56 SPLINES AND GEOMETRIC MODELING

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INTRODUCTION

Piecewise polynomials of fixed degree and continuously differentiable up to some order are known as ***splines*** or ***finite elements***. Splines are used in applications ranging from computer-aided design, computer graphics, data visualization, geometric modeling, and image processing to the solution of partial differential equations via finite element analysis. The spline-fitting problem of constructing a mesh of finite elements that interpolate or approximate data is by far the primary research problem in geometric modeling. ***Parametric splines*** are vectors of a set of multivariate polynomial (or rational) functions while ***implicit splines*** are zero contours of collections of multivariate polynomials. This chapter dwells mainly on spline surface fitting methods in real Euclidean space. We first discuss tensor product surfaces (Section 56.1), perhaps the most popular. The next sections cover generalized spline surfaces (Section 56.2), free-form surfaces (Section 56.3), and subdivision surfaces (Section 56.4). This classification is not strict, and some overlap exists. Interactive editing of surfaces is discussed in the final section (Section 56.5).

The various spline methods may be distinguished by several criteria:

- Implicit or parametric representations.
- Algebraic and geometric degree of the spline basis.
- Adaptivity and number of surface patches.
- Computation (time) and memory (space) required.
- Stability of fitting algorithms.
- Local or nonlocal interpolation.
- Spline Support and splitting of input mesh.
- Convexity of the input and solution.
- Fairness of the solution (first- and second-order variation).

These distinctions will guide the discussions throughout the chapter.

56.1 TENSOR PRODUCT SURFACES

Tensor product B-splines have emerged as the polynomial basis of choice for working with parametric surfaces [Boo68, CLR80]. The theory of tensor product splines or surface patches requires that the data have a rectangular geometry and that the parametrizations of opposite boundary curves be similar. It is based on the concept of bilinear interpolation. B-splines are generated by a rectangular (tensor product) mesh of control points. A reduced or decimated version of rectangular control points with T-junctions yields a T-spline tensor product [SZB$^+$03, DCL$^+$08,

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DLP13, Mou10] The most general results obtained to date are further summarized in Table 56.1.1, and will be discussed below.

GLOSSARY

Affine invariance: A property of a curve or surface generation scheme, implying invariance with respect to whether computation of a point on a curve or surface occurs before or after an affine map is applied to the input data.

A-spline: Collection of bivariate Bernstein-Bézier polynomials, each over a triangle and with prescribed geometric continuity, such that the zero contour of each polynomial defines a smooth and single-sheeted real algebraic curve segment. ("A" stands for "algebraic.")

A-patch: Smooth and "functional" zero contour of a Bernstein-Bézier polynomial over a tetrahedron.

Barycentric combination: A weighted average where the sum of the weights equals one.

Barycentric coordinates: A point in $\mathbb{R}^2$ may be written as a unique barycentric combination of three points. The coefficients in this combination are its barycentric coordinates. Similarly, a point in $\mathbb{R}^3$ may be written as a unique barycentric combination of four or more points. These latter are often referred to as generalized barycentric coordinates.

Basis function: Functions form linear spaces, which have bases. The elements of these bases are the basis functions.

Bernstein-Bézier form: Let $p_1,\ p_2,\ p_3,\ p_4 \in \mathbb{R}^3$ be affinely independent. Then the tetrahedron with these points as vertices is $V = [p_1p_2p_3p_4]$. Any polynomial $f(p)$ of degree n can be expressed in the Bernstein-Bézier (BB) form over V as

$$f(p) = \sum_{|\lambda|=n} b_\lambda\ B^n_\lambda(\alpha),\ \ \lambda \in \mathcal{Z}^4_+\ , \tag{56.1.1}$$

where

$$B^n_\lambda(\alpha) = \frac{n!}{\lambda_1!\lambda_2!\lambda_3!\lambda_4!}\ \alpha_1^{\lambda_1}\alpha_2^{\lambda_2}\alpha_3^{\lambda_3}\alpha_4^{\lambda_4}$$

are Bernstein polynomials, $|\lambda| = \sum_{i=1}^4 \lambda_i$ with $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)^T$, the barycentric coordinates of p are $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T$, $b_\lambda = b_{\lambda_1\lambda_2\lambda_3\lambda_4}$ are the *control points*, and $\mathcal{Z}^4_+$ is the set of all four-dimensional vectors with nonnegative integer components.

Bernstein polynomials: The basis functions for Bézier curves and surfaces.

Bézier curve: A curve whose points are determined by the parameter u in the equation $\sum_{i=0}^n B_i^n(u)P_i$, where the $B_i^n(u)$ are basis functions, and the P_i control points.

Bilinear interpolation: A tensor product of two orthogonal linear interpolants and the "simplest" surface defined by values at four points on a rectangle.

Blending functions: The basis functions used by interpolation schemes such as Gordon surfaces.

B-spline surface: Traditionally, a tensor product of curves defined using piecewise basis polynomials (B-spline basis). Any B-spline can be written in piecewise Bézier form. ("B" stands for "basis.")

C^k continuity: Smoothness defined in terms of matching of up to kth order derivatives along patch boundaries.

Control point: The coefficients in the expansion of a Bézier curve in terms of Bernstein polynomials.

Convex hull: The smallest convex set that contains a given set.

Convex set: A set such that the straight line segment connecting any two points of the set is completely contained within the set.

G^k continuity: Geometric continuity with smoothness defined in terms of matching of up to kth order derivatives allowing for reparametrization. For example, G^1 smoothness is defined in terms of matching tangent planes along patch boundaries.

Knots: A spline curve is defined over a partition of an interval of the real line. The points that define the partition are called knots.

Manifold Spline: A spline surface defined over a planar domain which can be extended to 2-manifolds in space, with arbitrary topology, and with or without boundaries.

Mesh: A decomposition of a geometric domain into finite elements; see Chapter 29.

Radial basis function: A real-valued function with value dependent on the distance from a point. Euclidean distance norm is typical. Quadric, multiquadric, poly-harmonic and thin-plate splines are based on radial basis functions.

Ruled (lofted) surface: A surface that interpolates two given curves using linear interpolation.

Subdivision surface: A surface that is iteratively refined from a surface mesh using splitting and averaging (linear) operations.

Tensor product surfaces: A surface represented with basis functions that are constructed as products of univariate basis functions. A tensor product Bézier surface is given by the equation $\sum_{i=0}^{n}\sum_{j=0}^{m} B_i^n(u)B_j^m(v)P_{ij}$, where the $B_i^n(u)$ and $B_j^m(v)$ are the univariate Bernstein polynomial basis functions, and the P_{ij} are control points.

T-spline surface: A reduced or decimated version of rectangular control points of B-splines with T-junctions produces or a reduced representation of the same tensor product B-spline.

Transfinite interpolation: Interpolating entire curves as opposed to values at discrete points.

Variation diminishing: A curve or surface scheme has this property if its output "wiggles less" than the control points from which it is constructed.

