graphs was given by Djokovic [Djo73], and the form in (i) is due to Avis (see [DL97]). Part (ii) is from Shpectorov [Shp93].

## ORDER OF CONGRUENCE

The isometric embeddability in  $\ell_1^2$  is characterized by 6-point subspaces (6 is best possible here), and can thus be tested in polynomial time (Bandelt and Chepoi [BC96]). The proof uses a result of Bandelt and Dress [BD92] of independent interest, about certain canonical decompositions of metric spaces (see also [DL97]).

On the other hand, for no  $d \geq 3$  it is known whether the order of congruence of  $\ell_1^d$  is finite; there is a lower bound of  $d^2$  (for odd  $d$ ) or  $d^2 - 1$  (for  $d$  even).

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### 8.1.3 THE OTHER $p$

The spaces  $\ell_\infty^d$  are the richest (and thus generally the most difficult to deal with); every  $n$ -point metric space  $(X, D)$  embeds isometrically in  $\ell_\infty^n$ . To see this, write  $X = \{x_1, x_2, \dots, x_n\}$  and define  $f: X \rightarrow \ell_\infty^n$  by  $f(x_i)_j = D(x_i, x_j)$ .

The other  $p \neq 1, 2, \infty$  are encountered less often, but it may be useful to know the cases where all  $\ell_p$  metrics embed with bounded distortion in  $\ell_q$ : This happens iff  $p = q$ , or  $p = 2$ , or  $q = \infty$ , or  $1 \leq q \leq p \leq 2$ . Isometric embeddings exist in all these cases. Moreover, for  $1 \leq q \leq p \leq 2$ , the whole of  $\ell_p^d$  can be  $(1+\varepsilon)$  embedded in  $\ell_q^{Cd}$  with a suitable  $C = C(p, q, \varepsilon)$  (so the dimension doesn't grow by much); see, e.g., [MS86]. These embeddings are probabilistic. The simplest one is  $\ell_2^d \rightarrow \ell_1^{Cd}$ , given by  $x \mapsto Ax$  for a random  $\pm 1$  matrix  $A$  of size  $Cd \times d$  (surprisingly, no good explicit embedding is known even in this case).

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## 8.2 APPROXIMATE EMBEDDINGS OF GENERAL METRICS IN $\ell_p$

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### 8.2.1 BOURGAIN'S EMBEDDING IN $\ell_2$

The mother of most embeddings mentioned in the next few sections, from both historical and “technological” points of view, is the following theorem.

**THEOREM 8.2.1** Bourgain [Bou85]

*Any  $n$ -point metric space  $(X, D)$  can be embedded in  $\ell_2$  (in fact, in every  $\ell_p$ ) with distortion  $O(\log n)$ .*

We describe the embedding, which is constructed probabilistically. We set  $m = \lfloor \log_2 n \rfloor$  and  $q = \lfloor C \log n \rfloor$  ( $C$  a suitable constant) and construct an embedding in  $\ell_2^{mq}$ , with the coordinates indexed by  $i = 1, 2, \dots, m$  and  $j = 1, 2, \dots, q$ . For each such  $i, j$ , we select a subset  $A_{ij} \subseteq X$  by putting each  $x \in X$  into  $A_{ij}$  with probability  $2^{-j}$ , all the random choices being mutually independent. Then we set  $f(x)_{ij} = D(x, A_{ij})$ . We thus obtain an embedding in  $\ell_2^{O(\log^2 n)}$  (Bourgain's original proof used exponential dimension; the possibility of reducing it was noted later), and it can be shown that the distortion is  $O(\log n)$  with high probability.

This yields an  $O(n^2 \log n)$  randomized algorithm for computing the desired embedding. The algorithm can be derandomized (preserving the polynomial time and the dimension bound) using the method of conditional probabilities; this result seems to be folklore. Alternatively, it can be derandomized using small sample spaces [LLR95]; this, however, uses dimension  $\Theta(n^2)$ . Finally, as was remarked above, an embedding of a given space in  $\ell_2$  with optimal distortion can be computed by semidefinite programming.

The  $O(\log n)$  distortion for embedding a general metric in  $\ell_2$  is tight [LLR95] (and similarly for  $\ell_p$ ,  $p < \infty$  fixed). Examples of metrics that cannot be embedded any better are the shortest-path metrics of constant-degree expanders. (An  $n$ -vertex graph is a **constant-degree expander** if all degrees are bounded by some constant  $r$  and each subset of  $k$  vertices has at least  $\beta k$  outgoing edges, for  $1 \leq k \leq \frac{n}{2}$  and for some constant  $\beta > 0$  independent of  $n$ .)

Another interesting lower bound is due to Linial et al. [LMN02]: The shortest-path metric of *any*  $k$ -regular graph ( $k \geq 3$ ) of girth  $g$  requires  $\Omega(\sqrt{g})$  distortion for embedding in  $\ell_2$ .

## 8.2.2 NEGATIVE TYPE METRICS

We say that a metric space  $(X, D)$  is of *negative type* if there exists a mapping  $f : X \rightarrow L_2$  such that  $\|f(x) - f(y)\|_2^2 = D(x, y)$ , for all  $x, y \in X$ .

In [ARV09] the Goemans-Linial semidefinite programming relaxation of the sparsest cut problem was used to obtain a  $O(\sqrt{\log n})$ -approximation for the uniform case of the problem. Building on this work and a previous bound from [CGR05], it was shown in [ALN08] that any  $n$ -point metric space of negative type admits an embedding into  $\ell_1$  with distortion  $O(\sqrt{\log n} \log \log n)$ . This implies a  $O(\sqrt{\log n} \log \log n)$ -approximation for the general sparsest cut problem. In [CKN09] it was shown that there exist  $n$ -point metric spaces of negative type that require distortion  $(\log n)^{\Omega(1)}$  to be embedded into  $\ell_1$ . This also implies that the integrality of the Goemans-Linial semidefinite programming relaxation is  $(\log n)^{\Omega(1)}$ . Very recently, that result was improved to  $\Omega(\sqrt{\log n})$ , matching the upper bound [NY17].

