62 RIGIDITY OF SYMMETRIC FRAMEWORKS

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INTRODUCTION

Since symmetry is ubiquitous in both man-made structures (e.g., buildings or mechanical linkages) and in structures found in nature (e.g., proteins or crystals), it is natural to consider the impact of symmetry on the rigidity and flexibility properties of frameworks. The special geometry induced by various symmetry groups often leads to added first-order (and sometimes even continuous) flexibility in the structure. These phenomena have been studied in the following two settings:

(1) **Forced Symmetry:** The framework is symmetric (with respect to a certain group) and must maintain this symmetry throughout its motions.

(2) **Incidental Symmetry:** The framework is symmetric (with respect to a certain group), but is allowed to move in unrestricted ways.

The key tool for analyzing the forced-symmetric rigidity properties of a symmetric framework is its corresponding group-labeled quotient graph (or "gain graph"). In particular, using very simple counts on the number of vertex and edge orbits under the group action (i.e., vertices and edges of the gain graph), we can often detect symmetry-preserving first-order flexibility in symmetric frameworks that are generically rigid without symmetry. For configurations which are regular modulo the given symmetry, these first-order flexes even extend to continuous flexes. By introducing (gain-)sparsity counts for all subgraphs of a gain graph, Laman-type combinatorial characterizations of all (symmetry-)regular forced-symmetric rigid frameworks have also been obtained for various symmetry groups. Moreover, these combinatorial results have been extended to "body-bar frameworks" with an arbitrary symmetry group in d-dimensional space.

Analyzing the rigidity of incidentally symmetric frameworks is more challenging and relies on tools from group representation theory. However, very simple necessary conditions for an incidentally symmetric framework to be first-order rigid can still be derived. These can be formulated in terms of counts on the number of vertices and edges that remain unshifted under the various symmetry operations of the framework. Similar techniques have also successfully been applied to the theory of scene analysis, and are expected to be of wider use in other areas of discrete geometry. A number of combinatorial characterizations of incidentally symmetric first-order rigid bar-joint and body-bar frameworks have also recently been obtained via an extension of some key tools from the forced-symmetric theory (such as the orbit rigidity matrix).

We discuss forced-symmetric frameworks in Section 62.1, and incidentally symmetric frameworks in Section 62.2. Finally, in Section 62.3, we discuss the rigidity of infinite periodic frameworks.

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62.1 FORCED-SYMMETRIC FRAMEWORKS

GLOSSARY

Automorphism of a graph: For a simple graph $G = (V, E)$, a permutation $\pi : V \to V$ such that $\{i, j\} \in E$ if and only if $\{\pi(i), \pi(j)\} \in E$. The group of all automorphisms of G is denoted by $\mathrm{Aut}(G)$.

Γ***-symmetric graph:*** For a group Γ, a simple graph $G = (V, E)$ for which there exists a group action $\theta : \Gamma \to \mathrm{Aut}(G)$. The action θ is ***free*** if $\theta(\gamma)(i) \neq i$ for all $i \in V$ and all non-trivial $\gamma \in \Gamma$.

Vertex orbit: For a Γ-symmetric graph $G = (V, E)$ and $i \in V$, the set $\Gamma i = \{\theta(\gamma)(i) \mid \gamma \in \Gamma\}$. Analogously, the ***edge orbit*** of G for $e = \{i, j\} \in E$, is the set $\Gamma e = \{\{\theta(\gamma)(i), \theta(\gamma)(j)\} \mid \gamma \in \Gamma\}$.

Quotient graph: For a Γ-symmetric graph $G = (V, E)$, the multigraph G/Γ with vertex set $V/\Gamma = \{\Gamma i \mid i \in V\}$ and edge set $E/\Gamma = \{\Gamma e \mid e \in E\}$.

Quotient Γ***-gain graph:*** Let $G = (V, E)$ be a Γ-symmetric graph, where the group action $\theta : \Gamma \to \mathrm{Aut}(G)$ is free. Each edge orbit Γe connecting Γi and Γj in the quotient graph G/Γ can be written as $\{\{\theta(\gamma)(i), \theta(\gamma) \circ \theta(\alpha)(j)\} \mid \gamma \in \Gamma\}$ for a unique $\alpha \in \Gamma$. For each Γe, orient Γe from Γi to Γj in G/Γ and assign to it the gain α. The resulting oriented quotient graph $G_0 = (V_0, E_0)$, together with the gain labeling $\psi : E_0 \to \Gamma$ described above, is the quotient Γ-gain graph (G_0, ψ) of G. (See also Figure 62.1.1.)
(Note that (G_0, ψ) is unique up to choices of representative vertices, and that the orientation is only used as a reference orientation and may be changed, provided that we also modify ψ so that if an edge has gain α in one orientation, then it has gain α^{-1} in the other direction.)

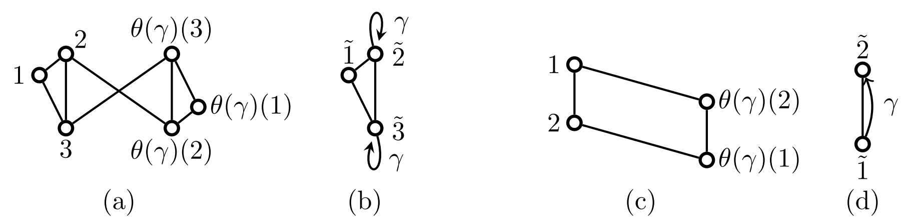

FIGURE 62.1.1
$\mathbb{Z}_2$-symmetric graphs (a,c), and their quotient $\mathbb{Z}_2$-gain graphs (b,d), where $\mathbb{Z}_2 = \{id, \gamma\}$. (The orientation and gain labeling is omitted for all edges with gain id.) The triangle in (b) is balanced, whereas any edge set containing a loop in (b) is unbalanced. The edge set in (d) is also unbalanced.

Gain of a closed walk: For a quotient Γ-gain graph (G_0, ψ) and a closed walk

$$W = \tilde{v}_1, \tilde{e}_1, \tilde{v}_2, \ldots, \tilde{v}_k, \tilde{e}_k, \tilde{v}_1$$

of (G_0, ψ), the group element $\psi(W) = \Pi_{i=1}^{k} \psi(\tilde{e}_i)^{\mathrm{sign}(\tilde{e}_i)}$, where $\mathrm{sign}(\tilde{e}_i) = 1$ if $\tilde{e}_i$ is directed from $\tilde{v}_i$ to $\tilde{v}_{i+1}$, and $\mathrm{sign}(\tilde{e}_i) = -1$ otherwise.

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Subgroup induced by an edge subset: For a quotient Γ-gain graph (G_0, ψ), a subset $F \subseteq E_0$, and a vertex $\tilde{i}$ of the vertex set $V(F) \subseteq V_0$ induced by F, the subgroup $\langle F \rangle_{\psi,\tilde{i}} = \{\psi(W) | W \in \mathcal{W}(F, \tilde{i})\}$ of Γ, where $\mathcal{W}(F, \tilde{i})$ is the set of closed walks starting at $\tilde{i}$ using only edges of F.

Balanced edge set: A (possibly disconnected) subset of the edge set of a quotient Γ-gain graph (G_0, ψ) with the property that all of its connected components are balanced, where a connected edge subset F of E_0 is balanced if $\langle F \rangle_{\psi,\tilde{i}} = \{\mathrm{id}\}$ for some $\tilde{i} \in V(F)$ (or equivalently, $\langle F \rangle_{\psi,\tilde{i}} = \{\mathrm{id}\}$ for all $\tilde{i} \in V(F)$). A subset of E_0 is called ***unbalanced*** if it is not balanced (that is, if it contains an unbalanced cycle).

Cyclic edge set: A (possibly disconnected) subset of the edge set of a quotient Γ-gain graph (G_0, ψ) with the property that all of its connected components are cyclic, where a connected edge subset F of E_0 is cyclic if $\langle F \rangle_{\psi,\tilde{i}}$ is a cyclic subgroup of Γ for some $\tilde{i} \in V(F)$ (or equivalently, for all $\tilde{i} \in V(F)$).

(k, ℓ, m)***-gain-sparse:*** For non-negative integers k, ℓ, m with $m \leq \ell$, a quotient Γ-gain graph (G_0, ψ) satisfying

$$|F| \leq \begin{cases} k|V(F)| - \ell, & \text{for all non-empty balanced } F \subseteq E_0, \\ k|V(F)| - m, & \text{for all non-empty } F \subseteq E_0. \end{cases}$$

If (G_0, ψ) also satisfies $|E_0| = k|V_0| - m$, then it is called (k, ℓ, m)***-gain-tight***.

For example, the $\mathbb{Z}_2$-gain graphs in Figure 62.1.1 (b) and (d) are $(2, 3, 1)$-gain-tight and $(2, 3, 2)$-gain-tight, respectively.

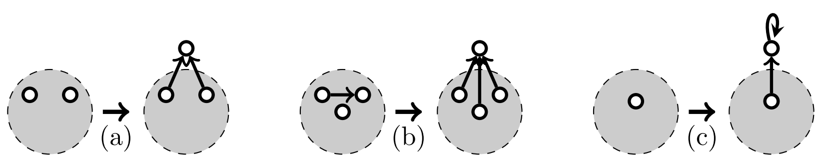


FIGURE 62.1.2

Examples of the three types of Γ-symmetric Henneberg construction moves. The gain labeling is omitted for all edges.

Γ***-symmetric Henneberg construction:*** For a quotient Γ-gain graph (G_0, ψ), a sequence $(H_1, \psi_1), \ldots, (H_n, \psi_n)$ of Γ-gain graphs such that:

(i) For each index $1 < j \leq n$, (H_j, ψ_j) is obtained from (H_{j-1}, ψ_{j-1}) by

vertex addition: attaching a new vertex $\tilde{v}$ by two new non-loop edges $\tilde{e}_1$ and $\tilde{e}_2$. If $\tilde{e}_1$ and $\tilde{e}_2$ are parallel, then $\psi_j(\tilde{e}_1) \neq \psi_j(\tilde{e}_2)$ (assuming that $\tilde{e}_1$ and $\tilde{e}_2$ are directed to $\tilde{v}$ (see Figure 62.1.2(a))).

edge splitting: replacing an edge (possibly a loop) $\tilde{e}$ of (H_{j-1}, ψ_{j-1}) with a new vertex $\tilde{v}$ joined to its end(s) by two new edges $\tilde{e}_1$ and $\tilde{e}_2$, such that the tail of $\tilde{e}_1$ is the tail of $\tilde{e}$ and the tail of $\tilde{e}_2$ is the head of $\tilde{e}$, and $\psi(\tilde{e}_1) \cdot \psi(\tilde{e}_2)^{-1} = \psi(\tilde{e})$, and finally adding a third edge $\tilde{e}_3$ oriented from a vertex $\tilde{z}$ of H_{j-1} to $\tilde{v}$ so that every two-cycle $\tilde{e}_i\tilde{e}_k$, if it exists, is unbalanced in (H_j, ψ_j) (see Figure 62.1.2(b)).