PARAMETRIC BÉZIER AND B-SPLINES

Tensor product Bézier surfaces are obtained by repeated applications of bilinear interpolation. Properties of tensor product Bézier patches include affine invariance, the "convex hull property," and the variation diminishing property. The boundary curves of a patch are polynomial curves that have their Bézier polygon given by the boundary polygons of the control net of the patch. Hence the four corners of the control net lie on the patch.

TABLE 56.1.1 Tensor product surfaces.

TYPE	INPUT	PROPERTIES
Piecewise Bézier and Hermite	rectangular grid of points, corner twists	C^1, initial global data survey data to determine the tangent and cross-derivative vectors at patch corners
Bicubic B-spline	rectangular grid of points	C^1
Coons patches	4 boundary curves	C^1
Gordon surfaces	rectangular network of curves	C^1, Gregory square
Biquadratic B-spline	limit of Doo-Sabin subdivision of rectangular faces	C^1
Bicubic B-spline	limit of Catmull-Clark subdivision of rectangular faces	C^1
Biquadratic splines	control points on mesh with arbitrary topology	G^1, system of linear equations for smoothness conditions around singular vertices
Biquartic splines	cubic curve mesh	C^1, interpolate second-order data at mesh points
Bisextic B-spline	rectangular network of cubic curves	C^1
Triquadratic/tricubic A-patches	rectilinear 3D grid points	C^1, local calculation of first-order cross derivatives
Triple products of B-splines	rectangular boxes	mixed orders possible
T-spline surface	reduced rectangular mesh with some T-junctions	C^1

Piecewise bicubic Bézier patches may be used to fit a C^1 surface through a rectangular grid of points. After the rectangular network of curves has been created, there are four coefficients left to determine the corner twists of each patch. These four corner twists cannot be specified independently and must satisfy a "compatibility constraint." Common twist estimation methods include zero twists, Adini's twist, Bessel twist, and Brunet's twist [Far98]. To obtain C^1 continuity between two patches the directions and lengths of the polyhedron edges must be matched across the common polyhedron boundary defining the common boundary curve. Piecewise bicubic Hermite patches are similar to the piecewise bicubic Bézier patches, but take points, partials, and mixed partials as input. The mixed partials affect only the interior shape of the patch, and are also called ***twist vectors***.

It is not possible to model a general closed surface or a surface with handles as a single non-degenerate B-spline. To represent free-form surfaces a significant amount of recent work has been done in the areas of geometric continuity, non-tensor product patches, and generalizing B-splines [CF83, Pet90a, Pet90b, GW91, DM83, GH87]. Common schemes include splitting, convex combinations of blending functions, subdivision, and local interpolation by construction [For95, HF84, MLL$^+$92, Pet93, Pet02, PR08].

IMPLICIT BÉZIER AND B-SPLINES

Patrikalakis and Kriezis [PK89] demonstrate how implicit algebraic surfaces can be manipulated in rectangular boxes as functions in a tensor product B-spline basis. This work, however, leaves open the problem of selecting weights or specifying knot sequences for C^1 meshes of tensor product implicit algebraic surface patches that fit given spatial data. Moore and Warren [MW91] extend the "marching cubes" scheme to compute a C^1 piecewise tensor product triquadratic approximation to scattered data using a Powell-Sabin-like split over subcubes. In [BBC$^+$99] an incremental and adaptive approach is used to construct C^1 spline functions defined over an octree subdivision that approximate a dense set of multiple volumetric scattered scalar values. Further details are provided in subsequent sub-sections on A-patches and implicit free-form surfaces.

COONS PATCHES AND GORDON SURFACES

Coons patches interpolate four boundary curves. They are constructed by composing two ruled, or lofted, surfaces and one bilinear surface, and hence are called ***bilinearly blended surfaces***. A Coons patch has four blending functions $f_i(u)$, $g_i(v)$, $i = 1, 2$. There are only two restrictions on the f_i and g_i: each pair must sum to one, and we must have $f_1(0) = g_1(0) = 1$ and $f_2(1) = g_2(1) = 0$ in order to interpolate.

A network of curves may be filled in with a C^1 surface using bicubically blended Coons patches. For this, the four twists at the data points and the four cross boundary derivatives must be computed. Compatibility problems may arise in computing the twists. If $\mathbf{x}(u, v)$ is twice differentiable, we have $\mathbf{x}_{uv} = \mathbf{x}_{vu}$, but this simplification does not apply here. One approach is to adjust the given data so that the incompatibilities disappear. Or if the data cannot be changed one can use a method known as ***Gregory's square*** that replaces the constant twist terms by variable twists that are computed from the cross boundary derivatives. The resulting surface does not have continuous twists at the corners and is rational parametric, which may not be acceptable geometry for certain geometric modeling systems.

Gordon surfaces are a generalization of Coons patches used to construct a surface that interpolates a rectangular network of curves. The idea is to take a univariate interpolation scheme, apply it to all curves, add the resulting surfaces, and subtract the tensor product interpolant that is defined by the univariate scheme. Polynomial interpolation or spline interpolation schemes may be used. Methods for Coons patches and Gordon surfaces can be formulated in terms of Boolean sums and projectors. This has also been generalized to create triangular Coons patches.

56.2 GENERALIZED SPLINE SURFACES

B-PATCHES

The B-patches developed by Seidel [Sei89, DMS92] are based on the study of symmetric recursive evaluation algorithms, and are defined by generalizing the deBoor algorithm for the evaluation of a B-spline segment from curves to surfaces. A polynomial surface that has a symmetric recursive evaluation algorithm is called a ***B-patch***. B-patches generalize Bézier patches over triangles, and are characterized by control points and a three-parameter family of knots. Every bivariate polynomial $F : \mathbb{R}^2 \to \mathbb{R}^d$ of degree n has a unique representation

$$F(U) = \sum_{|\vec{i}|=n} N_{\vec{i}}^n(U) P_{\vec{i}}, \qquad P_{\vec{i}} \in \mathbb{R}^d$$

as a B-patch, with parameters $\mathcal{K} = R_0, \ldots, R_{n-1},\ S_0, \ldots, S_{n-1},\ T_0, \ldots, T_{n-1}$ in $\mathbb{R}^2$, if the parameters (R_i, S_j, T_k) are affinely independent for $0 \le |\vec{i}| \le n-1$. The real-valued polynomials $N_{\vec{i}}^n(U)$ are called the ***normalized B-weights*** of degree n over $\mathcal{K}$.

MULTISIDED PATCHES

Multisided patches can be generated in basically two ways. Either the polygonal domain which is to be mapped into $\mathbb{R}^3$ is subdivided in the parametric plane, or one uniform equation is used as a combination of equations. In the former case, triangular or rectangular elements are put together or recursive subdivision is applied. In the latter case, either the known control point methods are generalized, or a weighted sum of interpolants is used. With constrained domain mapping, a domain point for an n-sided patch is represented by n dependent parameters. If the remainder of the parameters can be computed when any two parameters are independently chosen, it is called a *symmetric system of parameters*. The main results from multisided patch schemes obtained to date are summarized in Table 56.2.1.