## 8.2.3 THE DIMENSION OF EMBEDDINGS IN $\ell_\infty$

If we want to embed all  $n$ -point metrics in  $\ell_\infty^d$ , there is a tradeoff between the dimension  $d$  and the worst-case distortion. The following result was proved in [Mat96] by adapting Bourgain's technique.

### THEOREM 8.2.2

*For an integer  $b > 0$  set  $c = 2b - 1$ . Then any  $n$ -point metric space can be embedded in  $\ell_\infty^d$  with distortion  $c$ , where  $d = O(bn^{1/b} \log n)$ .*

An almost matching lower bound can be proved using graphs without short cycles, an idea also going back to [Bou85]. Let  $m(g, n)$  be the maximum possible number of edges of an  $n$ -vertex graph of girth  $g + 1$ . For every fixed  $c \geq 1$  and integer  $g > c$  there exists an  $n$ -point metric space such that any  $c$ -embedding in  $\ell_\infty^d$  has  $d = \Omega(m(g, n)/n)$  [Mat96]. The proof goes by counting: Fix a graph  $G_0$  witnessing  $m(g, n)$ , and let  $\mathcal{G}$  be the set of graphs (considered with the shortest-path metric) that can be obtained from  $G_0$  by deleting some edges. It turns out that if  $G, G' \in \mathcal{G}$  are distinct, then they cannot have “essentially the same”  $c$ -embeddings in  $\ell_\infty^d$ , and there are only “few” essentially different embeddings in  $\ell_\infty^d$  if  $d$  is small.

It is easy to show that  $m(g, n) = O(n^{1+1/\lfloor g/2 \rfloor})$  for all  $g$ , and this is conjectured to be the right order of magnitude [Erd64]. This has been verified for  $g \leq 7$  and for  $g = 10, 11$ , while only worse lower bounds are known for the other values of  $g$  (with exponent roughly  $1 + 4/3g$  for  $g$  large). Whenever the conjecture holds for some  $g = 2b - 1$ , the above theorem is tight up to a logarithmic factor for the corresponding  $b$ . Unfortunately, although explicit constructions of graphs of a given girth with many edges are known, the method doesn't provide explicit examples of badly embeddable spaces.

For special classes of metrics improved bounds on the dimension are possible. The shortest path metric of any graph that excludes some fixed minor admits a constant-distortion embedding into  $\ell_\infty^{O(\log n)}$  [KLMN05].

## DISTANCE ORACLES

An interesting algorithmic result, conceptually resembling the above theorem, was

obtained by Thorup and Zwick [TZ01]. They showed that for an integer  $b > 0$ , every  $n$ -point metric space can be stored in a data structure of size  $O(n^{1+1/b})$  (with preprocessing time of the same order) so that, within time  $O(b)$ , the distance between any two points can be approximated within a multiplicative factor of  $2b-1$ . Mendel and Naor [MN06] have obtained a data structure for the above problem of size  $O(n^{1+1/b})$ , with query time  $O(1)$  that approximates the distance between any two points within a multiplicative factor of  $O(b)$ .

## LOW DIMENSION

The other end of the tradeoff between distortion and dimension  $d$ , where  $d$  is fixed (and then all  $\ell_p$ -norms on  $\mathbb{R}^d$  are equivalent up to a constant) was investigated in [Mat90]. For all fixed  $d \geq 1$ , there are  $n$ -point metric spaces requiring distortion  $\Omega(n^{1/\lfloor(d+1)/2\rfloor})$  for embedding in  $\ell_2^d$  (for  $d = 2$ , an example is the shortest-path metric of  $K_5$  with every edge subdivided  $n/10$  times). On the other hand, every  $n$ -point space  $O(n)$ -embeds in  $\ell_2^1$  (the real line), and  $O(n^{2/d} \log^{3/2} n)$ -embeds in  $\ell_2^d$ ,  $d \geq 3$ .

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### 8.2.4 THE JOHNSON-LINDENSTRAUSS LEMMA: FLATTENING IN $\ell_2$

The  $n$ -point  $\ell_2$  metric with all distances equal to 1 requires dimension  $n - 1$  for isometric embedding in  $\ell_2$ . A somewhat surprising and extremely useful result shows that, in particular, this metric can be embedded in dimension only  $O(\log n)$  with distortion close to 1.

#### THEOREM 8.2.3 Johnson and Lindenstrauss [JL84]

For every  $\varepsilon > 0$ , any  $n$ -point  $\ell_2$  metric can be  $(1+\varepsilon)$ -embedded in  $\ell_2^{O(\log n/\varepsilon^2)}$ .

There is an almost matching lower bound for the necessary dimension, due to Alon (see [Mat02]):  $\Omega(\log n/(\varepsilon^2 \log(1/\varepsilon)))$ . For the special case of linear maps, a matching  $\Omega(\min\{n, \log n/\varepsilon^2\})$  lower bound has been obtained [LN16].

All known proofs (see, e.g., [Ach01] for references and an insightful discussion) first place the metric under consideration in  $\ell_2^n$  and then map it into  $\ell_2^d$  by a random linear map  $A: \ell_2^n \rightarrow \ell_2^d$ . Here  $A$  can be a random orthogonal projection (as in [JL84]). It can also be given by a random  $n \times d$  matrix with independent  $N(0, 1)$  entries [IM98], or even one with independent uniform random  $\pm 1$  entries. The proof in the last case, due to [Ach01], is considerably more difficult than the previous ones (which use spherically symmetric distributions), but this version has advantages in applications.