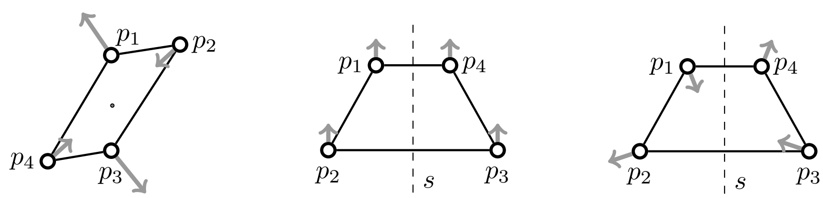
loop extension: attaching a new vertex $\tilde{v}$ to a vertex of H_{j-1} by a new edge with any gain, and adding a new loop $\tilde{l}$ incident to $\tilde{v}$ with $\psi(\tilde{l}) \neq id$ (see Figure 62.1.2(c)).

(ii) (H_1, ψ_1) is one vertex with one unbalanced loop, and $(H_n, \psi_n) = (G_0, \psi)$.

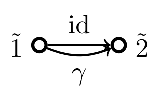
Γ*-symmetric framework:* For a graph $G = (V, E)$, a group action $\theta : \Gamma \to \mathrm{Aut}(G)$, and a homomorphism $\tau : \Gamma \to O(\mathbb{R}^d)$, a framework $G(p)$ (as defined in Chapter 61, with $p : V \to \mathbb{R}^d$ a configuration of points in $\mathbb{R}^d$) satisfying

$$\tau(\gamma)(p_i) = p_{\theta(\gamma)(i)} \qquad \text{for all } \gamma \in \Gamma \text{ and all } i \in V.$$

Symmetry group of a framework: For a Γ-symmetric framework, the group $\tau(\Gamma) = \{\tau(\gamma) \mid \gamma \in \Gamma\}$ of isometries of $\mathbb{R}^d$.

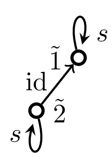


(a) (b) (c)



$$\begin{array}{rc} & \begin{array}{cc} \tilde{1} & \qquad\qquad \tilde{2} \end{array} \\ \begin{array}{r} ((\tilde{1},\tilde{2}),\mathrm{id}) \\ ((\tilde{1},\tilde{2}),\gamma) \end{array} & \begin{pmatrix} (a-c, b-d) & (c-a, d-b) \\ (a+c, b+d) & (c+a, d+b) \end{pmatrix} \end{array}$$

(d)



$$\begin{array}{rc} & \begin{array}{cc} \tilde{1} & \qquad\qquad \tilde{2} \end{array} \\ \begin{array}{r} ((\tilde{1},\tilde{2}),\mathrm{id}) \\ ((\tilde{1},\tilde{1}),s) \\ ((\tilde{2},\tilde{2}),s) \end{array} & \begin{pmatrix} (a-c, b-d) & (c-a, d-b) \\ (4a, 0) & 0\ 0 \\ 0\ 0 & (4c, 0) \end{pmatrix} \end{array}$$

(e)

FIGURE 62.1.3

First-order flexes of frameworks in $\mathbb{R}^2$ with $\mathcal{C}_2 = \{id, \gamma\}$ (half-turn) and $\mathcal{C}_s = \{id, s\}$ (reflection) symmetry: (a) a fully $\mathcal{C}_2$-symmetric non-trivial first-order flex; (b) a fully $\mathcal{C}_s$-symmetric trivial first-order flex; (c) a non-trivial first-order flex which is not fully $\mathcal{C}_s$-symmetric (but "anti-symmetric"); (d) The quotient $\mathbb{Z}_2$-gain graph corresponding to the framework in (a) and its orbit rigidity matrix (with $p_1 = (a, b), p_2 = (c, d), p_3 = (-a, -b)$, and $p_4 = (-c, -d)$); (e) the quotient $\mathbb{Z}_2$-gain graph corresponding to the framework in (b,c) and its orbit rigidity matrix (with $p_1 = (a, b), p_2 = (c, d), p_3 = (-c, d)$, and $p_4 = (-a, b)$).

Orbit rigidity matrix: For a Γ-symmetric framework $G(p)$ (with respect to the free action $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$), where (G_0, ψ) is the quotient Γ-gain graph of G, the $|E_0| \times d|V_0|$ matrix $O(G_0, \psi, p)$ defined as follows. Choose a representative vertex $\tilde{i}$ for each vertex Γi in V_0. The row corresponding to the edge $\tilde{e} = (\tilde{i}, \tilde{j})$, $\tilde{i} \neq \tilde{j}$, with gain $\psi(\tilde{e})$ in E_0 is then given by

$$\begin{pmatrix} 0 \ldots 0 & \overbrace{p(\tilde{i}) - \tau(\psi(\tilde{e}))p(\tilde{j})}^{\tilde{i}} & 0 \ldots 0 & \overbrace{p(\tilde{j}) - \tau(\psi(\tilde{e}))^{-1}p(\tilde{i})}^{\tilde{j}} & 0 \ldots 0 \end{pmatrix}.$$

If $\tilde{e} = (\tilde{i}, \tilde{i})$ is a loop at $\tilde{i}$, then the row corresponding to $\tilde{e}$ is given by

$$\begin{pmatrix} 0 \ldots 0 & \overbrace{2p(\tilde{i}) - \tau(\psi(\tilde{e}))p(\tilde{i}) - \tau(\psi(\tilde{e}))^{-1}p(\tilde{i})}^{\tilde{i}} & 0 \ldots 0 & 0 & 0 \ldots 0 \end{pmatrix}.$$

Fully Γ***-symmetric first-order flex:*** For a Γ-symmetric framework $G(p)$, a first-order flex $p' : V \to \mathbb{R}^d$ of $G(p)$ satisfying

$$\tau(\gamma)p'_i = p'_{\theta(\gamma)(i)} \qquad \text{for all } \gamma \in \Gamma \text{ and all } i \in V.$$

Fully Γ***-symmetric self-stress:*** For a Γ-symmetric framework $G(p)$, a self-stress ω_{ij} satisfying $\omega_e = \omega_f$ for all edges e, f in the same edge orbit Γe.

Forced Γ***-symmetric first-order rigid framework:*** A Γ-symmetric framework for which every fully Γ-symmetric first-order flex is trivial.

Forced Γ***-symmetric isostatic framework:*** A forced Γ-symmetric first-order rigid framework $G(p)$ whose orbit rigidity matrix $O(G_0, \psi, p)$ has independent rows (i.e., $G(p)$ has no fully Γ-symmetric self-stress).

Γ***-regular framework:*** A Γ-symmetric framework $G(p)$ (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$) whose orbit rigidity matrix has maximal rank among all Γ-symmetric frameworks $G(q)$ (with respect to θ and τ).

BASIC RESULTS

A key reason for the interest in forced Γ-symmetric first-order rigidity is that for almost all Γ-symmetric realizations of a given graph as a bar-joint framework, a fully Γ-symmetric first-order flex extends to a *continuous* flex which preserves the symmetry of the framework throughout the path [GF07, Sch10d]. Results on forced Γ-symmetric first-order rigidity therefore provide important tools for detecting hidden continuous flexibility in symmetric frameworks. (See, e.g., Figure 62.1.4.)

THEOREM 62.1.1 Γ-Regular Rigidity Theorem

A Γ*-regular framework* $G(p)$ *has a non-trivial fully* Γ*-symmetric first-order flex if and only if* $G(p)$ *has a non-trivial continuous flex which preserves the symmetry of* $G(p)$ *throughout the path.*

A fundamental tool for studying the forced Γ-symmetric first-order rigidity properties of a framework $G(p)$ is the orbit rigidity matrix [SW11].

THEOREM 62.1.2 The Orbit Rigidity Matrix

Let $G(p)$ *be a* Γ*-symmetric framework (with respect to* $\theta : \Gamma \to \mathrm{Aut}(G)$ *and* $\tau : \Gamma \to O(\mathbb{R}^d)$*). The kernel of the orbit rigidity matrix* $O(G_0, \psi, p)$ *is isomorphic to the space of fully* Γ*-symmetric first-order flexes of* $G(p)$*, and the kernel of* $O(G_0, \psi, p)^T$ *is isomorphic to the space of fully* Γ*-symmetric self-stresses of* $G(p)$*.*

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As an immediate consequence we obtain the following basic result.

THEOREM 62.1.3 Rank of the Orbit Rigidity Matrix

A Γ-symmetric framework (with respect to the free action $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$) with $|V| \geq d$ is forced Γ-symmetric first-order rigid if and only if rank $O(G_0, \psi, p) = d|V_0| - \mathrm{triv}_{\tau(\Gamma)}$, *where* $\mathrm{triv}_{\tau(\Gamma)}$ *is the dimension of the space of fully Γ-symmetric trivial first-order flexes of $G(p)$.*

Note that $\mathrm{triv}_{\tau(\Gamma)}$ can easily be computed for any symmetry group in any dimension. For $d = 2, 3$, $\mathrm{triv}_{\tau(\Gamma)}$ can also be read off directly from the character table of the symmetry group $\tau(\Gamma)$ [AH94, ACP70]. For example, for $d = 3$ and $\Gamma = \mathbb{Z}_2$, we have $\mathrm{triv}_{\tau(\Gamma)} = 2$ if $\tau(\Gamma) = \mathcal{C}_2$ (the velocities generated by a translation along the half-turn axis and a rotation about the axis form a basis), and $\mathrm{triv}_{\tau(\Gamma)} = 3$ if $\tau(\Gamma) = \mathcal{C}_s$ (the velocities generated by two independent translations along the mirror and a rotation about the axis perpendicular to the mirror form a basis).

The following result provides simple necessary counting conditions for a framework to be forced Γ-symmetric isostatic for all symmetry groups in all dimensions [JKT16].

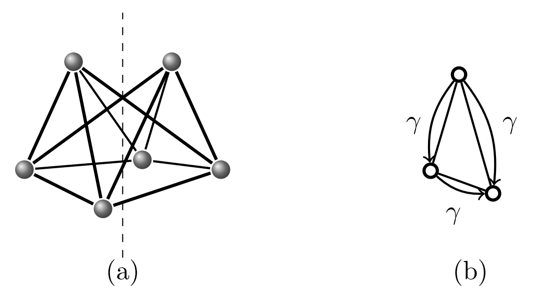


FIGURE 62.1.4

A flexible "Bricard octahedron" with half-turn symmetry (a) and its quotient $\mathbb{Z}_2$-gain graph, where $\mathbb{Z}_2 = \{id, \gamma\}$ (b). (The orientation and gain labeling is omitted for all edges with gain id). While generic realizations of the octahedral graph (without symmetry) are isostatic in 3-space, the symmetry-preserving continuous flexibility of the $\mathbb{Z}_2$-regular realization of this graph shown in (a) is easily detected using Theorems 62.1.1 and 62.1.4, since $|E_0| = 6$, $|V_0| = 3$, and hence $|E_0| = 6 < 7 = 3|V_0| - triv_{\mathcal{C}_2}$.