TABLE 56.2.1 Multisided schemes.

TYPE	LIMITATIONS	PROPERTIES	DOMAIN POINTS
Sabin	n=3,5	C^1	constrained domain mapping, symmetric system of parameters
Gregory/Charrot	n=3,5	C^1	barycentric coordinates
Hosaka/Kimura	$n \le 6$	C^1	constrained domain mapping, symmetric system of parameters
Varady		VC^1	$2n$ variables constrained along polygon sides
Base points	$n = 4, 5, 6$	rational Bézier surfaces	base points in the parametric domain map to rational curves in $\mathbb{R}^3$
S-patches	multisided	G^1 rational bi-quadratic and bicubic B-splines	embed n-sided domain polygon into simplex of dimension $n-1$
Multisided A-patches	"polynomial surfaces" boundary curves	C^1, C^2 implicit Bezier surfaces	Hermite interpolation of boundary curves
Generalized Barycentric Finite Elements	"functional" multisided	G^1 rational bi-quadratic and bicubic B-splines	defined on n-sided domain polygons

TRIANGULAR RATIONAL PATCHES WITH BASE POINTS

Another approach to creating multisided patches is to introduce base points into rational parametric functions. Base points are parameter values for which the homogeneous coordinates (x, y, z, w) are mapped to $(0, 0, 0, 0)$ by the rational parametrization. Gregory's patch [Gre83] is defined using a special collection of rational basis functions that evaluate to 0/0 at vertices of the parametric domain, and thus introduce base points in the resulting parametrization. Warren [War92] uses base points to create parametrizations of four-, five-, and six-sided surface patches using rational Bézier surfaces defined over triangular domains. Setting a triangle of weights to zero at one corner of the domain triangle produces a four-sided patch that is the image of the domain triangle.

S-PATCHES

Loop and DeRose [LD89, LD90] present generalizations of biquadratic and bicubic B-spline surfaces that are capable of representing surfaces of arbitrary topology by placing restrictions on the connectivity of the control mesh, relaxing C^1 continuity to G^1 (geometric) continuity, and allowing n-sided finite elements. This generalized view considers the spline surface to be a collection of possibly rational polynomial maps from independent n-sided polygonal domains, whose union possesses continuity of some number of geometric invariants, such as tangent planes. This more general view allows patches to be sewn together to describe free-form surfaces in more complex ways.

An n-sided S-patch S is constructed by embedding its n-sided domain polygon P into a simplex $\triangle$ whose dimension is one less than the number of sides of the polygon. The edges of the polygon map to edges of the simplex. A Bézier simplex $\mathbf{B}$ is then constructed using $\triangle$ as a domain. The patch representation S is obtained by restricting the Bézier simplex to the embedded domain polygon.

A-PATCHES

The A-patch technique provides simple ways to guarantee that a constructed implicit surface is single-sheeted and free of undesirable singularities. The technique uses the zero contouring surfaces of trivariate Bernstein-Bézier polynomials to construct a piecewise smooth surface. We call such iso-surfaces *A-patches*. Algorithms to fill an n-sided hole, using either a single multisided A-patch or a network of A-patches, are given in [BE95]. The blends may be C^0, C^1, or C^2 exact fits (interpolation), as well as C^1 or C^2 least squares fits (interpolation and approximation).

For degree-bounded patches, a triangular network of A-patches for the hole may be generated in two ways. First, the n-sided hole is projected onto a plane and the result of a planar triangulation is projected back onto the hole. Second, an initial multisided A-patch is created for the hole and then a coarse triangulation for the patch is generated using a rational spline approximation [BX94].

MULTIVARIATE SPLINES, SIMPLEX AND BOX SPLINES

Multivariate splines are a generalization of univariate B-splines to a multivariate setting [Dah80, DM83, Boo88, Hol82]. Multivariate splines have applications in data fitting, computer-aided design, the finite element method, and image analysis. Work on splines has traditionally been for a given planar triangulation using a polynomial function basis. Box splines are multivariate generalizations of B-splines with uniform knots. Many of the basis functions used in finite element calculations on uniform triangles occur as special instances of box splines. In general a box spline is a locally supported piecewise polynomial. One can define translates of box splines that form a negative partition of unity.

In the bivariate case, box splines correspond to surfaces defined over a regular tessellation of the plane. If the tessellation is composed of triangles, it is possible to represent the surface as a collection of Bernstein-Bézier patches. The two most commonly used special tessellations arise from a rectangular grid by drawing in lines in north-easterly diagonals in each subrectangle or by drawing in both diagonals for each subrectangle. For these special triangulations there is an elegant way to construct locally supported splines.

Multivariate splines defined as projections of simplices are called ***simplex splines***. Auerbach [AGM+91] constructs approximations with simplex splines over irregular triangles. Bivariate quadratic simplicial B-splines defined by their corresponding sets of knots derived from a (suboptimal) constrained Delaunayi triangulation of the domain are employed to obtain a C^1 surface. This approach is well suited for scattered data.

Fong and Seidel [FS86, FS92] construct multivariate B-splines for quadratics and cubics by matching B-patches with simplex splines. The surface scheme is an approximation scheme based on blending functions and control points and allows the modeling of C^{k-1} continuous piecewise polynomial surfaces of degree k over arbitrary triangulations of the parameter plane. The resulting surfaces are defined as linear combinations of the blending functions, and are parametric piecewise polynomials over a triangulation of the parameter plane whose shape is determined by their control points.

SPHERICAL SPLINES

Spherical splines are piecewise representations of functions on the sphere. These can be applied to data fitting problems: (a) the input data is scattered on a unit sphere, (b) a boundary value approximation where the boundary is a unit sphere, (c) solving spherical partial differential equations. The splines can be defined using BB-basis polynomials on spherical triangles, or spherical quads, and are linearly independent and form a basis for functions on the sphere [LS07].

56.3 FREE-FORM SURFACES

The representation of free-form surfaces is one of the major issues in geometric modeling. These surfaces are generally defined in a piecewise manner by smoothly joining several, mostly four-sided, patches. Common approaches to constructing surfaces over irregular meshes are local construction, blending polynomial pieces, and splitting.

GLOSSARY

Blending polynomial pieces: Constructing k pieces for a k-sided mesh facet such that each piece matches a part of the facet data, and a convex combination of the pieces matches the whole.

Vertex enclosure constraint: Not every mesh of polynomial curves with a well-defined tangent plane at the mesh points can be interpolated by a smooth regularly parametrized surface with one polynomial piece per facet. This constraint on the mesh is a necessary and sufficient condition to guarantee the existence of such an interpolant [Pet91]. Rational patches, singular parametrizations, and the splitting of patches are techniques to enforce the vertex enclosure constraint.