An embedding as in the theorem can be computed deterministically in time  $O(n^2 d (\log n + 1/\varepsilon)^{O(1)})$  [EIO02] (also see [Siv02]).

The aforementioned embeddings use random dense  $n \times d$  matrices  $A$ , which means that mapping a point  $p$  into  $Ap$  takes rectangular (i.e.,  $\Theta(nd)$ ) time. In order to reduce the running time several approaches were proposed. The *Fast Johnson-Lindenstrauss transform* approach [AC06, AL13, KW11] uses the product of a diagonal matrix, a Fourier matrix and a projection matrix, which makes it possible to evaluate the matrix-vector product  $Ap$  in sub-rectangular time. In particular, the algorithm of [KW11] runs in  $O(n \log n)$  time, albeit the reduced dimension  $d$  is  $O(\log(n) \log^4(d)/\varepsilon^2)$ .

Another approach to faster dimensionality reduction is *Feature Hashing*, also

known as the *Sparse Johnson-Lindenstrauss transform* [WDL<sup>+</sup>09, DKS10, KN14]. Here, the speedup is achieved by showing that one can make the matrix  $A$  sparse without sacrificing the bound on the reduced dimension. In particular [KN14] shows a distribution over  $n \times d$  matrices  $A$  that have only  $\varepsilon nd$  non-zeros and achieve the same bound for  $d$  as given in Theorem 8.2.3.

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## 8.2.5 IMPOSSIBILITY OF DIMENSIONALITY REDUCTION IN $\ell_1$

It has been shown that no analogue of the Johnson–Lindenstrauss result holds in  $\ell_1$ . The first result of this type was obtained by Brinkman and Charikar [BC05] and was later simplified by Lee and Naor [LN04]. The best known lower bound is due to Andoni et al. [ACNN11] who showed that for any  $\varepsilon > 0$  and  $n > 0$  there exist some  $n$ -point subset  $X$  of  $\ell_1$  such that any embedding of  $X$  into  $\ell_1$  with distortion  $1 + \varepsilon$  requires dimension at least  $n^{1-O(1/\log(1/\varepsilon))}$ .

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## 8.2.6 VOLUME-RESPECTING EMBEDDINGS

Feige [Fei00] introduced the notion of *volume-respecting* embeddings in  $\ell_2$ , with impressive algorithmic applications. While the distortion of a mapping depends only on pairs of points, the volume-respecting condition takes into account the behavior of  $k$ -tuples. For an arbitrary  $k$ -point metric space  $(S, D)$ , we set  $\text{Vol}(S) = \sup_{\text{nonexpanding } f: S \rightarrow \ell_2} \text{Evol}(f(S))$ , where  $\text{Evol}(P)$  is the  $(k-1)$ -dimensional volume of the convex hull of  $P$  (in  $\ell_2$ ). Given a nonexpanding  $f: X \rightarrow \ell_2$  for some metric space  $(X, D)$  with  $|X| \geq k$ , we define the  $k$ -distortion of  $f$  to be

$$\sup_{S \subseteq X, |S|=k} \left( \frac{\text{Vol}(S)}{\text{Evol}(f(S))} \right)^{1/(k-1)}.$$

If the  $k$ -distortion of  $f$  is  $\Delta$ , we call  $f$   $(k, \Delta)$ -*volume-respecting*.

If  $f: X \rightarrow \ell_2$  is an embedding scaled so that it is nonexpanding but just so, the 2-distortion coincides with the usual distortion. But note that for  $k > 2$ , the isometric “straight” embedding of a path in  $\ell_2$  is not volume-respecting at all. In fact, it is known that for any  $k > 2$ , no  $(k, o(\sqrt{\log n}))$ -volume-respecting embedding of a *line* exists [DV01].

Extending Bourgain’s technique, Feige proved that for every  $k > 2$ , every  $n$ -point metric space has a  $(k, O(\log n + \sqrt{k \log n \log k}))$ -volume-respecting embedding in  $\ell_2$ . Magen and Zouzias [MZ08] have obtained volume-respecting dimensionality reduction for finite subsets of Euclidean space. They show that any  $n$ -point subset of  $\ell_2$  admits a  $O(k, 1 + \varepsilon)$ -volume-respecting embedding into  $\ell_2^d$ , for some  $d = O(\max\{k/\varepsilon, \varepsilon^{-2} \log n\})$ .

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## 8.3 APPROXIMATE EMBEDDING OF SPECIAL METRICS IN $\ell_p$

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### GLOSSARY

*$\mathcal{G}$ -metric:* Let  $\mathcal{G}$  be a class of graphs and let  $G \in \mathcal{G}$ . Each positive weight

function  $w: E(G) \rightarrow (0, \infty)$  defines a metric  $D_w$  on  $V(G)$ , namely, the shortest-path metric, where the length of a path is the sum of the weights of its edges. A metric space is a  $\mathcal{G}$ -metric if it is isometric to a subspace of  $(V(G), D_w)$  for some  $G \in \mathcal{G}$  and some  $w$ .

**Tree metric, planar-graph metric:** A  $\mathcal{G}$ -metric for  $\mathcal{G}$ , the class of all trees or all planar graphs, respectively.

**Minor:** A graph  $G$  is a minor of a graph  $H$  if it can be obtained from  $H$  by repeated deletions of edges and contractions of edges.

**Doubling metric:** A metric  $(X, D)$  such that for all  $r > 0$  any ball of radius  $r$  can be covered by a constant number of balls of radius  $r/2$ .

### 8.3.1 TREE METRICS, PLANAR-GRAPH METRICS, AND FORBIDDEN MINORS

A major research direction has been improving Bourgain's embedding in  $\ell_2$  for restricted families of metric spaces.