THEOREM 62.1.4 Necessary Counting Conditions

Let $G(p)$ be a forced Γ-symmetric isostatic framework with respect to the free action $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$, where $|V| \geq d$. Then the quotient Γ-gain graph (G_0, ψ) of G satisfies

(a) $|E_0| = d|V_0| - \mathrm{triv}_{\tau(\Gamma)}$

(b) $|F| \leq d|V(F)| - \mathrm{triv}_{\tau(\langle F\rangle_{\psi,\tilde{i}})}(p(F))$ *for all $F \subseteq E_0$ and all $\tilde{i} \in V(F)$,*

where $\tilde{i} \in V_0$ is identified with its representative vertex, and $\mathrm{triv}_{\tau(\langle F\rangle_{\psi,\tilde{i}})}(p(F))$ *is the dimension of the space of fully $(\langle F\rangle_{\psi,\tilde{i}})$-symmetric trivial first-order flexes of the configuration $p(F) = \{\tau(\gamma)(p(\tilde{i})) \mid \tilde{i} \in V(F), \gamma \in \Gamma\}$.*

For a number of symmetry groups $\tau(\Gamma)$ in the plane, these counts have also been shown to be sufficient for a Γ-regular framework to be forced Γ-symmetric isostatic (see Theorems 62.1.5 and 62.1.6).

Finally, we note that while the orbit rigidity matrix $O(G_0, \psi, p)$ has a particularly simple form if the action $\theta : \Gamma \to \mathrm{Aut}(G)$ is free, it can also be constructed for frameworks $G(p)$, where θ is not free [SW11]. In this case, the counts in Theorem 62.1.4 need to be adjusted accordingly. For example, for half-turn symmetry $\mathcal{C}_2$ in 3-space, the count in Theorem 62.1.4(a) becomes $|E_0| = 3|V_0 \setminus V_0'| + |V_0'| - \mathrm{triv}_{\mathcal{C}_2}$, where V_0' is the set of vertices that are fixed by the half-turn. This is because each vertex in V_0' is in an orbit on its own and has only one degree of freedom, as it must remain on the half-turn axis. These adjustments are straightforward, but they lead to significantly messier gain-sparsity counts in Theorem 62.1.4. While all of the results in this section are expected to extend to frameworks where the action θ is not free, these problems have not yet been fully investigated.

COMBINATORIAL RESULTS

All Γ-regular realizations of G (i.e., *almost all* Γ-symmetric realizations of G) share the same fully Γ-symmetric rigidity properties. Therefore, for Γ-regular frameworks, forced Γ-symmetric first-order rigidity is a purely combinatorial concept, and hence a property of the underlying quotient Γ-gain graph. For forced-symmetric rigidity in the plane, Laman-type theorems have been established for all cyclic groups and all dihedral groups of order $2n$, where n is odd [JKT16, MT11, MT12, MT15].

THEOREM 62.1.5 Reflectional or Rotational Symmetry in the Plane

Let $n \geq 2$, and let $G(p)$ be a $\mathbb{Z}_n$-regular framework with respect to the free action $\theta : \mathbb{Z}_n \to \mathrm{Aut}(G)$ and $\tau : \mathbb{Z}_n \to O(\mathbb{R}^2)$. Then the following are equivalent:

(a) *$G(p)$ is forced $\mathbb{Z}_n$-symmetric isostatic;*

(b) *the quotient $\mathbb{Z}_n$-gain graph (G_0, ψ) of G is $(2, 3, 1)$-gain-tight;*

(c) *(G_0, ψ) has a Γ-symmetric Henneberg construction.*

Note that the count $|E_0| = 2|V_0| - 1$ reflects the fact that $\mathrm{triv}_{\tau(\mathbb{Z}_n)} = 1$ for all cyclic groups $\mathbb{Z}_n$, $n \geq 2$. If $\tau(\mathbb{Z}_n)$ describes rotational symmetry, then a first-order rotation about the origin forms a basis, and if $\tau(\mathbb{Z}_2)$ describes mirror symmetry, then a first-order translation along the mirror line forms a basis (see also Figure 62.1.3 (b)).

Examples of $(2, 3, 1)$-gain-tight $\mathbb{Z}_2$-gain graphs are shown in Figures 62.1.1 (b) and 62.1.3 (e). By Theorem 62.1.5, $\mathbb{Z}_2$-regular realizations of the corresponding "covering graphs" are forced $\mathbb{Z}_2$-symmetric isostatic (but still flexible). See also Figures 62.1.1 (a) (and 62.2.2 (b) with one edge removed) and 62.1.3 (c), respectively.

For the dihedral groups of order $2n$, where n is odd, we have the following result [JKT16].

THEOREM 62.1.6 Dihedral Symmetry in the Plane

Let $G(p)$ be a D_{2n}-regular framework with respect to the free action $\theta : D_{2n} \to \mathrm{Aut}(G)$ and $\tau : D_{2n} \to O(\mathbb{R}^2)$, where $n \geq 3$ is an odd integer, and $\mathcal{C}_{nv} = \tau(D_{2n})$ describes the dihedral symmetry group of order $2n$ in $\mathbb{R}^2$. Then $G(p)$ is forced

D_{2n}-symmetric isostatic if and only if the quotient D_{2n}-gain graph (G_0, ψ) of G satisfies

(a) $|E_0| = 2|V_0|$

(b) $|F| \leq \begin{cases} 2|V(F)| - 3 & \text{for all non-empty balanced } F \subseteq E_0, \\ 2|V(F)| - 1 & \text{for all non-empty unbalanced and cyclic } F \subseteq E_0, \\ 2|V(F)| & \text{for all } F \subseteq E_0. \end{cases}$

Analogous to Theorem 62.1.5, the D_{2n}-gain graphs satisfying the counts in Theorem 62.1.6 can also be characterized via an inductive Henneberg-type construction sequence. However, this construction sequence requires some additional base graphs and some additional gain-graph operations [JKT16].

For the dihedral groups D_{2n}, where n is an even integer, combinatorial characterizations for forced D_{2n}-symmetric rigidity have not yet been obtained. A famous example which shows that the counts in the above theorem are not sufficient for a D_{2n}-regular framework to be forced D_{2n}-symmetric isostatic is the realization of the complete bipartite graph $K_{4,4}$ shown in Figure 62.1.5 [SW11]. (The motion of this framework is also known as Bottema's mechanism in the engineering literature.) See [JKT16] for further examples.

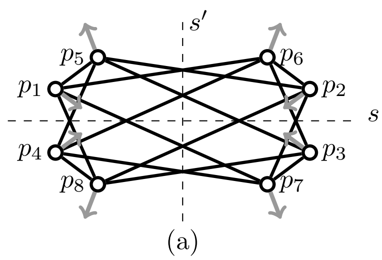


FIGURE 62.1.5
A D_{2n}-regular realization of $K_{4,4}$ and its quotient D_{2n}-gain graph, where $n = 2$. The fully D_{2n}-symmetric first-order flex extends to a symmetry-preserving continuous flex.

Since a combinatorial characterization of isostatic generic bar-joint frameworks (without symmetry) in dimension 3 and higher has not yet been established, there are also no known characterizations of forced Γ-regular isostatic frameworks for any symmetry group in dimension 3 or higher. However, such combinatorial characterizations have been obtained for any symmetry group in any dimension for the special class of "body-bar frameworks" (rigid full-dimensional bodies, connected in pairs by stiff bars) [Tan15]. The underlying combinatorial structure of a body-bar framework is a multigraph whose vertices and edges represent the rigid bodies and stiff bars, respectively. The notions of a Γ-symmetric graph and a Γ-symmetric bar-joint framework can naturally be extended to multigraphs and body-bar frameworks.

THEOREM 62.1.7 Body-Bar Frameworks in d-Space

Let G be a multigraph which is Γ-symmetric with respect to the free action $\theta : \Gamma \to \mathrm{Aut}(G)$. Further, let $\tau : \Gamma \to O(\mathbb{R}^d)$ be a homomorphism. Then all Γ-regular body-bar realizations $G(q)$ of G are forced Γ-symmetric isostatic if and only if the quotient Γ-gain graph (G_0, ψ) of G satisfies

(a) $|E_0| = \binom{d+1}{2}|V_0| - \mathrm{triv}_{\tau(\Gamma)}$

(b) $|F| \leq \binom{d+1}{2}|V(F)| - \mathrm{triv}_{\tau(\langle F\rangle_{\psi,\tilde{\imath}})}(q(F))$ *for all* $F \subseteq E_0$.

Analogous to the non-symmetric situation (recall Section 61.1.3), Γ-regular body-bar, body-hinge, and even molecular realizations of G are conjectured to share the same forced Γ-symmetric rigidity properties [PRS+14]. This has so far only been verified for body-bar and body-hinge frameworks with $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ symmetry, where the group acts freely on the vertices and edges [ST14] (see also Theorem 62.2.8).

CONJECTURE 62.1.8 Symmetric Molecular Conjecture (Version I)

The orbit rigidity matrix of a Γ*-regular body-bar realization of a multigraph* G *has the same rank as the orbit rigidity matrix of a* Γ*-regular molecular realization of* G*.*

This conjecture has important practical applications, as many proteins exhibit non-trivial (rotational) symmetries. The most common ones are half-turn symmetry ($\mathcal{C}_2$) and dihedral symmetry of order 4 and 6 generated by two half-turns ($\mathcal{D}_2$) and a half-turn and a three-fold rotation ($\mathcal{D}_3$), respectively. It turns out these three groups are also the symmetries which give rise to (symmetry-preserving) flexibility in body-bar frameworks which have the minimal number of edges to satisfy the necessary Maxwell count $|E| \geq 6|V| - 6$ for rigidity (under the assumption that the group acts freely on the vertices and edges) [SSW14]. See Table 62.1.1 for details. It is therefore intended to refine the ProFlex/FIRST algorithms for testing protein flexibility based on this conjecture.

Note that there also exists a second version of the Symmetric Molecular Conjecture, which concerns the first-order rigidity of incidentally symmetric frameworks (see Section 62.2).

TABLE 62.1.1 Symmetry-induced flexibility in body-bar frameworks with $\mathcal{C}_2$, $\mathcal{D}_2$ and $\mathcal{D}_3$ symmetry. (We use the standard Schoenflies notation for symmetry groups.)