MAIN RESULTS

Blending approaches prescribe a mesh of boundary curves and their normal derivatives. For this approach, however, the existence of a well-defined tangent plane at the data points is not sufficient to guarantee the existence of a C^1 mesh interpolant, because the mixed derivatives p_{uv} and p_{vu} are given independently at any point p. Splitting approaches, on the other hand, expect to be given at least tangent vectors at the data points, and sometimes the complete boundary. Mann et al. [MLL$^+$92] conclude that local polynomial interpolants generally produce unsatisfactory shapes.

With splitting schemes, every triangle in the triangulation of the data points (also called a *macro-triangle*) is split into several *mini-triangles*. Split-triangle interpolants do not require derivative information of higher order than the continuity of the desired interpolant. The simplest of the split-triangle interpolants is the C^1 Clough-Tocher interpolant. Each vertex is joined to the centroid, and the macro-triangle is split into three mini-triangles. The first-order data that this interpolant requires are position and gradient value at the macro-triangle vertices, plus some cross-boundary derivative at the midpoint of each edge. There are twelve data per macro-triangle, and cubic polynomials are used over each mini-triangle. The C^1 Powell-Sabin interpolants produce C^1 piecewise quadratic interpolants to C^1 data at the vertices of a triangulated data set. Each macro-triangle is split into six or twelve mini-triangles.

PARAMETRIC PATCH SCHEMES

These patches are given in vector-valued parametric form, generally mapping a rectangular or triangular parametric domain into $\mathbb{R}^3$. Parametric free-form surface patch schemes are summarized in Table 56.3.1.

TABLE 56.3.1 Free-form parametric schemes.

DEGREE	SCHEME	INPUT	PROPERTIES
Piecewise biquartic	local interpolation	cubic curve mesh	C^1, interpolate second-order data at mesh points
Piecewise biquadratic	G-edges	control points on a mesh with arbitrary topology	G^1, system of linear eqns for smoothness conditions around singular vertices
Sextic triangular pieces	approximation, no local splitting	triangular control mesh	G^1
Quadratic/cubic triangular pieces	splitting, subdivision	irregular mesh of points	C^1, refine mesh by Doo-Sabin to isolate regions of irregular points

IMPLICIT PATCH SCHEMES

While it is possible to model a general closed surface of arbitrary genus as a single implicit surface patch, the geometry of such a global surface is difficult to specify, interactively control, and polygonize. The main difficulties stem from the fact that implicit representations are iso-contours which generally have multiple real sheets, self-intersections, and several other undesirable singularities. Looking on the bright side, implicit polynomial splines of the same geometric degree have more degrees of freedom compared with parametric splines, and hence potentially are more flexible for approximating a complicated surface with fewer pieces and for achieving a higher order of smoothness. The potential of implicits remains largely latent: virtually all commercial and many research modeling systems are based on the parametric representation. An exception is SHASTRA, which allows modeling with both implicit and parametric splines [Baj93]. Implicit free-form surface schemes are summarized in Table 56.3.2.

TABLE 56.3.2 Free-form implicit schemes.

DEGREE	SCHEME	INPUT	PROPERTIES
5, 7	local interpolation, no splitting	curve mesh from spatial triangulation	C^1 interpolate or approximate
2	simplicial hull construction	spatial triangulation	
3	simplicial hull construction, Clough-Tocher split	spatial triangulation	C^1
3	simplicial hull construction, Clough-Tocher split of coplanar faces	spatial triangulation	C^1 A-patches, 3 or 4 sides
5	simplicial hull construction, Clough-Tocher split of coplanar faces	spatial triangulation	C^2 A-patches

A-SPLINES

An *A-spline* is a piecewise G^k-continuous chain of real algebraic curve segments, such that each curve segment is a smooth and single-sheeted zero contour of a bivariate Bernstein-Bézier polynomial (called a regular curve segment). A-splines are a suitable polynomial form for working with piecewise implicit polynomial curves. A characterization of A-splines defined over triangles or quadrilaterals is available [BX99, XB00], as is a detailing of their applications in curve design and fitting [BX01a].

CURVILINEAR MESH SCHEMES

Bajaj and Ihm [BI92a] construct implicit surfaces to solve the scattered data-fitting problem. The resulting surfaces approximate or contain with C^1 continuity any collection of points and algebraic space curves with derivative information. Their Hermite interpolation algorithm solves a homogeneous linear system of equations to compute the coefficients of the polynomial defining the algebraic surface. This idea has been extended to C^k (rescaling continuity) interpolate or least squares approximate implicit or parametric curves in space [BIW93]. This problem is formulated as a constrained quadratic minimization problem, where the algebraic distance is minimized instead of the geometric distance.

In a *curvilinear-mesh-based* scheme, Bajaj and Ihm [BI92b] construct low-degree implicit polynomial spline surfaces by interpolating a mesh of curves in space using the techniques of [BI92a, BIW93]. They consider an arbitrary spatial triangulation $\mathcal{T}$ consisting of vertices in $\mathbb{R}^3$ (or more generally, a simplicial polyhedron $\mathcal{P}$ when the triangulation is closed), with possibly normal vectors at the vertex points. Their algorithm constructs a C^1 mesh of real implicit algebraic surface patches over $\mathcal{T}$ or $\mathcal{P}$. The scheme is local (each patch has independent free parameters) and there is no local splitting. The algorithm first converts the given triangulation or polyhedron into a curvilinear wire frame, with at most cubic parametric curves which C^1 interpolate all the vertices. The curvilinear wire frame is then fleshed to produce a single implicit surface patch of degree at most 7 for each triangular face $\mathcal{T}$ of $\mathcal{P}$. If the triangulation is convex then the degree is at most 5. Similar techniques exist for parametrics [Pet91, Sar87]; however, the geometric degrees of the solution surfaces tend to be prohibitively high.

SIMPLEX- AND BOX-BASED SCHEMES

In a ***simplex-based*** approach, one first constructs a tetrahedral mesh (called the simplicial hull) conforming to a surface triangulation $\mathcal{T}$ of a polyhedron $\mathcal{P}$. The implicit piecewise polynomial surface consists of the zero set of a Bernstein-Bézier polynomial, defined within each tetrahedron (simplex) of the simplicial hull. A simplex-based approach enforces continuity between adjacent patches by enforcing that vertex/edge/face-adjacent trivariate polynomials are continuous with one another.

Similar to the trivariate interpolation case, Powell-Sabin or Clough-Tocher splits are used to introduce degree-bounded vertices to prevent the continuity system from propagating globally. Such splitting, however, could result in a large number of patches. However, as only the zero set of the polynomial is of interest, one does not need a complete mesh covering the entire space.

Sederberg [Sed85] showed how various smooth implicit algebraic surfaces, represented in trivariate Bernstein basis form, can be manipulated as functions in Bézier control tetrahedra with finite weights. He showed that if the coefficients of the Bernstein-Bézier form of the trivariate polynomial on the lines that parallel one edge, say L, of the tetrahedron all increase (or decrease) monotonically in the same direction, then any line parallel to L will intersect the zero contour algebraic surface patch at most once.