#### TREE METRICS

It is easy to show that any tree metric embeds isometrically in  $\ell_1$ . Any  $n$ -point tree metric can also be embedded isometrically in  $\ell_\infty^{O(\log n)}$  [LLR95]. For  $\ell_p$  embeddings, the situation is rather delicate:

#### THEOREM 8.3.1

*Distortion of order  $(\log \log n)^{\min(1/2, 1/p)}$  is sufficient for embedding any  $n$ -vertex tree metric in  $\ell_p$  ( $p \in (1, \infty)$  fixed) [Mat99], and it is also necessary in the worst case (for the complete binary tree; [Bou86]).*

Gupta [Gup00] proved that any  $n$ -point tree metric  $O(n^{1/(d-1)})$ -embeds in  $\ell_2^d$  ( $d \geq 1$  fixed), and for  $d = 2$  and trees with unit-length edges, Babilon et al. [BMMV02] improved this to  $O(\sqrt{n})$ . Bădoiu et al. [BCIS06] have shown that any  $n$ -point ultrametric  $O(n^{1/d})$ -embeds in  $\ell_2^d$ .

#### PLANAR-GRAPH METRICS AND OTHER CLASSES WITH A FORBIDDEN MINOR

The following result was proved by Rao, building on the work of Klein, Plotkin, and Rao.

#### THEOREM 8.3.2 Rao [Rao99]

*Any  $n$ -point planar-graph metric can be embedded in  $\ell_2$  with distortion  $O(\sqrt{\log n})$ . More generally, let  $H$  be an arbitrary fixed graph and let  $\mathcal{G}$  be the class of all graphs not containing  $H$  as a minor; then any  $n$ -point  $\mathcal{G}$ -metric can be embedded in  $\ell_2$  with distortion  $O(\sqrt{\log n})$ .*

This bound is tight even for series-parallel graphs (no  $K_4$  minor) [NR02]; the example is obtained by starting with a 4-cycle and repeatedly replacing each edge by two paths of length 2.

A challenging conjecture, one that would have significant algorithmic consequences, states that under the conditions of Rao's theorem, all  $\mathcal{G}$ -metrics can be

$c$ -embedded in  $\ell_1$  for some  $c$  depending only on  $\mathcal{G}$  (but not on the number of points). Apparently, this conjecture was first published in [GNRS04], where it was verified for the forbidden minors  $K_4$  (series-parallel graphs) and  $K_{2,3}$  (outerplanar graphs). It has also been verified for graphs that exclude the 4-wheel [CJLV08] and for graphs of bounded pathwidth (equivalently, for graphs that exclude some forest) [LS13]. It has also been shown that the conjecture can be reduced to the conjunction of two apparently simpler problems: the special case of planar graphs (also known as the planar embedding conjecture) and the so-called  $k$ -sum conjecture which asserts that constant-distortion embeddability into  $\ell_1$  is closed under bounded clique-sums [LS09]. The latter conjecture has been verified special case of edge-sums over graphs of bounded size [LP13].

## DOUBLING METRICS

It has been shown by Gupta et al. [GKL03] that any  $n$ -point doubling metric admits a  $O(\sqrt{\log n})$ -embedding into  $\ell_2$  and that there exists a doubling metric that requires distortion  $\Omega(\sqrt{\log n})$ . Since  $\ell_2$  embeds into  $\ell_1$  isometrically, the above upper bound also holds for embedding into  $\ell_1$ . It has been shown that there exists an  $n$ -point doubling metric that requires distortion  $\Omega(\sqrt{\log n / \log \log n})$  to be embedded into  $\ell_1$  [LS11].

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### 8.3.2 METRICS DERIVED FROM OTHER METRICS

In this section we focus on metrics derived from other metrics, e.g., by defining a distance between two *sets* or *sequences* of points from the underlying metric.

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## GLOSSARY

**Uniform metric:** For any set  $X$ , the metric  $(X, D)$  is uniform if  $D(p, q) = 1$  for all  $p \neq q, p, q \in X$ .

**Hausdorff distance:** For a metric space  $(X, D)$ , the Hausdorff metric  $H$  on the set  $2^X$  of all subsets of  $X$  is given by  $H(A, B) = \min(\vec{H}(A, B), \vec{H}(B, A))$ , where  $\vec{H}(A, B) = \sup_{a \in A} \inf_{b \in B} D(a, b)$ .

**Earth-mover distance:** For a metric space  $(X, D)$  and an integer  $d \geq 1$ , the earth-mover distance of two  $d$ -element sets  $A, B \subseteq X$  is the minimum weight of a perfect matching between  $A$  and  $B$ ; that is,  $\min_{\text{bijective } \pi: A \rightarrow B} \sum_{a \in A} D(a, \pi(a))$ .

**Levenshtein distance (or edit distance):** For a metric space  $M = (\Sigma, D)$ , the distance between two strings  $w, w' \in \Sigma^*$  is the minimum cost of a sequence of operations that transforms  $w$  into  $w'$ . The allowed operations are: character insertion (of cost 1), character deletion (of cost 1), or replacement of a symbol  $a$  by another symbol  $b$  (of cost  $D(a, b)$ ), where  $a, b \in \Sigma$ . The total cost of the sequence of operations is the sum of all operation costs.

**Fréchet distance:** For a metric space  $M = (X, D)$ , the Fréchet distance (also called the *dogkeeper's distance*) between two functions  $f, g: [0, 1] \rightarrow X$  is defined as

$$\inf_{\pi: [0,1] \rightarrow [0,1]} \sup_{t \in [0,1]} D(f(t), g(\pi(t)))$$

where  $\pi$  is continuous, monotone increasing, and such that  $\pi(0) = 0, \pi(1) = 1$ .

## HAUSDORFF DISTANCE

The Hausdorff distance is often used in computer vision for comparing geometric shapes, represented as sets of points. However, even computing a single distance  $H(A, B)$  is a nontrivial task. As noted in [FCI99], for any  $n$ -point metric space  $(X, D)$ , the Hausdorff metric on  $2^X$  can be isometrically embedded in  $\ell_\infty^n$ .