$\tau(\Gamma)$	$\mathrm{triv}_{\tau(\Gamma)}$	$\lvert E\rvert$	$\lvert E_0\rvert$	$6\lvert V_0\rvert - \mathrm{triv}_{\tau(\Gamma)}$	$6\lvert V_0\rvert - \mathrm{triv}_{\tau(\Gamma)} - \lvert E_0\rvert$
$\mathcal{C}_1$	6	$6\lvert V\rvert - 6$	$6\lvert V_0\rvert - 6$	$6\lvert V_0\rvert - 6$	0
$\mathcal{C}_2$	2	$6\lvert V\rvert - 6$	$6\lvert V_0\rvert - 3$	$6\lvert V_0\rvert - 2$	1
$\mathcal{C}_3$	2	$6\lvert V\rvert - 6$	$6\lvert V_0\rvert - 2$	$6\lvert V_0\rvert - 2$	0
$\mathcal{C}_6$	2	$6\lvert V\rvert - 6$	$6\lvert V_0\rvert - 1$	$6\lvert V_0\rvert - 2$	-1
$\mathcal{D}_2$	0	$6\lvert V\rvert - 4$	$6\lvert V_0\rvert - 1$	$6\lvert V_0\rvert$	1
$\mathcal{D}_3$	0	$6\lvert V\rvert - 6$	$6\lvert V_0\rvert - 1$	$6\lvert V_0\rvert$	1

GEOMETRIC RESULTS

The forced Γ-symmetric rigidity properties of frameworks can also be transferred to other metric spaces, such as the spherical or hyperbolic space, via the technique of coning [SW12]. Recall from Section 61.1.3 that the cone graph of a graph G is the graph $G * u$ obtained from G by adding the new vertex u and the edges $\{u, v_i\}$ for all vertices $v_i \in V$.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

THEOREM 62.1.9 Symmetric Coning

Let $G(p)$ be a framework in $\mathbb{R}^d$, and embed $G(p)$ into the hyperplane $x_{d+1} = 1$ of $\mathbb{R}^{d+1}$ via $\overline{p}_i = (p_i, 1) \in \mathbb{R}^{d+1}$. Further, let $(G * u)(\overline{p}^*)$ be the framework obtained from $G(p)$ by coning G and placing the new cone vertex at the origin of $\mathbb{R}^{d+1}$. Then

(a) *$G(p)$ is Γ-symmetric with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$ if and only if $(G * u)(\overline{p}^*)$ is Γ-symmetric with respect to $\theta^* : \Gamma \to \mathrm{Aut}(G * u)$ defined by $\theta^*(\gamma)|_V = \theta(\gamma)$ and $\theta^*(\gamma)(u) = u$ for all $\gamma \in \Gamma$, and $\tau^* : \Gamma \to O(\mathbb{R}^{d+1})$ defined by $\tau^*(\gamma) = \begin{pmatrix} \tau(\gamma) & 0 \\ 0 & 1 \end{pmatrix}$ for all $\gamma \in \Gamma$.*

(b) *$G(p)$ has a non-trivial fully Γ-symmetric first-order flex (self-stress) in $\mathbb{R}^d$ if and only if $(G * u)(\overline{p}^*)$ has a non-trivial fully Γ-symmetric first-order flex (self-stress) in $\mathbb{R}^{d+1}$.*

(c) *If $(G * u)(q)$ is a Γ-symmetric framework (with respect to θ^* and τ^*) obtained from $(G * u)(\overline{p}^*)$ by moving the vertices of a vertex orbit of G along their corresponding cone rays to $p(u)$ (the origin), then $G(p)$ has a non-trivial fully Γ-symmetric first-order flex (self-stress) if and only if $(G * u)(q)$ has a non-trivial fully Γ-symmetric first-order flex (self-stress).*

As a simple corollary of Theorem 62.1.9 we obtain the following result.

THEOREM 62.1.10 Transfer between Euclidean and Spherical Space

Let q be a configuration of points on the unit sphere $\mathbb{S}^d$ (with no points on the equator) such that the projection $\pi(q)$ of the points from the origin (the center of the sphere) onto the hyperplane $x_{d+1} = 1$ of $\mathbb{R}^{d+1}$ (and then projected back to $\mathbb{R}^d$), is equal to the configuration p. Then $G(p)$ has a non-trivial fully Γ-symmetric first-order flex (self-stress) in $\mathbb{R}^d$ if and only if $G(q)$ has a non-trivial fully Γ-symmetric first-order flex (self-stress) in $\mathbb{S}^d$.

It turns out that we may even transfer *continuous* flexibility between metrics via the technique of symmetric coning.

THEOREM 62.1.11 Transfer of Continuous Symmetry-Preserving Flexes

If $G(p)$ is a Γ-regular framework in $\mathbb{R}^d$, and $G(q)$ is a Γ-symmetric framework in $\mathbb{R}^{d+1}$ such that the projection $\pi(q)$ of q (as defined above) is equal to p, then $G(p)$ has a non-trivial symmetry-preserving continuous flex if and only if $G(q)$ does.

In particular, Theorem 62.1.11 allows us to transfer continuous flexibility between the d-sphere (with no points on the equator) and Euclidean d-space.

The transfer of fully Γ-symmetric first-order (and continuous) rigidity and flexibility properties from Euclidean space to other Cayley-Klein metrics, such as hyperbolic space, is carried out analogously [SW12].

62.2 INCIDENTALLY SYMMETRIC FRAMEWORKS

GLOSSARY

Group representation: For a group Γ and a linear space X, a homomorphism

$\rho : \Gamma \to \mathrm{GL}(X)$. The space X is called the ***representation space*** of ρ. (Note that two representations are considered equivalent if they are similar.)

ρ-invariant subspace: For a representation $\rho : \Gamma \to \mathrm{GL}(X)$, a subspace $U \subseteq X$ satisfying $\rho(\gamma)(U) \subseteq U$ for all $\gamma \in \Gamma$.

Irreducible representation: A group representation $\rho : \Gamma \to \mathrm{GL}(X)$ with the property that X and $\{0\}$ are the only ρ-invariant subspaces of X.

Intertwining map: For two representations ρ_1 and ρ_2 of a group Γ (with respective representation spaces X and Y), a linear map $T : X \to Y$ such that $T\rho_1(\gamma) = \rho_2(\gamma)T$ for all $\gamma \in \Gamma$. The set of all intertwining maps of ρ_1 and ρ_2 forms a linear space which is denoted by $\mathrm{Hom}_\Gamma(\rho_1, \rho_2)$.

Tensor product: For two representations ρ_1 and ρ_2 of a group Γ, the representation $\rho_1 \otimes \rho_2$ defined by $\rho_1 \otimes \rho_2(\gamma) = \rho_1(\gamma) \otimes \rho_2(\gamma)$ for all $\gamma \in \Gamma$.

External representation: For a Γ-symmetric framework $G(p)$ (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$) the representation $\tau \otimes P_V : \Gamma \to \mathbb{R}^{d|V|}$, where $P_V : \Gamma \to \mathrm{GL}(\mathbb{R}^{|V|})$ assigns to $\gamma \in \Gamma$ the permutation matrix of the permutation $\theta(\gamma)$ of V; that is, $P_V(\gamma) = [\delta_{i,\theta(\gamma)(j)}]_{i,j}$, where δ denotes the Kronecker delta.

Internal representation: For a Γ-symmetric graph G (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$), the representation $P_E : \Gamma \to \mathrm{GL}(\mathbb{R}^{|E|})$ which assigns to $\gamma \in \Gamma$ the permutation matrix of the permutation $\theta(\gamma)$ of E.

Fixed vertex: For a Γ-symmetric graph G (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$) and an element $\gamma \in \Gamma$, a vertex i with $\theta(\gamma)(i) = i$. Similarly, an edge $e = \{i, j\}$ of G is fixed by γ if $\theta(\gamma)(e) = e$, i.e., if either $\theta(\gamma)(i) = i$ and $\theta(\gamma)(j) = j$ or $\theta(\gamma)(i) = j$ and $\theta(\gamma)(j) = i$.

Inc-Γ-regular framework: A Γ-symmetric framework $G(p)$ (with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$) whose rigidity matrix has maximal rank among all Γ-symmetric frameworks $G(q)$ (with respect to θ and τ).

Character: For a representation ρ of a group Γ, the row vector $\chi(\rho)$ whose ith component is the trace of $\rho(\gamma_i)$ for some fixed ordering $\gamma_1, \ldots, \gamma_{|\Gamma|}$ of the elements of Γ.

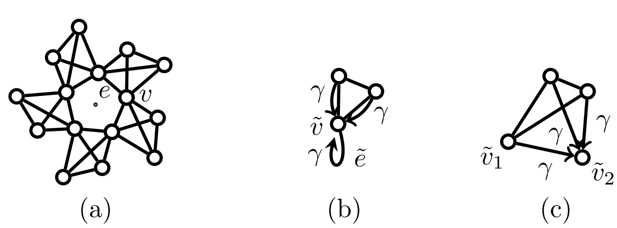


FIGURE 62.2.1

A $\mathbb{Z}_5$-symmetric graph (a) and its corresponding quotient $\mathbb{Z}_5$-gain graph (b) whose edge set is near-balanced. A balanced split is shown in (c). The orientation and gain labeling is omitted for all edges with gain id, and γ denotes rotation by $2\pi/5$.

Near-balanced edge set: For a quotient Γ-gain graph (G_0, ψ), a vertex $\tilde{v}$ of (G_0, ψ), and a partition $\{E_1, E_2, E_{12}\}$ of the edges of (G_0, ψ) incident with $\tilde{v}$, where E_{12} is the set of loops at $\tilde{v}$, a ***split*** of (G_0, ψ) is a quotient Γ-gain graph (G_0', ψ) obtained from (G_0, ψ) by splitting $\tilde{v}$ into two vertices $\tilde{v}_1$ and $\tilde{v}_2$ so that

$\tilde{v}_i$ is incident to the edges in E_i for $i = 1, 2$, and the loops in E_{12} are replaced by directed edges from $\tilde{v}_1$ to $\tilde{v}_2$, without changing any gains. (By the definition of a quotient Γ-gain graph, the gain of a loop is freely invertible, so we may choose the original gain or its inverse for any edge replacing a loop.) A connected subset F of (G_0, ψ) is near-balanced if it is unbalanced and there is a split of (G_0, ψ) in which F becomes a balanced set. See also Figure 62.2.1.

SYMMETRY-ADAPTED COUNTING RULES

Using methods from group representation theory, the rigidity matrix of a Γ-symmetric framework $G(p)$ can be transformed into a block-diagonalized form [KG00, Sch09, Sch10a]. This is a fundamental result, as it can be used to break up the rigidity analysis of a symmetric framework into a number of independent subproblems, one for each block of the rigidity matrix. The block-diagonalization of the rigidity matrix is obtained by showing that it intertwines two representations of the group Γ associated with the edges and vertices of the graph G (also known as the external and internal representation in the engineering community).