Guo [Guo91] used cubics to create free-form geometric models and enforced monotonicity conditions on a cubic polynomial along the direction from one vertex to a point of the face opposite the vertex. A Clough-Tocher split is used to subdivide each tetrahedron of the simplicial hull. Dahmen and Thamm-Scharr [DTS93] utilize a single cubic patch per tetrahedron, except for tetrahedra on coplanar faces.

Lodha [Lod92] constructed low degree surfaces with both parametric and implicit representations and investigated their properties. A method is described for creating quadratic triangular Bézier surface patches that lie on implicit quadric surfaces. Another method is described for creating biquadratic tensor product Bézier surface patches that lie on implicit cubic surfaces. The resulting patches satisfy all the standard properties of parametric Bézier surfaces, including interpolation of the corners of the control polyhedron and the convex hull property.

Bajaj and Ihm, Guo, and Dahmen [BI92b, Guo91, Guo93, Dah89] provide heuristics based on monotonicity and least square approximation to circumvent the multiple-sheeted and singularity problems of implicit patches.

Bajaj, Chen, and Xu [BCX95] construct 3- and 4-sided A-patches that are implicit surfaces in Bernstein-Bézier (BB) form and that are smooth and single-sheeted. They give sufficiency conditions for the BB form of a trivariate polynomial within a tetrahedron, such that the zero contour of the polynomial is a single-sheeted nonsingular surface within the tetrahedron, and its cubic-mesh complex for the polyhedron $\mathcal{P}$ is guaranteed to be both nonsingular and single-sheeted. They distinguish between convex and non-convex facets and edges of the triangulation. A double-sided tetrahedron is built for nonconvex facets and edges, and single-sided tetrahedra are built for convex facets and edges. A generalization of Sederberg's condition is given for a three-sided j-patch where any line segment passing through the jth vertex of the tetrahedron and its opposite face intersects the patch only once. Instead of having coefficients be monotonically increasing or decreasing there is a single sign change condition. There are also free parameters for both local and global shape control.

Reconstructing surfaces and scalar fields defined over the surface from scattered data using implicit Bézier splines is described in [BBX95, BX01b]. See also Chapter 35.

GENERALIZED BARYCENTRIC FINITE ELEMENTS

A ***generalized-barycentric*** scheme for the construction of free-form curve or surface elements, is an extension of the afore-mentioned simplex and box based schemes to meshes of convex polytopes, especially in dimensions 2 and 3 [War96, FHK06, GRB16]. It is well known that on simplicial and tensor product meshes, standard barycentric coordinates provide a local basis for scalar-valued finite element spaces, commonly called the Lagrange interpolation elements. Using generalized barycentric coordinates on can construct local interpolation bases with the same global continuity and polynomial reproduction properties as their simplicial counterparts. Further, local bases for the lowest-order vector-valued Brezzi-Douglas-Marini [BDM85], Raviart-Thomas [RT77], and Nedelec [BDD$^+$87, Néd80, Néd86] finite element spaces on simplices can also be defined in a canonical fashion from an associated set of standard barycentric functions.

In [GRB12, RGB13] linear order, scalar-valued interpolants on polygonal meshes are considered use four different types of generalized barycentric coordinates: Wachspress [Wac11], Sibson [Far90, Sib80], harmonic [Chr08, JMD$^+$07, MKB$^+$08], and mean value [Flo03, FHK06, FKR05]. The analysis was extended by Gillette, Floater, and Sukumar in the case of Wachspress coordinates to convex polytopes in any dimension [FGS14], based on work by Warren and colleagues [JSW$^+$05, War96, WSH$^+$07]. In a related vein, it has also been shown how taking pairwise products of generalized barycentric coordinates can be used to construct quadratic order methods on polygons [RGB14]. Applications of generalized barycentric coordinates to finite element methods have primarily focused on scalar-valued PDE problems [MP08, RS06, SM06, ST04, WBG07] though extensions to vector-valued interpolants on dual meshes is the method of choice for solution of mixed or coupled PDEs [AFW09].

56.4 MULTIRESOLUTION SPLINE SURFACES

SUBDIVISION SURFACES

Subdivision techniques can be used to produce generally pleasing surfaces from arbitrary control meshes. The faces of the mesh need not be planar, nor need the vertices lie on a topologically regular mesh. Subdivision consists of splitting and averaging. Each edge or face is split, and each new vertex introduced by the splitting is positioned at a fixed affine combination of its neighbor's weights. Subdivision schemes are summarized in Table 56.4.1.

TABLE 56.4.1 Subdivision schemes.

TYPE	PROPERTIES
Doo-Sabin; Catmull-Clark	C^1, interpolate centroids of all faces at each step
Nasri	interpolate points/normals on irregular networks
Loop	C^1, split each triangle of a triangular mesh into 4 triangles
Hoppe et al.	extends Loop's method to incorporate shape edges in limit surfaces; initial vertices belong to vertex, edge, or face of limit surface
Storry and Ball	C^1 n-sided B-spline patch to fit in bicubic surface, one dof
Dyn, Levin, and Gregory	interpolatory butterfly subdivision, modify set of deterministic rules for subdivision
Bajaj, Chen, and Xu	approximation, one step subdivision to build simplicial hull, C^1 cubic and C^2 quintic A-patches
Reif	regularity conditions

MAIN ALGORITHMS

Subdivision algorithms start with a polyhedral configuration of points, edges, and faces. The control mesh will in general consist of large regular regions and isolated singular regions. Subdivision enlarges the regular regions of the control net and shrinks the singular regions. Each application of the subdivision algorithm constructs a refined polyhedron, consisting of more points and smaller faces, tending in the limit to a smooth surface. In general the new control points are computed as linear combinations of old control points. The associated matrix is called the ***subdivision matrix***. Except for some special cases, the limiting surface does not have an explicit analytic representation. If each face of the polyhedron is a rectangle, the Doo-Sabin subdivision rules generate biquadratic tensor product B-splines, and the Catmull-Clark subdivision rules generate bicubic tensor product B-splines. Also, the subdivision technique of Loop generates three-direction box splines.

Reif [Rei92] presents a unified approach to subdivision algorithms for meshes with arbitrary topology and gives a sufficient condition for the regularity of the surface. The existence of a smooth regular parametrization for the generated surface near the point is determined from the leading eigenvalues of the subdivision matrix and an associated characteristic map. Details and further discussion of recent subdivision schemes are available from [WW02].

APPROXIMATING SCHEMES

Bajaj, Chen, and Xu [BCX94] construct an "inner" simplicial hull after one step of subdivision of the input polyhedron $\mathcal{P}$. As in traditional subdivision schemes, $\mathcal{P}$ is used as a control mesh for free-form modeling, while an inner surface triangulation $\mathcal{T}$ of the hull can be considered as the second-level mesh. Both a C^1 mesh with cubic A-patches and a C^2 mesh with quintic patches can be constructed to approximate the polyhedron $\mathcal{P}$ [XBE01].