The dimension of the host norm can be further reduced if we focus on embedding particular Hausdorff metrics. In particular, let  $H_M^s$  be the Hausdorff metric over all  $s$ -subsets of  $M$ . Farach-Colton and Indyk [FCI99] showed that if  $M = (\{1, \dots, \Delta\}^k, \ell_p)$ , then  $H_M^s$  can be embedded in  $\ell_\infty^{d'}$  with distortion  $1 + \varepsilon$ , where  $d' = O(s^2(1/\varepsilon)^{O(k)} \log \Delta)$ . A different tradeoff was obtained in [BS16], where it is shown that for  $M = (\{1, \dots, \Delta\}^k, \ell_p)$ , where  $k, s$  are constant,  $H_M^s$  embeds into  $\ell_\infty^{f(k,s)}$  with  $g(k, s)$  distortion for some functions  $f$  and  $g$ . For a general (finite) metric space  $M = (X, D)$  they show that  $H_M^s$  can be embedded in  $\ell_\infty^{s^{O(1)}|X|^\alpha \log \Delta}$  for any  $\alpha > 0$  with constant distortion, where  $\Delta = (\min_{p \neq q \in X} D(p, q)) / (\max_{p, q \in X} D(p, q))$ .

## EARTH-MOVER DISTANCE (EMD)

A very interesting relation between embedding EMD in normed spaces and embeddings in probabilistic trees (discussed below in Section 8.4.1) was discovered in [Cha02]: If a finite metric space can be embedded in a convex combination of dominating trees with distortion  $c$ , then the EMD over it can be embedded in  $\ell_1$  with distortion  $O(c)$ . Consequently, the EMD over subsets of  $(\{1, \dots, \Delta\}^k, \ell_p)$  can be embedded in  $\ell_1$  with distortion  $O(k \log \Delta)$ . Lower bounds have been proven in [KN06, NS07].

## LEVENSHTEIN DISTANCE AND ITS VARIANTS

The Levenshtein distance is used in text processing and computational biology. The best algorithm computing the Levenshtein distance of two strings  $w, w'$ , even approximately, has running time of order  $|w| \cdot |w'|$  (for a constant-size  $\Sigma$ ). In the simplest (but nevertheless quite common) case of the uniform metric over  $\Sigma = \{0, 1\}$ , Levenshtein distance over strings of length  $d$  admits a  $2^{O(\sqrt{\log d \log \log d})}$ -embedding into  $\ell_1$  [OR07]. A lower bound of  $\Omega(\log d)$  has also been obtained for this case [KR09, KN06].

If we modify the definition of the distance by permitting the movement of an arbitrarily long contiguous block of characters as a single operation, and if the underlying metric is uniform, then the resulting **block-edit** metric can be embedded in  $\ell_1$  with distortion  $O(\log l \cdot \log^* l)$ , where  $l$  is the length of the embedded strings (see [MS00, CM02] and references therein). The modified metric has applications in computational biology and in string compression. The embedding of a given string can be computed in almost linear time, which yields a very fast approximation algorithm for computing the distance between two strings (the exact distance computation is NP-hard!).

## FRÉCHET METRIC

The Fréchet metric is an interesting metric measuring the distances between two *curves*. From the applications perspective, it is interesting to investigate the case where  $M = \ell_2^k$  and  $f, g$  are continuous, closed polygonal chains, consisting of (say)

at most  $d$  segments each. Denote the set of such curves by  $C_d^k$ . It is not known whether  $C_d^k$ , under Fréchet distance, can be embedded in  $\ell_\infty$  with finite dimension (for infinite dimension, an isometric embedding follows from the universality of the  $\ell_\infty$  norm). On the other hand, it is easy to check that for any bounded set  $S \subset \ell_\infty^d$ , there is an isometry  $f: S \rightarrow C_{3d}^1$ .

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### 8.3.3 OTHER SPECIAL METRICS

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#### GLOSSARY

**(1, 2)-B metric:** A metric space  $(X, D)$  such that for any  $x \in X$  the number of points  $y$  with  $D(x, y) = 1$  is at most  $B$ , and all other distances are equal to 2.

**Transposition distance:** The (unfortunately named) metric  $D_T$  on the set of all permutations of  $\{1, 2, \dots, n\}$ ;  $D_T(\pi_1, \pi_2)$  is the minimum number of moves of contiguous subsequences to arbitrary positions needed to transform  $\pi_1$  into  $\pi_2$ .

#### BOUNDED DISTANCE METRICS

Trevisan [Tre01] considered approximate embeddings of  $(1, 2)$ - $B$  metrics in  $\ell_p^d$  (in a sense somewhat different from low-distortion embeddings). Guruswami and Indyk [GI03] proved that any  $(1, 2)$ - $B$  metric can be isometrically embedded in  $\ell_\infty^{O(B \log n)}$ .

#### PERMUTATION METRICS

It was shown in [CMS01] that  $D_T$  can be  $O(1)$ -embedded in  $\ell_1$ ; similar results were obtained for other metrics on permutations, including reversal distance and permutation edit distance.

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## 8.4 APPROXIMATE EMBEDDINGS IN RESTRICTED METRICS

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#### GLOSSARY

**Dominating metric:** Let  $D, D'$  be metrics on the same set  $X$ . Then  $D'$  dominates  $D$  if  $D(x, y) \geq D'(x, y)$  for all  $x, y \in X$ .

**Convex combination of metrics:** Let  $X$  be a set,  $T_1, T_2, \dots, T_k$  metrics on it, and  $\alpha_1, \dots, \alpha_k$  nonnegative reals summing to 1. The convex combination of the  $T_i$  (with coefficients  $\alpha_i$ ) is the metric  $D$  given by  $D(x, y) = \sum_{i=1}^k \alpha_i T_i(x, y)$ ,  $x, y \in X$ .