THEOREM 62.2.1 Intertwining Property of the Rigidity Matrix

Let $G(p)$ be a Γ-symmetric framework with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$. Then the rigidity matrix of $G(p)$, $R_G(p)$, lies in $\mathrm{Hom}_\Gamma(\tau \otimes P_V, P_E)$.

By Theorem 62.2.1 and Schur's lemma, there exist invertible matrices S and T such that $T^\top R_G(p)S$ is block-diagonalized. More precisely, if $\rho_0, \dots, \rho_r$ are the irreducible representations of Γ, then for an appropriate choice of symmetry-adapted bases, the rigidity matrix takes on the following block form

$$T^\top R_G(p)S := \widetilde{R}_G(p) = \begin{pmatrix} \widetilde{R}_0(G(p)) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \widetilde{R}_r(G(p)) \end{pmatrix},$$

where the submatrix block $\widetilde{R}_i(G(p))$ corresponds to the irreducible representation ρ_i of Γ. This block-diagonalization of the rigidity matrix corresponds to a decomposition $\mathbb{R}^{d|V|} = X_0 \oplus \cdots \oplus X_r$ of the space $\mathbb{R}^{d|V|}$ into a direct sum of $(\tau \otimes P_V)$-invariant subspaces X_i, and a decomposition $\mathbb{R}^{|E|} = Y_0 \oplus \cdots \oplus Y_r$ of the space $\mathbb{R}^{|E|}$ into a direct sum of P_E-invariant subspaces Y_i. The spaces X_i and Y_i are associated with ρ_i, and the submatrix $\widetilde{R}_i(G(p))$ is of size $\dim (Y_i) \times \dim (X_i)$.

Note that the submatrix block $\widetilde{R}_0(G(p))$ which corresponds to the trivial irreducible representation ρ_0 (with $\rho_0(\gamma) = 1$ for all $\gamma \in \Gamma$) is equivalent to the orbit rigidity matrix discussed in the previous section. The entries of the orbit rigidity matrix can be written down explicitly (see Section 62.1) without using any methods from group representation theory.

THEOREM 62.2.2 $(\tau \otimes P_V)$-Invariance of the Trivial Flex Space

Let $G(p)$ be a Γ-symmetric framework with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$. Then the space of trivial first-order flexes $\mathcal{T}(G, p)$ of $G(p)$ is a $(\tau \otimes P_V)$-invariant subspace of $\mathbb{R}^{d|V|}$.

We denote by $(\tau\otimes P_V)^{(\mathcal{T})}$ the subrepresentation of $\tau\otimes P_V$ with representation space $\mathcal{T}(G,p)$. The space $\mathcal{T}(G,p)$ may now also be written as a direct sum $\mathcal{T}=T_0\oplus\cdots\oplus T_r$ of $(\tau\otimes P_V)$-invariant subspaces, and for each $i=1,\ldots,r$, we obtain the necessary condition dim $(Y_i)=$ dim $(X_i)-$ dim (T_i) for a Γ-symmetric framework to be isostatic. Using basic results from character theory, these conditions can be written in a more succinct form as follows [FG00, OP10, Sch09, Sch10a].

THEOREM 62.2.3 Symmetry-Extended Necessary Counting Conditions

Let $G(p)$ be an isostatic framework which is Γ-symmetric with respect to θ and τ. Then the following character equation holds.

$$\chi(P_E)=\chi(\tau\otimes P_V)-\chi((\tau\otimes P_V)^{(\mathcal{T})}).$$

It is well known from group representation theory that the character of any representation of Γ can be written uniquely as the linear combination of the characters of the irreducible representations of Γ. So suppose that $\chi(P_E)=\alpha_0\chi(\rho_0)+\cdots+\alpha_r\chi(\rho_r)$, and $\chi(\tau\otimes P_V)-\chi((\tau\otimes P_V)^{(\mathcal{T})})=\beta_0\chi(\rho_0)+\cdots+\beta_r\chi(\rho_r)$, where $\alpha_i\,\beta_i\in\mathbb{N}\cup\{0\}$ for all $i=0,\ldots,r$. If a Γ-symmetric framework $G(p)$ is not isostatic, then it follows from Theorem 62.2.3 that $\alpha_i\neq\beta_i$ for some i. If $\alpha_i<\beta_i$, then $G(p)$ has a non-trivial "ρ_i-symmetric" first-order flex belonging to the space X_i, and if $\alpha_i>\beta_i$, then $G(p)$ has a non-zero "ρ_i-symmetric" self-stress belonging to the space Y_i.

Consider, for example, the framework $G(p)$ with half-turn symmetry $\mathcal{C}_2$ in Figure 62.2.2 (b). We have $\chi(P_E)=(9,3)$, $\chi(P_V)=(6,0)$, $\chi(\tau)=(2,-2)$, and $\chi(\tau\otimes P_V)=(12,0)$. Moreover, $\chi((\tau\otimes P_V)^{(\mathcal{T})})=(3,-1)$ (see Table 62.2.1). Thus,

$$\chi(P_E)=(9,3)\neq(9,1)=(12,0)-(3,-1)=\chi(\tau\otimes P_V)-\chi((\tau\otimes P_V)^{(\mathcal{T})}),$$

and hence, by Theorem 62.2.3, $G(p)$ is not isostatic. Let ρ_0 be the trivial ("fully-symmetric") irreducible representation of $\mathcal{C}_2$ (which assigns 1 to both the identity and the half-turn), and let ρ_1 be the non-trivial ("anti-symmetric") irreducible representation of $\mathcal{C}_2$ (which assigns 1 to the identity and -1 to the half-turn). Then we have $(9,3)=6\rho_0+3\rho_1$ and $(9,1)=5\rho_0+4\rho_1$, and hence we may conclude that $G(p)$ has an anti-symmetric first-order flex and a fully-symmetric self-stress.

(a) (b)

FIGURE 62.2.2

Realizations of the triangular prism graph with half-turn symmetry in the plane. The framework in (a) is isostatic, whereas the framework in (b) is first-order flexible, as detected by Theorem 62.2.3.

The equation in Theorem 62.2.3 comprises of one equation for each $\gamma\in\Gamma$. If we consider each of these equations independently, then we may obtain very simple necessary conditions for a Γ-symmetric framework $G(p)$ to be isostatic in terms of the number of vertices and edges of G that are fixed by the elements of Γ [CFG+09].

THEOREM 62.2.4 Conditions for Individual Group Elements

Let $G(p)$ be an isostatic framework which is Γ-symmetric with respect to θ and τ, and let $|V_\gamma|$ and $|E_\gamma|$ denote the number of vertices and edges of G that are fixed by γ, respectively. Then, for every $\gamma \in \Gamma$, we have

$$|E_\gamma| = \mathrm{trace}(\tau(\gamma)) \cdot |V_\gamma| - \mathrm{trace}((\tau \otimes P_V)^{(\mathcal{T})}(\gamma)).$$

By considering standard bases for the spaces of first-order translations and rotations, the numbers $\mathrm{trace}((\tau \otimes P_V)^{(\mathcal{T})}(\gamma))$, $\gamma \in \Gamma$, can easily be computed for any symmetry group $\tau(\Gamma)$ [Sch09]. The calculations of characters for the symmetry-extended counting rule for isostatic frameworks in the plane, for example, are shown in Table 62.2.1. In this table we again use the Schoenflies notation for symmetric structures; in particular, the symbols s and C_n denote a reflection and a rotation by $2\pi/n$, respectively.

TABLE 62.2.1 Calculations of characters for the symmetry-extended counting rule for isostatic frameworks in the plane.

	Id	C_n, $n>2$	C_2	s
$\chi(P_E)$	$\|E\|$	$\|E_{C_n}\|$	$\|E_{C_2}\|$	$\|E_s\|$
$\chi(\tau \otimes P_V)$	$2\|V\|$	$(2\cos\frac{2\pi}{n})\|V_{C_n}\|$	$-2\|V_{C_2}\|$	0
$\chi((\tau \otimes P_V)^{(\mathcal{T})})$	3	$2\cos\frac{2\pi}{n}+1$	-1	-1

For the identity element of Γ, Theorem 62.2.4 simply recovers the standard non-symmetric count. For the non-trivial symmetry operations, however, we obtain additional necessary conditions. In particular, for isostatic frameworks in the plane, we have the following result.

THEOREM 62.2.5 Restrictions on Fixed Structural Elements in the Plane

Let $G(p)$ be an isostatic framework which is Γ-symmetric with respect to θ and τ. Then the following hold.

(a) *If $C_2 \in \tau(\Gamma)$, then $|V_{C_2}| = 0$ and $|E_{C_2}| = 1$;*

(b) *if $C_3 \in \tau(\Gamma)$, then $|V_{C_3}| = 0$;*

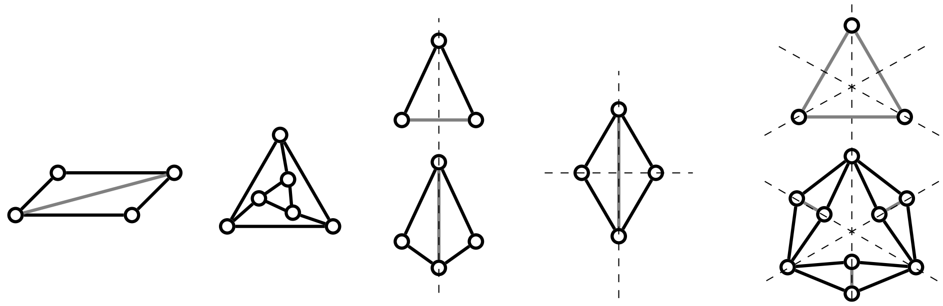
(c) *if $s \in \tau(\Gamma)$, then $|E_s| = 1$;*

(d) *there does not exist a rotation $C_n \in \tau(\Gamma)$ with $n > 3$.*

By Theorem 62.2.5 there are only 5 non-trivial symmetry groups which allow an isostatic framework in the plane, namely the rotational groups $\mathcal{C}_2$ and $\mathcal{C}_3$, the reflectional group $\mathcal{C}_s$, and the dihedral groups $\mathcal{C}_{2v}$ and $\mathcal{C}_{3v}$ of order 4 and 6.

For 3-dimensional frameworks, all symmetry groups are possible. In fact, there exist infinite families of triangulated convex polyhedra for every symmetry group in 3-space, and the 1-skeleta of these structures are isostatic by Cauchy-Dehn's rigidity theorem (recall Chapter 61). However, restrictions to the placement of structural components still apply.

Analogous symmetry-extended counting rules have also been established for various other types of geometric constraint systems, such as body-bar frameworks



($\mathcal{C}_2$) ($\mathcal{C}_3$) ($\mathcal{C}_s$) ($\mathcal{C}_{2v}$) ($\mathcal{C}_{3v}$)

FIGURE 62.2.3
Examples of symmetric isostatic frameworks in the plane, where fixed edges are shown in gray colour.