INTERPOLATING SCHEMES

There are two key approaches to constructing interpolating subdivision surfaces. One approach is to first compute a new configuration of vertices, edges, and faces with the same topology such that the vertices of the new configuration converge to the given vertices in the limit. The subdivision technique is then applied to this new configuration The other approach is to modify the deterministic subdivision rules so that the limiting surface interpolates the vertices.

HIERARCHICAL SPLINES

Hierarchical splines are a multiresolution approach to the representation and manipulation of free-form surfaces. A hierarchical B-spline is constructed from a base surface (level 0) and a series of overlays are derived from the immediate parent in the hierarchy. Forsey and Bartels [FB88] present a refinement scheme that uses a hierarchy of rectangular B-spline overlays to produce C^2 surfaces. Overlays can be added manually to add detail to the surface, and local or global changes to the surface can be made by manipulating control points at different levels.

Forsey and Wang [FW93] create hierarchical bicubic B-spline approximations to scanned cylindrical data. The resulting hierarchical spline surface is interactively modifiable using editing capabilities of the hierarchical surface representation, allowing either local or global changes to surface shape while retaining the details of the scanned data. Oscillations occur, however, when the data have high-amplitude or high-frequency regions. Forsey and Bartels use a hierarchical wavelet-based representation for fitting tensor product parametric spline surfaces to gridded data in [FB95]. The multiresolution representation is extend to include arbitrary meshes in [EDD$^+$95]. The method is based on approximating an arbitrary mesh by a special type of mesh and using a continuous parametrization of the arbitrary mesh over a simple domain mesh.

Multiresolution representations for spherical splines or spherical wavelets is given in [SS95] Further discussion of wavelet based multiresolution schemes and some of their applications is available from [SDS96].

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56.5 PHYSICALLY BASED APPROACHES TO SURFACE MODELING

ENERGY-BASED SPLINES

A group of researchers [TF88, PB88, TPB+87, WFB87, BHN99] have presented discrete models which are based extensively on the theory of elasticity and plasticity, using energy fields to define and enforce constraints [AFW06, AFW10]. Haumann [Hau87] used the same approach but used a triangulated model and a simpler physical model based on points, springs, and hinges. Thingvold and Cohen [TC90] defined a model of elastic and plastic B-spline surfaces which supports both animation and design operations. The basis for the physical model is a generalized point-mass/spring/hinge model that has been adapted into a simultaneous refinement of the geometric/physical model. Always having a sculptured surface representation as well as the physical hinge/spring/mesh model allows the user to intertwine physical-based operations, such as force application, with geometrical modeling. Refinement operations for spring and hinge B-spline models are compatible with the physics and mathematics of B-spline models. The models of elasticity and plasticity are written in terms of springs and hinges, and can be implemented with standard integration techniques to model realistic motions of elastic and plastic surfaces. These motions are controlled by the physical properties assigned and by kinematic constraints on various portions of the surface. Terzopoulos and Qin [TQ94] develop a dynamic generalization of the nonuniform rational B-spline (NURBS) model. They present a physics-based model that incorporates mass distributions, internal deformation energies, and other physical quantities into the NURBS geometric substrate. These dynamic NURBS can be used in applications such as rounding of solids, optimal surface fitting to unstructured data, surface design from cross-sections, and free-form deformations.

DIFFERENTIAL EQUATIONS AND SURFACE SPLINES

Early research on using partial differential equations (PDEs) to handle surface modeling problems traces back to Bloor et al.'s work at the end of the 1980s ([BW89a, BW89b, BW90]). The basic idea of these papers is the use of biharmonic equations on a rectangular domain to solve blending and hole filling problems. One of the advantages of using the biharmonic equation is that it is linear, and therefore easier to solve. However, the solution of the equation depends on the surface parametrization.

The evolution technique, based on the heat equation $\partial_t x - \Delta x = 0$, has been extensively used in the area of image processing (see [PM87, PR99, Wei98], where Δ is a $2D$ Laplace operator. This was extended later to smoothing or fairing noisy surfaces (see [CDR00, DMS+99, DMS+03]). For a surface M, the counterpart of the Laplacian Δ is the Laplace-Beltrami operator Δ_M (see [Car92]). One then obtains the geometric diffusion equation $\partial_t x - \Delta_M x = 0$ for a surface point $x(t)$ on the surface $M(t)$. Taubin [Tau95] discusses the discretized operator of the Laplacian and related approaches in the context of generalized frequencies on meshes. Kobbelt

[Kob96] considers discrete approximations of the Laplacian in the construction of fair interpolatory subdivision schemes. This work was extended in [KCV+98] to arbitrary connectivity for purposes of multi-resolution interactive editing. Desbrun et al. [DMS+99] used an implicit discretization of geometric diffusion to obtain a strongly stable numerical smoothing scheme. The same strategy of discretization is also adopted and analyzed by Deckelnick and Dziuk [DD02] with the conclusion that this scheme is unconditionally stable. Clarenz et al. [CDR00] introduced anisotropic geometric diffusion to enhance features while smoothing. Ohtake et al. [OBB00] combined an inner fairness mechanism in their fairing process to increase the mesh regularity. Bajaj and Xu [BX03] smooth both surfaces and functions on surfaces, in a C^2 smooth function space defined by the limit of triangular subdivision surfaces (quartic Box splines).

Similar to surface diffusion using the Laplacian, a more general class of PDE based methods called ***flow surface techniques*** have been developed which simulate different kinds of flows on surfaces (see [WJE00] for references) using the equation $\partial_t x - v(x,t) = 0$, where $v(x,t)$ represents the instantaneous stationary velocity field.

Level set methods were also used in surface fairing and surface reconstruction; see [BCO+01, BSC+00, CS99, MBW+02, OF03, WB98, ZOF01, ZOM+00]. In these methods, surfaces are formulated as iso-surfaces (level surfaces) of 3D functions, which are usually defined from the signed distance over Cartesian grids of a volume. An evolution PDE on the volume governs the behavior of the level surface. These level-set methods have several attractive features including, ease of implementation, arbitrary topology [BW01] and a growing body of theoretical results. Often, fine surface structures are not captured by level sets, although it is possible to use adaptive [PR99] and triangulated grids as well as Hermite data [JLS+02, KBS+01]. To reduce the computationally complexity, Bertalmio et al. [BCO+01, BSC+00] solve the PDE in a narrow band for deforming vectorial functions on surfaces (with a fixed surface represented by the level surface).

Recently, surface diffusion flow has been used to solve the surface blending problem and free-form surface design problem. In [SK00], fair meshes with G^1 conditions are created in the special case where the meshes are assumed to have subdivision connectivity. In this work, local surface parametrization is still used to estimate the surface curvatures. A later paper [SK01] uses the same equation for smoothing meshes while satisfying G^1 boundary conditions. Outer fairness (the smoothness in the classical sense) and inner fairness (the regularity of the vertex distribution) criteria are used in their fairing process.