**Hierarchically well-separated tree (k-HST):** A 1-HST is exactly an *ultra-metric*; that is, the shortest-path metric on the leaves of a rooted tree  $T$  (with weighted edges) such that all leaves have the same distance from the root. For a  $k$ -HST with  $k > 1$  we require that, moreover,  $\Delta(v) \leq \Delta(u)/k$  whenever  $v$  is a child of  $u$  in  $T$ , where  $\Delta(v)$  denotes the diameter of the subtree rooted at  $v$  (w.l.o.g. we may assume that each non-leaf has degree at least 2, and so  $\Delta(v)$

equals the distance of  $v$  to the nearest leaves). *Warning:* This is a newer definition introduced in [BBM06]. Older papers, such as [Bar96, Bar98], used another definition, but the difference is merely technical, and the notion remains essentially the same.

### 8.4.1 PROBABILISTIC EMBEDDINGS IN TREES

A convex combination  $\overline{D} = \sum_{i=1}^r \alpha_i T_i$  of some metrics  $T_1, \dots, T_r$  on  $X$  can be thought of as a **probabilistic metric** (this concept was suggested by Karp). Namely,  $\overline{D}(x, y)$  is the expectation of  $T_i(x, y)$  for  $i \in \{1, 2, \dots, r\}$  chosen at random according to the distribution given by the  $\alpha_i$ . Of particular interest are embeddings in convex combinations of *dominating* metrics. The domination requirement is crucial for many applications. In particular, it enables one to solve many problems over the original metric  $(X, D)$  by solving them on a (simple) metric chosen at random from  $T_1, \dots, T_r$  according to the distribution defined by the  $\alpha_i$ .

The usefulness of probabilistic metrics comes from the fact that a sum of metrics is much more powerful than each individual metric. For example, it is not difficult to show that there are metrics (e.g., cycles [RR98, Gup01]) that cannot be embedded in tree metrics with  $o(n)$  distortion. In contrast, we have the following result:

**THEOREM 8.4.1** Fakcharoenphol, Rao, and Talwar [FRT03]

*Let  $(X, D)$  be any  $n$ -point metric space. For every  $k > 1$ , there exist a natural number  $r$ ,  $k$ -HST metrics  $T_1, T_2, \dots, T_r$  on  $X$ , and coefficients  $\alpha_1, \dots, \alpha_r > 0$  summing to 1, such that each  $T_i$  dominates  $D$  and the (identity) embedding of  $(X, D)$  in  $(X, \overline{D})$ , where  $\overline{D} = \sum_{i=1}^r \alpha_i T_i$ , has distortion  $O(k \log n)$ .*

The first result of this type was obtained by Alon et al [AKPW95]. Their embedding has distortion  $2^{O(\sqrt{\log n \log \log n})}$ . A few years later Bartal [Bar96] improved the distortion bound considerably, to  $O(\log^2 n)$  and later even to  $O(\log n \log \log n)$  [Bar98]. Finally, Fakcharoenphol, Rao, and Talwar [FRT03] obtained an embedding with distortion  $O(\log n)$ . This embeddings uses convex combination of very simple tree metrics (i.e., HST's), which further simplifies the design of algorithms. The  $O(\log n)$  distortion is the best possible in general (since any convex combination of tree metrics embeds isometrically in  $\ell_1$ ). Embedding the  $\sqrt{n} \times \sqrt{n}$  grid into a convex combination of tree metrics requires distortion  $\Omega(\log n)$  [AKPW95].

The constructions in [Bar96, Bar98, FRT03] generate trees with Steiner nodes (i.e., nodes that do not belong to  $X$ ). However, one can get rid of such nodes in *any* tree while increasing the distortion by at most 8 [Gup01]. A lower bound of  $8 - o(1)$  on the distortion has also been obtained [CXKR06]. The problem of removing Steiner nodes has also been considered for the case of general graphs. It has been shown that for any graph  $G$  with edge weights  $w$  and  $T \subseteq V(G)$  there exists a some graph  $G'$  with edge weights  $w'$  and  $V(G) = T$  such that  $G'$  is isomorphic to a minor of  $G$  and for all  $u, v \in T$ ,  $D_w(u, v) \leq D_{w'}(u, v) \leq O(\log^5 |T|) D_w(u, v)$  [KKN15].

An interesting extra feature of the construction of Alon et al. mentioned above is that if the metric  $D$  is given as the shortest-path metric of a (weighted) graph  $G$  on the vertex set  $X$ , then all the  $T_i$  are spanning trees of this  $G$ . None of the constructions in [Bar96, Bar98, FRT03] share this property. However, more recent work [EEST08] showed an embedding with distortion  $O(\log^2 n \log \log n)$ , which was

later improved to  $O(\log n \log \log n (\log \log \log n)^3)$  [ABN08].

The embedding algorithms in Bartal's papers [Bar96, Bar98] are randomized and run in polynomial time. A deterministic algorithm for the same problem was given in [CCG<sup>+</sup>98]. The latter algorithm constructs a distribution over  $O(n \log n)$  trees (the number of trees in Bartal's construction was exponential in  $n$ ).

Probabilistic embeddings into other classes of graph metrics have also been considered. It has been shown that the metric of any graph of genus  $g > 0$  admits an embedding into a convex combination of planar graph metrics with distortion  $O(\log(g+1))$  [Sid10]. Similarly, the metric of any graph of bounded genus and with a bounded number of apices admits a  $O(1)$ -embedding into a convex combination of planar graph metrics [LS09]. In contrast, it has been shown that for any  $k \geq 2$  there exist graphs of treewidth  $k+1$  such that any embedding into a convex combination of graphs of treewidth  $k$  has distortion  $\Omega(\log n)$  [CJLV08].