[GSW10], body-hinge frameworks [GF05, SFG14], and infinite periodic frameworks [GF14], as well as for frameworks in non-Euclidean normed spaces [KiS15]. Moreover, similar methods have recently also been applied to analyze liftings of symmetric pictures to polyhedral scenes [KaS17].

COMBINATORIAL RESULTS

It is easy to see that the set of all inc-Γ-regular realizations of a graph G (as a bar-joint framework) forms a dense open subset of all Γ-symmetric realizations of G, and that all inc-Γ-regular realizations share the same first-order rigidity properties. It was conjectured in [CFG$^+$09] that for the five non-trivial symmetry groups which allow isostatic frameworks in the plane, the standard Laman counts, together with the additional conditions in Theorem 62.2.4, are also sufficient for inc-Γ-regular realizations of G to be isostatic. This conjecture has been proved for the groups $\mathcal{C}_2$, $\mathcal{C}_3$, and $\mathcal{C}_s$ [Sch10b, Sch10c], but it remains open for the dihedral groups.

THEOREM 62.2.6 Symmetric Laman's Theorem

Let $G(p)$ be an inc-Γ-regular framework with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^2)$. Then $G(p)$ is isostatic if and only if $|E| = 2|V| - 3$ and for every subgraph (V', E') with $|V'| \geq 2$ vertices, $|E'| \leq 2|V'| - 3$ (Laman's conditions), and

(a) *for $\tau(\Gamma) = \mathcal{C}_2$, we have $|V_{C_2}| = 0$ and $|E_{C_2}| = 1$;*

(b) *for $\tau(\Gamma) = \mathcal{C}_3$, we have $|V_{C_3}| = 0$;*

(c) *for $\tau(\Gamma) = \mathcal{C}_s$, we have $|E_s| = 1$.*

There also exist alternative characterizations for inc-Γ-regular isostaticity for these three groups. These are given in terms of symmetric Henneberg-type inductive construction sequences and in terms of symmetric 3Tree2 partitions (recall Section 61.1.2). See [Sch10b, Sch10c] for details.

Since an isostatic Γ-symmetric framework must obey certain restrictions on the number of vertices and edges that are fixed by the various elements in Γ, a first-order rigid Γ-symmetric framework usually does not contain an isostatic Γ-symmetric subframework on the same vertex set (see the frameworks in Figure 62.2.4, for example). Consequently, Theorem 62.2.6 can in general not be used to decide whether a given inc-Γ-regular framework $G(p)$ is first-order rigid.

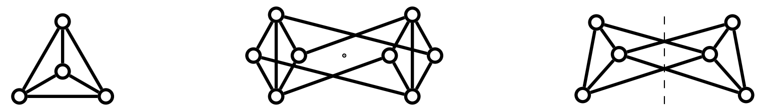


FIGURE 62.2.4
First-order rigid Γ-symmetric frameworks in $\mathbb{R}^2$ with respective symmetry groups $\mathcal{C}_3$, $\mathcal{C}_2$, and $\mathcal{C}_s$, which do not contain a Γ-symmetric isostatic subframework on the same vertex set.

However, this problem may be solved by analyzing each of the block matrices of the block-decomposed rigidity matrix $\widetilde{R}_G(p)$. Clearly, if for every $i = 0, \dots, r$, $G(p)$ does not have any ρ_i-symmetric non-trivial first-order flex, then the block-diagonalization of $\widetilde{R}_G(p)$ and the corresponding decomposition of the infinitesimal flex space guarantees that $G(p)$ is first-order rigid. To make each of the block-matrices $\widetilde{R}_i(G(p))$ combinatorially accessible, an equivalent ρ_i-symmetric orbit rigidity matrix has recently been described in explicit form for each i [ST15]. This approach has provided complete combinatorial characterizations of first-order rigid inc-Γ-regular frameworks for a variety of groups, both in the plane and in higher dimensions. These results are important, because symmetric first-order rigid (rather than isostatic) frameworks are ubiquitous in human designs, and in some natural settings, such as proteins.

THEOREM 62.2.7 Inc-$\mathbb{Z}_2$-Regular First-Order Rigidity in the Plane

Let $G(p)$ be an inc-$\mathbb{Z}_2$-regular framework with respect to the free action $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ and $\tau : \mathbb{Z}_2 \to O(\mathbb{R}^2)$, where $\tau(\mathbb{Z}_2) = \mathcal{C}_2$ or $\mathcal{C}_s$. Then $G(p)$ is first-order rigid if and only if the quotient Γ-gain graph of G contains a spanning $(2, 3, i)$-gain-tight subgraph (H_i, ψ_i) for each $i = 1, 2$.

For the group $\mathbb{Z}_3$, first-order rigid inc-$\mathbb{Z}_3$-regular frameworks in the plane (where the action $\theta : \mathbb{Z}_3 \to \mathrm{Aut}(G)$ is free) can be characterized in terms of spanning isostatic $\mathbb{Z}_3$-symmetric subframeworks using Theorem 62.2.6(c).

For any cyclic group $\mathbb{Z}_k$ of order $k \geq 4$, there always exists a non-trivial irreducible representation ρ_i for which there does not exist any ρ_i-symmetric trivial first-order flex. Moreover, the ρ_0-symmetric trivial first-order flex space is of dimension 1 for all of these groups (recall Theorem 62.1.5). Therefore, for $k \geq 4$, the quotient $\mathbb{Z}_k$-gain graph (G_0, ψ) of a $\mathbb{Z}_k$-symmetric first-order rigid framework $G(p)$ in the plane (where $\theta : \mathbb{Z}_k \to \mathrm{Aut}(G)$ is free) must contain a spanning $(2, 3, 0)$-gain-tight subgraph and a spanning $(2, 3, 1)$-gain-tight subgraph. Further necessary conditions were established in [Ike15, IT15, IT13, ST15]. In particular, the edge set of the $(2, 3, 0)$-gain-tight subgraph cannot be near-balanced (see Figure 62.2.1).

It was also shown in [Ike15, IT13] that for an inc-$\mathbb{Z}_k$-regular framework $G(p)$, where $k \geq 6$, the existence of a spanning generically 2-isostatic subgraph is not sufficient for $G(p)$ to be first-order rigid (see Figure 62.2.5). However, if for the graph G, we have $|E| = 2|V|$, and $k \geq 5$ is a prime number less than 1000, then $G(p)$ is in fact first-order rigid if and only if G contains a spanning generically 2-

isostatic subgraph. For complete combinatorial characterizations of inc-$\mathbb{Z}_k$-regular first-order rigid frameworks in the plane, where k is odd, see [Ike15, IT15, IT13].

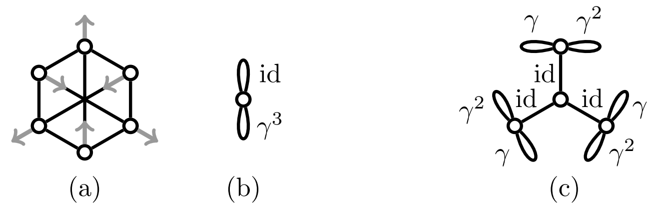


FIGURE 62.2.5
$\mathbb{Z}_6$-symmetric realizations of the generically 2-isostatic graph $K_{3,3}$ are first-order flexible (a); the quotient $\mathbb{Z}_6$-gain graph of $K_{3,3}$ is shown in (b), where γ denotes rotation by $2\pi/6$. (c) A quotient $\mathbb{Z}_k$-gain graph with $\mathbb{Z}_k = \langle\gamma\rangle$ whose covering graph is generically rigid without symmetry, but becomes first-order flexible for $\mathbb{Z}_k$-symmetric realizations, where $k \geq 7$. The directions of edges are omitted in the gain graphs.

For the remaining symmetry groups $\tau(\Gamma)$ in the plane, analogous combinatorial descriptions of inc-Γ-regular first-order rigid bar-joint frameworks have not yet been established. Due to the well-known difficulties for the non-symmetric case, there are also no extensions of these results to bar-joint frameworks in dimension $d \geq 3$. However, combinatorial characterizations for inc-Γ-regular first-order rigidity have been obtained for *body-bar* and *body-hinge* frameworks in arbitrary dimension for the groups $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ [ST14]. These characterizations are given in terms of gain-sparsity counts as well as in terms of subgraph packing conditions for the corresponding quotient gain graphs. Since the statements of these theorems are fairly complex, we only provide a sample result for body-hinge frameworks in 3-space with half-turn symmetry, the most common symmetry found in proteins.

THEOREM 62.2.8 Inc-$\mathbb{Z}_2$-Regular Body-Hinge Frameworks in 3-Space

Let $G(h)$ be an inc-$\mathbb{Z}_2$-regular body-hinge framework in $\mathbb{R}^3$ with respect to the action $\theta : \mathbb{Z}_2 \to \mathrm{Aut}(G)$ which is free on the vertex set and the edge set of the multigraph G, and $\tau : \mathbb{Z}_2 \to O(\mathbb{R}^3)$, where $\tau(\mathbb{Z}_2) = \mathcal{C}_2$. Then $G(h)$ is first-order rigid if and only if the quotient $\mathbb{Z}_2$-gain graph of G contains six edge-disjoint subgraphs, two of which are spanning trees, and each of the other four has the property that each connected component contains exactly one cycle, which is unbalanced.

The results in [ST14] suggest that inc-Γ-regular body-bar and body-hinge realizations of the same multigraph share the same first-order rigidity properties. Moreover, it is conjectured that the following stronger version of Conjecture 62.1.8 for incidentally symmetric structures also holds [PRS+14].

CONJECTURE 62.2.9 Symmetric Molecular Conjecture (Version II)

The rigidity matrix of an inc-Γ-regular body-bar realization of a multigraph G has the same rank as the rigidity matrix of an inc-Γ-regular molecular realization of G.

GEOMETRIC RESULTS

It follows immediately from Theorem 61.1.20 (Coning Theorem) that a Γ-symmetric framework $G(p)$ with respect to $\theta : \Gamma \to \mathrm{Aut}(G)$ and $\tau : \Gamma \to O(\mathbb{R}^d)$ is first-order rigid if and only if the Γ-symmetric cone framework $(G * u)(\overline{p}^*)$ with respect to $\theta^* : \Gamma \to \mathrm{Aut}(G * u)$ and $\tau^* : \Gamma \to O(\mathbb{R}^{d+1})$ (as defined in Section 62.1) is first-order rigid.

In particular, this allows us to transfer first-order rigidity and flexibility of incidentally symmetric frameworks between Euclidean, spherical, and hyperbolic space. Note, however, that for a nontrivial group Γ, a nontrivial first-order flex of an inc-Γ-regular framework usually does not extend to a continuous flex.