Another category of surface fairing research is based on utilizing optimization techniques. In this category, one constructs an optimization problem that minimizes certain objective functions [Gre94, GL10, HG00, MS92, Sap94, Wah90, WW92], such as thin plate energy, membrane energy [KCV+98], total curvature [KHP+97, WW94], or sum of distances [Mal92]. Using local interpolation or fitting, or replacing differential operators with divided difference operators, the optimization problems are discretized to arrive at finite dimensional linear or nonlinear systems. Approximate solutions are then obtained by solving the constructed systems. In general, such an approach is quite computationally intensive.

56.6 SOURCES AND RELATED MATERIAL

SURVEYS

All results not given an explicit reference above may be traced in these surveys.

[Alf89]: Scattered data fitting and multivariate splines.

[Baj97]: Summary of data fitting with implicit algebraic splines.

[BBB87]: Application of B-splines.

[Dau92]: An introduction to wavelets.

[BHR93]: An introduction to Box splines.

[DM83]: Scattered data fitting and multivariate splines.

[Far98]: Summary of the history of triangular Bernstein-Bézier patches.

[GL93]: An introduction to Knot manipulation techniques in splines.

[Hol82]: Scattered data fitting and multivariate splines.

[SDS96]: Application of wavelet representations.

[Wah90]: Penalized regression splines.

[WW02, PR08]: Subdivision techniques.

RELATED CHAPTERS

Chapter 29: Triangulations and mesh generation
Chapter 37: Computational and quantitative real algebraic geometry
Chapter 52: Computer graphics
Chapter 57: Solid modeling

REFERENCES

[AFW06] D. Arnold, R. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. *Acta Numer.*, pages 1–155, 2006.

[AFW09] D. Arnold, R. Falk, and R. Winther. Geometric decompositions and local bases for spaces of finite element differential forms. *Comput. Methods Appl. Mech. Engrg.*, 198:1660–1672, 2009.

[AFW10] D. Arnold, R. Falk, and R. Winther. Finite element exterior calculus: from Hodge theory to numerical stability. *Bull. Amer. Math. Soc.*, 47:281–354, 2010.

[AGM+91] S. Auerbach, R.H.J. Gmelig Meyling, M. Neamtu, and H. Schaeben. Approximation and geometric modeling with simplex B-splines associated with irregular triangles. *Comput. Aided Geom. Design*, 8:67–87, 1991.

[Alf89] P. Alfeld. Scattered data interpolation in three or more variables. In T. Lyche and L.L. Schumaker, editors, *Mathematical Methods in Computer Aided Geometric Design*, pages 1–34, Academic Press, San Diego, 1989.

[Baj93] C.L. Bajaj. The emergence of algebraic curves and surfaces in geometric design. In R. Martin, editor, *Directions in Geometric Computing*, pages 1–29, Information Geometers, Winchester, 1993.

[Baj97] C.L. Bajaj. Implicit surface patches. In J. Bloomenthal, editor, *Introduction to Implicit Surfaces*, pages 98–125, Morgan Kaufman, San Francisco, 1997.

[BBB87] R. Bartels, J. Beatty, and B. Barsky. *An Introduction to Splines for Use in Computer Graphics and Geometric Modeling.* Morgan Kaufmann, San Francisco, 1987.

[BBC+99] F. Bernardini, C.L. Bajaj, J. Chen, and D. Schikore. Automatic reconstruction of 3D CAD models from digital scans. *Internat. J. Comput. Geom. Appl.*, 9:327–369, 1999.

[BBX95] C.L. Bajaj, F. Bernardini, and G. Xu. Automatic reconstruction of surfaces and scalar fields from 3D scans. *Proc. 22nd Conf. Comput. Graph. Interactive Tech.*, pages 109–118, ACM Press, 1995.

[BCO+01] M. Bertalmio, L.-T. Cheng, S. Osher, and G. Sapiro. Variational problems and partial differential equations on implicit surfaces. *J. Comput. Physics*, 174:759–780, 2001.

[BCX94] C.L. Bajaj, J. Chen, and G. Xu. Smooth low degree approximations of polyhedra. Comput. Sci. Tech. Rep., CSD-TR-94-002, Purdue Univ., 1994.

[BCX95] C.L. Bajaj, J. Chen, and G. Xu. Modeling with cubic A-patches. *ACM Trans. Graph.*, 14:103–133, 1995.

[BDD+87] F. Brezzi, J. Douglas Jr., R. Durán, and M. Fortin. Mixed finite elements for second order elliptic problems in three variables. *Numer. Math.*, 51:237–250, 1987.

[BDM85] F. Brezzi, J. Douglas Jr., and L.D. Marini. Two families of mixed finite elements for second order elliptic problems. *Numer. Math.*, 47:217–235, 1985.

[BE95] C.L. Bajaj and S.B. Evans. Smooth multi-sided blends with A-patches. Presented at *4th SIAM Conf. Geom. Design*, 1995.

[BHN99] C.L. Bajaj, R. Holt, and A. Netravali. Energy formulations for A-splines. *Comput. Aided Geom. Design*, 16:39–59, 1999.

[BHR93] C. de Boor, K. Hollig, and S. Riemenschneider. *Box Splines.* Springer-Verlag, New York, 1993.

[BI92a] C.L. Bajaj and I. Ihm. Algebraic surface design with Hermite interpolation. *ACM Trans. Graph.*, 11:61–91, 1992.

[BI92b] C.L. Bajaj and I. Ihm. C^1 Smoothing of polyhedra with implicit algebraic splines. *SIGGRAPH Comput. Graph.*, 26:79–88, 1992.

[BIW93] C.L. Bajaj, I. Ihm, and J. Warren. Higher-order interpolation and least-squares approximation using implicit algebraic surfaces. *ACM Trans. Graph.*, 12:327–347, 1993.

[Boo68] C. de Boor. On calculating with B-splines. *J. Approx. Theory*, 1:219–235, 1968.

[Boo88] C. de Boor. What is a multivariate spline? In *Proc 1st Internat. Conf. Industrial Appl. Math.*, pages 91–101, SIAM, Philadelphia, 1988.

[BSC+00] M. Bertalmio, G. Sapiro, L.T. Cheng, and S. Osher. A framework for solving surface partial differential equations for computer graphics applications. CAM Report 00-43, UCLA, Math. Dept., 2000.

[BW89a] M.I.G. Bloor and M.J. Wilson. Generating blend surfaces using partial differential equations. *Comput. Aided Design*, 21:165–171, 1989.

[BW89b] M.I.G. Bloor and M.J. Wilson. Generating N-sided patches with partial differential equations. In *Advances in Comput. Graph.*, pages 129–145. Springer-Verlag, Berlin, 1989.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

[BW90] M.I.G. Bloor and M.J. Wilson. Using partial differential equations to generate free-form surfaces. *Comput. Aided Design*, 22:221–234, 1990.

[BW01] D. Breen and R. Whitaker. A level-set approach for the metamorphosis of solid models. *IEEE Trans. Vis. Comput. Graph.*, 7:173–192, 2001.