For general graphs the following probabilistic embedding has also been obtained: Let  $G$  be a graph with edge weights  $w$  and let  $T \subseteq V(G)$ . Then there exists a  $O(\log n \log \log n)$ -embedding of  $(T, D_w)$  into a convex combination of graph metrics  $H_1, \dots, H_r$  on  $T$  where each  $H_i$  is minor of  $G$  [EGK<sup>+</sup>10].

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## 8.4.2 RAMSEY-TYPE THEOREMS

Many Ramsey-type questions can be asked in connection with low-distortion embeddings of metric spaces. For example, given classes  $\mathcal{X}$  and  $\mathcal{Y}$  of finite metric spaces, one can ask whether for every  $n$ -point space  $Y \in \mathcal{Y}$  there is an  $m$ -point  $X \in \mathcal{X}$  such that  $X$  can be  $\alpha$ -embedded in  $Y$ , for given  $n, m, \alpha$ .

Important results were obtained in [BBM06], and later greatly improved and extended in [BLMN03], for  $\mathcal{X}$  the class of all  $k$ -HST and  $\mathcal{Y}$  the class of all finite metric spaces; they were used for a lower bound in a significant algorithmic problem (metrical task systems). Let us quote some of the numerous results of Bartal et al.:

**THEOREM 8.4.2** Bartal, Linial, Mendel, and Naor [BLMN03]

*Let  $R_{\text{UM}}(n, \alpha)$  denote the largest  $m$  such that for every  $n$ -point metric space  $Y$  there exists an  $m$ -point 1-HST (i.e., ultrametric) that  $\alpha$ -embeds in  $Y$ , and let  $R_2(n, \alpha)$  be defined similarly with “ultrametric” replaced with “Euclidean metric.”*

- (i) *There are positive constants  $C, C_1, c$  such that for every  $\alpha > 2$  and all  $n$ ,*

$$n^{1-C_1(\log \alpha)/\alpha} \leq R_{\text{UM}}(n, \alpha) \leq R_2(n, \alpha) \leq Cn^{1-c/\alpha}.$$

- (ii) *(Sharp threshold at distortion 2) For every  $\alpha > 2$ , there exists  $c(\alpha) > 0$  such that  $R_2(n, \alpha) \geq R_{\text{UM}}(n, \alpha) \geq n^{c(\alpha)}$  for all  $n$ , while for every  $\alpha \in (1, 2)$ , we have  $c'(\alpha) \log n \leq R_{\text{UM}}(n, \alpha) \leq R_2(n, \alpha) \leq 2 \log n + C'(\alpha)$  for all  $n$ , with suitable positive  $c'(\alpha)$  and  $C'(\alpha)$ .*

For embedding a  $k$ -HST in a given space, one can use the fact that every ultrametric is  $k$ -equivalent to a  $k$ -HST. For an earlier result similar to the second part of (ii), showing that the largest Euclidean subspace  $(1+\varepsilon)$ -embeddable in a general  $n$ -point metric space has size  $\Theta(\log n)$  for all sufficiently small fixed  $\varepsilon > 0$ , see [BFM86].

TABLE 8.4.1 A summary of approximate embeddings

FROM	TO	DISTORTION	REFERENCE
any	$\ell_p, 1 \leq p < \infty$	$O(\log n)$	[Bou85]
constant-degree expander	$\ell_p, p < \infty$ fixed	$\Omega(\log n)$	[LLR95]
$k$ -reg. graph, $k \geq 3$ , girth $g$	$\ell_2$	$\Omega(\sqrt{g})$	[LMN02]
any	$\ell_\infty^{O(bn^{1/b} \log n)}$	$2b-1, b=1, 2, \dots$	[Mat96]
some	$\Omega(n^{1/b})$ -dim'l. normed space	$2b-1, b=1, 2, \dots$ (Erdős's conj.!)	[Mat96]
any	$\ell_1^1$	$\Theta(n)$	[Mat90]
any	$\ell_p^d, d$ fixed	$O(n^{2/d} \log^{3/2} n), \Omega(n^{1/\lfloor (d+1)/2 \rfloor})$	[Mat90]
$\ell_2$ metric	$\ell_2^{O(\log n/\varepsilon^2)}$	$1 + \varepsilon$	[JL84]
$\ell_1$ metric	$\ell_1^{n^{1-O(1/\log(1/\varepsilon))}}$	$1 + \varepsilon$	[ACNN11]
planar or forbidden minor series-parallel	$\ell_2$	$O(\sqrt{\log n})$	[Rao99]
planar or forbidden minor outerplanar or series-parallel	$\ell_2$	$\Omega(\sqrt{\log n})$	[NR02]
tree	$\ell_\infty^{O(\log n)}$	$O(1)$	[KLMN05]
tree	$\ell_1$	$O(1)$	[GNRS04]
tree	$\ell_1$	1	(folklore)
tree	$\ell_1^{c(\varepsilon) \log n}$	$1 + \varepsilon$	[LMM13]
bounded pathwidth	$\ell_1$	$O(1)$	[LS09]
tree	$\ell_2$	$\Theta((\log \log n)^{1/2})$	[Bou86, Mat99]
tree	$\ell_2^d$	$O(n^{1/(d-1)})$	[Gup00]
outerplanar, unit edges	$\ell_2^2$	$\Theta(n^{1/2})$	[BMMV02, BDHM07]
planar	$\ell_2^2$	$\Omega(n^{2/3})$	[BDHM07]
ultrametric	$\ell_2^d$	$O(n^{1/d})$	[BCIS06]
doubling metric	$\ell_2$	$\Theta(\sqrt{\log n})$	[GKL03]
doubling metric	$\ell_1$	$\Omega(\sqrt{\log n / \log \log n})$	[LS11]
Hausdorff metric over $(X, D)$	$\ell_\infty^{ X }$	1	[FCI99]
Hausd. over $s$ -subsets of $(X, D)$	$\ell_\infty^{s^{O(1)}  X ^\alpha \log \Delta}$	$c(\alpha)$	[FCI99]
Hausd. over $s$ -subsets of $\ell_p^k$	$\ell_\infty^{s^2(1/\varepsilon)^{O(k)} \log \Delta}$	$1 + \varepsilon$	[FCI99]
Hausd. over $s$ -subsets of $\ell_p^k$	$\ell_\infty^{f(k,s)}$	$g(k, s)$	[BS16]
EMD over $(X, D)$	$\ell_1$	$O(\log  X )$	[Cha02]
EMD over $\{0 \dots n\}^2$	$\ell_1$	$\Omega(\sqrt{\log n})$	[NS07]
EMD over $\{0, 1\}^k$	$\ell_1$	$\Omega(k)$	[KN06]
Levenshtein metric over $\{0, 1\}^d$	$\ell_1$	$2^{O(\sqrt{\log d \log \log d})}$	[OR07]
Levenshtein metric over $\{0, 1\}^d$	$\ell_1$	$\Omega(\log d)$	[KR09]
block-edit metric over $\Sigma^d$	$\ell_1$	$O(\log d \cdot \log^* d)$	[MS00, CM02]
(1,2)-B metric	$\ell_\infty^{O(B \log n)}$	1	[GI03]; for $\ell_p$ cf. [Tre01]
any	convex comb. of dom. trees (HSTs)	$O(\log n)$	[Bar98, FRT03]
genus- $g$ graph	convex comb. of planar graphs	$O(\log(g+1))$	[Sid10]
any	convex comb. of spanning trees	$O(\log n \log \log n (\log \log \log n)^3)$	[ABN08]