ALGORITHMS

Using simple modifications of the algorithms presented in Chapter 61, we can check whether inc-Γ-regular realizations of a graph in $\mathbb{R}^2$ are isostatic in polynomial time. For example, we may check Laman's conditions in $O(|V|^2)$ time and then check the additional symmetry conditions on the number of fixed structural components (see Theorem 62.2.6) in constant time. Alternatively, we may use an $O(|V|^2)$ matroid partition algorithm to check whether there exists a proper symmetric 3Tree2 partition.

To verify whether a Γ-symmetric graph is incidentally symmetric first-order rigid (or just forced Γ-symmetric rigid), we need to check the corresponding (k, ℓ, m)-gain-sparsity counts for its quotient Γ-gain graph (recall Theorems 62.1.5, 62.1.6 and 62.2.7, for example). This can be done in polynomial time as follows, provided that $0 \leq \ell \leq 2k - 1$ (see also [BHM$^+$11, JKT16]).

In the first step, we may verify that the gain graph is $(k, 0)$-sparse (or (k, m)-sparse if $m > 0$) using a standard "pebble game algorithm" [BJ03, LS08]. In the second step, we then need to test whether every edge set violating the (k, ℓ)-sparsity count induces a certain subgroup of Γ (or whether it is near-balanced, if necessary). It suffices to test this for every circuit in the matroid induced by the (k, ℓ)-sparsity count, and these circuits can be enumerated in polynomial time (see [Sey94] e.g.).

As we will see in the next section (Section 62.3), this algorithm may also be used to decide the rigidity of infinite periodic frameworks.

Recall that for regular (finite) bar-joint frameworks in dimension $d \geq 3$, we do not have a polynomial time algorithm to test for rigidity. However, polynomial time algorithms do exist for deciding the rigidity of regular *body-bar* or *body-hinge* frameworks in d-space. The same is true in the symmetric situation, where we can check the conditions in Theorem 62.2.8, for example, in $O(|V_0|^{5/2}|E_0|)$ time via a matroid partition algorithm [ST14].

Finally, we note that for a given Γ-symmetric framework $G(p)$, there exists a randomized polynomial time algorithm to check whether $G(p)$ is Γ-regular or inc-Γ-regular.

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62.3 PERIODIC FRAMEWORKS

GLOSSARY

Periodic graph: For a free abelian group Γ of rank d, a simple infinite graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with finite degree at every vertex for which there exists a group action $\theta : \Gamma \to \mathrm{Aut}(\tilde{G})$ which is free on the vertex set of $\tilde{G}$ and such that the quotient graph $\tilde{G}/\Gamma$ is finite. (Note that Γ is isomorphic to $\mathbb{Z}^d$.)

L-periodic framework: For a periodic graph $\tilde{G}$ (with respect to $\theta : \Gamma \to \mathrm{Aut}(\tilde{G})$), the pair $(\tilde{G}, \tilde{p})$, also denoted by $\tilde{G}(\tilde{p})$, where $\tilde{p} : \tilde{V} \to \mathbb{R}^d$ is a map, $\mathcal{T}(\mathbb{R}^d)$ is the group of translations of $\mathbb{R}^d$ (which is identified with the space $\mathbb{R}^d$ of translation vectors), and $L : \Gamma \to \mathcal{T}(\mathbb{R}^d)$ is a faithful representation with the property that

$$\tilde{p}_i + L(\gamma) = \tilde{p}_{\theta(\gamma)(i)} \qquad \text{for all } \gamma \in \Gamma \text{ and all } i \in \tilde{V}.$$

Periodic first-order flex: Fix an isomorphism $\Gamma \simeq \mathbb{Z}^d$, and let $\tilde{G}(\tilde{p})$ be an L-periodic framework (with respect to $\theta : \Gamma \to \mathrm{Aut}(\tilde{G})$). Choose a set of representatives, $v_1, \dots, v_a$, for the vertex orbits of $\tilde{G}$, and a set of representatives, $(v_i, \theta(\gamma_\beta)v_j)$, for the b edge orbits of $\tilde{G}$. Further, let $x_i = \tilde{p}(v_i)$ for each i, and let $\mu_k = L(\gamma_k)$, $k = 1, \dots, d$, be the translation vectors in $\mathbb{R}^d$ which correspond to the standard basis $\gamma_1, \dots, \gamma_d$ of Γ. Let $\gamma_\beta = \sum_{k=1}^d c_\beta^k \gamma_k$ for $c_\beta^k \in \mathbb{Z}$, and $\mu(\beta) = \sum_{k=1}^d c_\beta^k \mu_k$. A vector $(y_1, \dots, y_a, \nu_1, \dots, \nu_d) \in \mathbb{R}^{da+d^2}$ is a periodic first-order flex of $\tilde{G}(\tilde{p})$ if

$$\langle (x_j + \mu(\beta)) - x_i, (y_j + \nu(\beta)) - y_i \rangle = 0 \quad \text{for } \beta = 1, \dots, b,$$

where $\langle \cdot, \cdot \rangle$ represents the inner product, and $\nu(\beta) = \sum_{k=1}^d c_\beta^k \nu_k$. We refer to the matrix corresponding to the linear system above as the ***periodic rigidity matrix*** of $\tilde{G}(\tilde{p})$.

First-order rigid periodic framework: An L-periodic framework $\tilde{G}(\tilde{p})$ for which every periodic first-order flex is trivial. A first-order rigid periodic framework with no redundant constraints is called ***isostatic***.

$\mathbb{Z}^d$-regular: An L-periodic framework $\tilde{G}(\tilde{p})$ in $\mathbb{R}^d$ whose periodic rigidity matrix has maximal rank among all L'-periodic realizations $\tilde{G}(\tilde{p}')$ of $\tilde{G}$ in $\mathbb{R}^d$ (with any choice of L').

BASIC RESULTS

Rigidity analyses of infinite periodic frameworks have found numerous applications in both mathematics (in the theory of periodic packings, e.g.) and in other scientific areas such as crystallography, materials science, and engineering. Most of the theoretical work has focused on *forced-periodic* frameworks, i.e., periodic frameworks which must maintain the periodicity throughout their motions. In the following, we will therefore restrict our attention to this setting. However, the rigidity and

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flexibility of infinite structures is currently a highly active research area and new tools and methods for more general rigidity analyses of such structures are developing quickly. In particular, new insights into the rigidity and flexibility properties of periodic or crystallographic frameworks have recently been gained via novel applications of methods from real and functional analysis [OP11, Pow14a, Pow14b].

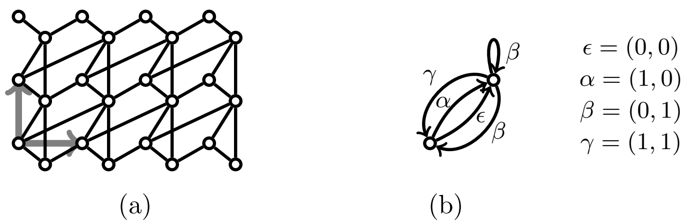

FIGURE 62.3.1
Part of a first-order rigid periodic framework in the plane (a) and its quotient $\mathbb{Z}^2$-gain graph (b).

The description of the first-order rigidity of periodic frameworks provided in the previous section is based on the approach taken in [BS10] (see also [GH03, Pow14a]). An alternative mathematical formulation is given in [Ros11, Ros14a]. The main difference between the two models is that in [BS10] a periodic framework is considered as a realization of an infinite graph with a periodic group action, whereas in [Ros11, Ros14a] it is considered as a realization of a finite graph within a fundamental domain of $\mathbb{R}^d$. Since in the latter model, the orientation of the periodic lattice is fixed, rotations have been eliminated from the space of trivial motions. In other words, realizations of finite graphs in a fundamental domain constitute equivalence classes of L-periodic frameworks $\tilde{G}(\tilde{p})$ under rotation.

When studying the first-order rigidity of periodic frameworks, we may allow different types of flexibility in the lattice representation of the periodicity group. The definitions provided in the previous section assume that the lattice representation is fully flexible. While this seems to be the most natural set-up, motions of periodic structures with a fixed lattice representation or with various types of partially flexible lattice representations are also of interest in applications. In particular, these results are useful to analyze motions of structures at shorter time scales.

The following basic result is the analog of Theorem 62.1.1. While it is stated for periodic frameworks with a fully flexible lattice representation, it also holds for other types of lattice flexibility, including the fixed lattice [MT13, Ros11].

THEOREM 62.3.1 $\mathbb{Z}^d$-Regular Rigidity Theorem

A $\mathbb{Z}^d$-regular L-periodic framework $\tilde{G}(\tilde{p})$ has a non-trivial periodic first-order flex if and only if $\tilde{G}(\tilde{p})$ has a non-trivial continuous flex which preserves the periodicity throughout the motion.

Using the periodic rigidity matrix (and its counterparts for the fixed or the partially flexible lattice representations), we may immediately derive some basic necessary counting conditions for a periodic framework in d-space to be first-order rigid. The following theorem states these conditions for the fully flexible and the fixed lattice representation.

THEOREM 62.3.2 Necessary Counting Conditions for Periodic Rigidity

Let $\tilde{G}(\tilde{p})$ be an L-periodic isostatic framework in d-space. Then the quotient graph $\tilde{G}_0$ of $\tilde{G}$ satisfies

(a) *for $L(\mathbb{Z}^d)$ fully flexible: $|\tilde{E}_0| = d|\tilde{V}_0| + \binom{d}{2}$;*

(b) *for $L(\mathbb{Z}^d)$ fixed: $|\tilde{E}_0| = d|\tilde{V}_0| - d$.*

TABLE 62.3.1 Types of lattice deformation in 2- and 3-space.

Lattice Deformation	l (for d=2)	l (for d=3)
flexible	3	6
distortional	2	5
scaling	2	3
hydrostatic	1	1
fixed	0	0

More generally, for an L-periodic framework $\tilde{G}(\tilde{p})$ in d-space to be isostatic, its quotient graph must satisfy the count $|\tilde{E}_0| = d|\tilde{V}_0| + l - d$, where l is the dimension of the space of permissible lattice deformations. Table 62.3.1 summarizes these counts for some fundamental types of lattice deformations in dimensions 2 and 3. A "distortional change" in the lattice keeps the volume of the fundamental domain fixed but allows the shape of the lattice to change, a "scaling change" keeps the angles of the fundamental domain fixed but allows the scale of the translations to change independently, and finally, a "hydrostatic change" keeps the shape of the lattice unchanged but allows scalings to change the volume.

Analogous counts for various types of "crystallographic frameworks" (i.e., periodic frameworks with additional symmetry) can be found in [RSW11]. Note that in addition to these overall counts we may also derive further necessary counting conditions for first-order rigidity by considering all edge-induced subgraphs of the given quotient graph. However, as in the case of finite symmetric frameworks, these gain-sparsity counts are more complex, as we will see in the next section.

COMBINATORIAL RESULTS

For a given periodicity group Γ, combinatorial characterizations of first-order rigid periodic frameworks have been studied at multiple different levels. At the simplest level, we may ask whether a given quotient graph $\tilde{G}/\Gamma$ is the quotient graph of a first-order rigid periodic framework for *some* gain assignment of the edges of $\tilde{G}/\Gamma$. In other words, we seek a characterization for first-order rigidity up to generic liftings of edges from $\tilde{G}/\Gamma$ to a covering periodic graph. The following theorem summarizes the key results for this problem in the case where the lattice representation of Γ is either fully flexible or fixed [BS11, Whi88].

THEOREM 62.3.3 Periodic Rigidity for Generic Liftings

A quotient graph $\tilde{G}/\Gamma$ is the quotient graph of an isostatic L-periodic framework $\tilde{G}(\tilde{p})$ in d-space for some gain assignment of the edges of $\tilde{G}/\Gamma$ if and only if

(a) *for $L(\mathbb{Z}^d)$ fully flexible:* $\tilde{G}/\Gamma$ *satisfies* $|\tilde{E}_0| = d|\tilde{V}_0| + \binom{d}{2}$ *and contains a spanning subgraph with* $d|\tilde{V}_0| - d$ *edges which has the property that every subgraph with* m *edges and* n *vertices satisfies* $m \leq dn - d$.

(b) *for $L(\mathbb{Z}^d)$ fixed:* $\tilde{G}/\Gamma$ *satisfies* $|\tilde{E}_0| = d|\tilde{V}_0| - d$ *and every subgraph of* $\tilde{G}/\Gamma$ *with* m *edges and* n *vertices satisfies* $m \leq dn - d$.

Note that the condition in Theorem 62.3.3(b) is equivalent to the condition that $\tilde{G}/\Gamma$ is the union of d edge-disjoint spanning trees.

For the fully flexible lattice, Theorem 62.3.3(a) has also been extended to body-bar frameworks [BST15].

THEOREM 62.3.4 Generic Liftings to Periodic Body-Bar Frameworks

A quotient graph $\tilde{G}/\Gamma$ *with* $|\tilde{E}_0| = \binom{d+1}{2}(|\tilde{V}_0| - 1) + d^2$ *is the quotient graph of an isostatic periodic body-bar framework in* d*-space for some gain assignment of the edges of* $\tilde{G}/\Gamma$ *if and only if it satisfies one and hence both of the following equivalent conditions.*

(a) $\tilde{G}/\Gamma$ *decomposes into the disjoint union of two spanning subgraphs, one with* $d|\tilde{V}_0| - d$ *edges and the property that every subgraph with* m *edges and* n *vertices satisfies* $m \leq dn - d$*, and the other one with* $\binom{d}{2}|\tilde{V}_0| + \binom{d+1}{2}$ *edges and the property that every subgraph with* m *edges and* n *vertices satisfies* $m \leq \binom{d}{2}n + \binom{d}{2}$.

(b) $\tilde{G}/\Gamma$ *contains the disjoint union of two spanning subgraphs, one with* $d|\tilde{V}_0| - d$ *edges and the property that every subgraph with* m *edges and* n *vertices satisfies* $m \leq dn - d$*, and the other one with* $\binom{d}{2}|\tilde{V}_0|$ *edges and the property that every subgraph with* m *edges and* n *vertices satisfies* $m \leq \binom{d}{2}n$.

Next we will consider combinatorial characterizations of first-order rigid periodic frameworks at a more discerning level. Specifically, for a given periodicity group Γ, we will study Γ-regular first-order rigidity by analyzing *quotient* Γ*-gain graphs*, i.e., quotient graphs that are equipped with an orientation and a gain labeling of the edges, rather than just quotient graphs. In other words, the gain assignment of the edges, and hence the lifting of the edges from the quotient graph to the covering periodic graph is now part of the initial data. Note that the formal definition of a quotient Γ-gain graph is completely analogous to the one given in Section 62.1 for finite symmetric frameworks (see also Figure 62.3.1). We begin by summarizing the main results for bar-joint frameworks in the plane. For the fully flexible lattice representation, the following result was obtained using periodic direction networks [MT13].

THEOREM 62.3.5 Periodic Rigidity for the Fully Flexible Lattice in the Plane

Let $\tilde{G}(\tilde{p})$ *be a* $\mathbb{Z}^2$*-regular* L*-periodic framework in* $\mathbb{R}^2$*. Then* $\tilde{G}(\tilde{p})$ *is isostatic if and only if the quotient* $\mathbb{Z}^2$*-gain graph of* $\tilde{G}$ *satisfies*

(a) $|\tilde{E}_0| = 2|\tilde{V}_0| + 1$;

(b) $|F| \leq 2|V(F)| - 3 + 2k(F) - 2(c(F) - 1)$ for all non-empty $F \subseteq \tilde{E}_0$,

where $k(F)$ *is the* $\mathbb{Z}^2$*-rank of* F*, i.e., the rank of the subgroup of* $\mathbb{Z}^2$ *induced by* F *(which is defined analogously as for finite groups (see Section 62.1), and* $c(F)$ *is the number of connected components of the subgraph induced by* F.

Using a Henneberg-type recursive construction sequence for quotient $\mathbb{Z}^2$-gain graphs, the following analogous result for the fixed lattice was established in [Ros15].

THEOREM 62.3.6 Periodic Rigidity for the Fixed Lattice in the Plane

Let $\tilde{G}(\tilde{p})$ be a $\mathbb{Z}^2$-regular L-periodic framework in $\mathbb{R}^2$, where $L(\mathbb{Z}^2)$ is non-singular and has to remain fixed. Then $\tilde{G}(\tilde{p})$ is isostatic if and only if the quotient $\mathbb{Z}^2$-gain graph of $\tilde{G}$ satisfies

(a) $|\tilde{E}_0| = 2|\tilde{V}_0| - 2$;

(b) $|F| \leq 2|V(F)| - 3$ *for all non-empty $F \subseteq \tilde{E}_0$ with $\mathbb{Z}^2$-rank equal to* 0;

(c) $|F| \leq 2|V(F)| - 2$ *for all non-empty $F \subseteq \tilde{E}_0$.*

Similar results also exist for periodic frameworks with partially flexible lattice representations (such as lattice representations with one degree of freedom [NR15] or with a fixed angle or fixed area fundamental domain [MT14a]).

There also exist various extensions of these results to crystallographic frameworks which must maintain the full crystallographic symmetry throughout any motion. In particular, for crystallographic groups Γ generated by translations and rotations in the plane, combinatorial characterizations for forced Γ-symmetric first-order rigidity (with a fully flexible lattice representation) are presented in [MT14b]. To obtain these results, one needs to carefully keep track of the types of lattice flexibility that are compatible with the subgroups of Γ induced by the various edge sets.

For crystallographic *body-bar frameworks* with a *fixed* lattice representation in an arbitrary-dimensional Euclidean space, combinatorial characterizations for forced Γ-symmetric first-order rigidity were established in [Ros14b, Tan15]. Analogous to Theorem 62.1.7, these characterizations are given in terms of gain-sparsity counts for the underlying multigraphs. In particular, for *periodic* body-bar frameworks, we have the following result.

THEOREM 62.3.7 Periodic Fixed-Lattice Body-Bar Frameworks in d-Space

Let $\tilde{G}(\tilde{b})$ be a $\mathbb{Z}^d$-regular L-periodic body-bar framework $\mathbb{R}^d$, where $L(\mathbb{Z}^d)$ is non-singular and has to remain fixed. Then $\tilde{G}(\tilde{b})$ is isostatic if and only if the quotient $\mathbb{Z}^d$-gain graph of $\tilde{G}$ satisfies

(a) $|\tilde{E}_0| = \binom{d+1}{2}|\tilde{V}_0| - d$;

(b) $|F| \leq \binom{d+1}{2}|V(F)| - \binom{d+1}{2} + \sum_{i=1}^{k(F)}(d-i)$ *for all non-empty $F \subseteq \tilde{E}_0$,*

where $k(F)$ is the $\mathbb{Z}^d$-rank of F, i.e., the rank of the subgroup of $\mathbb{Z}^d$ induced by F.

Extensions of this result to periodic body-hinge or molecular structures have not yet been investigated.

In order to gain an understanding of "incidentally periodic" first-order rigid frameworks, some recent work has also investigated the problem of characterizing periodic frameworks which are first-order rigid *for any choice of the periodicity lattice*. Such frameworks are called "ultra-rigid." An algebraic characterization for ultra-rigidity in arbitrary-dimensional Euclidean space has been obtained in [MT14c]. For the special case when the number of edge orbits is as small as possible for ultra-rigidity in dimension 2, a combinatorial characterization is also given in [MT14c]. All of these results apply to both the fully flexible and the fixed lattice representation.

Finally, when we handle and analyze a real crystal, it is finite. It is a *fragment* that might embed into a theoretical, infinite periodic structure. A recent paper [Whi14] examines when the rigidity, or flexibility, of a sufficiently large fragment will extend to some, or all, of the infinite periodic structures into which the fragment might be embedded.

62.4 SOURCES AND RELATED MATERIALS

BASIC SOURCES

The following are some key references for the various aspects of the study of rigid and flexible structures with symmetry:

[SW11]: A basic introduction to forced-symmetric rigidity and the orbit rigidity matrix.

[JKT16]: A key source for basic properties of gain graphs and combinatorial characterizations for forced-symmetric rigidity in the plane.

[SW12]: A description of the transfer of forced-symmetric rigidity properties between Euclidean and spherical space (as well as other Cayley-Klein metrics).

[CG16]: A new textbook on rigidity theory which contains a thorough discussion of the first-order rigidity analysis of incidentally symmetric frameworks.

[FG00]: A (somewhat informal) description of the symmetry-extended necessary counting conditions for a symmetric bar-joint framework to be isostatic. A rigorous mathematical treatment of this theory can be found in [Sch10a], for example.

[Sch10b]: An article on combinatorial characterizations of (incidentally) symmetry-regular isostatic bar-joint frameworks. In particular, it establishes a symmetry-extended Laman's theorem for the simplest group in the plane (three-fold rotational symmetry).

[ST14]: An introduction to "phase-symmetric orbit rigidity matrices" and the theory of incidentally symmetric first-order rigid frameworks.

[BS10]: A basic introduction to the rigidity and flexibility of periodic frameworks. See [Ros11] for an alternative approach.

[MT13]: A detailed description of combinatorial characterizations of $\mathbb{Z}^2$-regular rigid periodic frameworks.

RELATED CHAPTERS

Chapter 9: Geometry and topology of polygonal linkages
Chapter 18: Symmetry of polytopes and polyhedra
Chapter 34: Geometric reconstruction problems
Chapter 51: Robotics
Chapter 55: Graph drawing
Chapter 60: Geometric applications of the Grassmann-Cayley algebra

Chapter 61: Rigidity and scene analysis
Chapter 63: Global rigidity
Chapter 64: Crystals, periodic and aperiodic

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