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### 8.4.3 APPROXIMATION BY SPARSE GRAPHS

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#### GLOSSARY

***t*-Spanner:** A subgraph  $H$  of a graph  $G$  (possibly with weighted edges) is a *t*-spanner of  $G$  if  $D_H(u, v) \leq t \cdot D_G(u, v)$  for every  $u, v \in V(G)$ .

Sparse spanners are useful as a more economic representation of a given graph (note that if  $H$  is a *t*-spanner of  $G$ , then the identity map  $V(G) \rightarrow V(H)$  is a *t*-embedding).

**THEOREM 8.4.3** Althöfer et al. [ADD<sup>+</sup>93]

*For every integer  $t \geq 2$ , every  $n$ -vertex graph  $G$  has a  $t$ -spanner with at most  $m(t, n)$  edges, where  $m(g, n) = O(n^{1+1/\lfloor g/2 \rfloor})$  is the maximum possible number of edges of an  $n$ -vertex graph of girth  $g + 1$ .*

The proof is extremely simple: Start with empty  $H$ , consider the edges of  $G$  one by one from the shortest to the longest, and insert each edge into the current  $H$  unless it creates a cycle with at most  $t$  edges. It is also immediately seen that the bound  $m(t, n)$  is the best possible in the worst case.

Rabinovich and Raz [RR98] proved that there are (unweighted)  $n$ -vertex graphs  $G$  that cannot be *t*-embedded in graphs (possibly weighted) with fewer than  $m(\Omega(t), n)$  edges (for  $t$  sufficiently large and  $n$  sufficiently large in terms of  $t$ ). Their main tool is the following lemma, proved by elementary topological considerations: If  $H$  is a simple unweighted connected  $n$ -vertex graph of girth  $g$  and  $G$  is a (possibly weighted) graph on at least  $n$  vertices with  $\chi(G) < \chi(H)$ , then  $H$  cannot be *c*-embedded in  $G$  for  $c < g/4 - 3/2$ ; here  $\chi(G)$  denotes the **Euler characteristic** of a graph  $G$ , which, for  $G$  connected, equals  $|E(G)| - |V(G)| + 1$ .

Euclidean spanners are spanners of the complete graph on a given point set in  $\ell_2^d$  with edge weights given by the Euclidean distances. Many papers were devoted to computing sparse Euclidean spanners in small (fixed) dimensions. A strong result, subsuming most of the previous work, is due to Arya et al. [ADM<sup>+</sup>95]: For any fixed  $\varepsilon > 0$  and  $d > 0$  and for any  $n$ -point set in  $\ell_2^d$ , a  $(1 + \varepsilon)$ -spanner of maximum degree  $O(1)$  can be computed in  $O(n \log n)$  time. Improved spanner constructions have also been obtained in Euclidean space of arbitrary dimension: Any  $n$ -point set in  $\ell_2$  admits a  $O(\sqrt{\log n})$ -spanner with  $O(n \log n \log \log n)$  edges [HPIS13].

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## 8.5 OPEN PROBLEMS AND WORK IN PROGRESS

Since the first edition of this chapter (2002) many of the problems in the area of low-distortion embeddings of metric spaces have been resolved and have given rise to several new algorithmic tools. Many of the most challenging problems remain open. Instead of stating open problems here, we refer to a list compiled by the second author [MN11] which is available on the Web.

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## 8.6 SOURCES AND RELATED MATERIAL

Discrete metric spaces have been studied from many different points of view, and the area is quite wide and diverse. The low-distortion embeddings treated in this chapter constitute only one particular (although very significant) direction. For recent results in some other directions the reader may consult [Cam00, DDL98, DD96], for instance. For more detailed overviews of the topics surveyed here, with many more references, the reader is referred to [Mat02][Chapter 15] (including proofs of basic results) and [Ind01] (with emphasis on algorithmic applications). Approximate embeddings of normed spaces are treated, e.g., in [MS86]. A general reference for isometric embeddings, especially embeddings in  $\ell_1$ , is [DL97].

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### RELATED CHAPTERS

Chapter 32: Proximity algorithms

Chapter 43: Nearest neighbors in high-dimensional spaces

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