[BX94] C.L. Bajaj and G. Xu. Rational spline approximations of real algebraic curves and surfaces. In H.P. Dikshit and C. Micchelli, editors, *Advances in Computational Mathematics*, pages 73–85, World Scientific, Singapore, 1994.

[BX99] C.L. Bajaj and G. Xu. A-splines: Local interpolation and approximation using G^k-continuous piecewise real algebraic curves. *Comput. Aided Geom. Design*, 16:557–578, 1999.

[BX01a] C.L. Bajaj and G. Xu. Regular algebraic curve segments (III)—applications in interactive design and data fitting. *Comput. Aided Geom. Design*, 18:149–173, 2001.

[BX01b] C.L. Bajaj and G. Xu. Smooth shell construction with mixed prism fat surfaces. In *Geometric Modeling*, pages 19–36, Springer, Berlin, 2001.

[BX03] C.L. Bajaj and G. Xu. Anisotropic diffusion of surfaces and functions on surfaces. *ACM Trans. Graph.*, 22:4–32, 2003.

[Car92] M. do Carmo. *Riemannian Geometry*. Birkhäuser, Boston, 1992.

[CDR00] U. Clarenz, U. Diewald, and M. Rumpf. Anisotropic geometric diffusion in surface processing. In *Proc. IEEE Visualization*, pages 397–505, 2000.

[CF83] H. Chiyokura and K. Fumihiko. Design of solids with free form surfaces. *SIGGRAPH Comput. Graph.*, 17:289–298, 1983.

[Chr08] S.H. Christiansen. A construction of spaces of compatible differential forms on cellular complexes. *Math. Models Methods Appl. Sci.*, 18:739–757, 2008.

[CLR80] E. Cohen, T. Lyche, and R. Riesenfeld. Discrete B-splines and subdivision techniques in computer aided geometric design and computer graphics. *Comput. Graph. Image Process.*, 14:87–111, 1980.

[CS99] D.L. Chopp and J.A. Sethian. Motion by intrinsic Laplacian of curvature. *Interfaces and Free Boundaries*, 1:1–18, 1999.

[Dah80] W. Dahmen. On multivariate B-splines. *SIAM J. Numer. Anal.*, 17:179–191, 1980.

[Dah89] W. Dahmen. Smooth piecewise quadratic surfaces. In T. Lyche and L.L. Schumaker, editors, *Mathematical Methods in Computer Aided Geometric Design*, pages 181–193, Academic Press, Boston, 1989.

[Dau92] I. Daubechies. *Ten Lectures on Wavelets*. SIAM, Philadelphia, 1992.

[DCL+08] J. Deng, F. Chen, X. Li, C. Hu, W. Tong, Z. Yang, and Y. Feng. Polynomial splines over hierarchical T-meshes. *Graphical Models*, 70:76–86, 2008.

[DD02] K. Deckelnick and G. Dziuk. A fully discrete numerical scheme for weighted mean curvature flow. *Numer. Math.*, 91:423–452, 2002.

[DLP13] T. Dokken, T. Lyche, and K. Pettersen. Polynomial splines over locally refined box partitions. *Comput. Aided Geom. Design*, 30:331–356, 2013.

[DM83] W. Dahmen and C. Micchelli. Recent progress in multivariate splines. In L. Shumaker, C. Chui, and J. Word, editors, *Approximation Theory IV*, pages 27–121, Academic Press, 1983.

[DMS92] W. Dahmen, C. Micchelli, and H.-P. Seidel. Blossoming begets B-spline bases built better by B-patches. *Mathematics of Computation*, 59:97–115, 1992.

[DMS+99] M. Desbrun, M. Meyer, P. Schröder, and A.H. Barr. Implicit fairing of irregular meshes using diffusion and curvature flow. In *Proc. 26th Conf. Comput. Graph. Interactive Tech*, pages 317–324, ACM Press, 1999.

[DMS+03] M. Meyer, M. Desbrun, P. Schröder, A.H. Barr. Discrete differential-geometry operators for triangulated 2-manifolds. In H.-C. Hege and K. Polthier, editors, *Visualization and Mathematics III*, pages 35–57, Springer, Berlin, 2003.

[DTS93] W. Dahmen and T-M. Thamm-Schaar. Cubicoids: Modeling and visualization. *Comput. Aided Geom. Design*, 10:89–108, 1993.

[EDD+95] M. Eck, T. DeRose, T. Duchamp, H. Hoppe, M. Lounsbery, and W. Stuetzle. Multiresolution analysis of arbitrary meshes. In *Proc. 22nd Conf. Comput. Graph. Interactive Tech*, pages 173–180, ACM Press, 1995.

[Far90] G. Farin. Surfaces over Dirichlet tessellations. *Comput. Aided Geom. Design*, 7:281–292, 1990.

[Far98] G. Farin. *Curves and Surfaces for Computer Aided Geometric Design: A Practical Guide*. Academic Press, Boston, 1998.

[FB88] D. Forsey and R. Bartels. Hierarchical B-spline refinement. *SIGGRAPH Comput. Graph.*, 22:205–212, 1988.

[FB95] D. Forsey and R. Bartels. Surface fitting with hierarchical splines. *ACM Trans. Graph.*, 14:134–161, 1995.

[FGS14] M.S. Floater, A. Gillette, and N. Sukumar. Gradient bounds for Wachspress coordinates on polytopes. *SIAM J. Numer. Anal.*, 52:515–532, 2014.

[FHK06] M. Floater, K. Hormann, and G. Kós. A general construction of barycentric coordinates over convex polygons. *Adv. Comput. Math.*, 24:311–331, 2006.

[FKR05] M. Floater, G. Kós, and M. Reimers. Mean value coordinates in 3D. *Comput. Aided Geom. Design*, 22:623–631, 2005.

[Flo03] M. Floater. Mean value coordinates. *Comput. Aided Geom. Design*, 20:19–27, 2003.

[For95] D. Forsey. Surface fitting with hierarchical splines. *ACM Trans. Graph.*, 14:134–161, 1995.

[FS86] P. Fong and H.-P. Seidel. Control points for multivariate B-spline surfaces over arbitrary triangulations. *Comput. Graphics Forum*, 10:309–317, 1986.

[FS92] P. Fong and H.-P. Seidel. An implementation of multivariate B-spline surfaces over arbitrary triangulations. In *Proc. Graphics Interface*, pages 1–10, Vancouver, 1992.

[FW93] D. Forsey and L. Wang. Multi-resolution surface approximation for animation. In *Proc. Graphics Interface*, pages 192–199, Toronto, 1993.

[GH87] J.A. Gregory and J. Hahn. Geometric continuity and convex combination patches. *Comput. Aided Geom. Design*, 4:79–89, 1987.

[GL93] R.N. Goldman and T. Lyche. *Knot Insertion and Deletion Algorithms for B-spline Curves and Surfaces*. SIAM, Philadelphia, 1993.

[GL10] S. Guillas and M.-J. Lai. Bivariate splines for spatial functional regression models. *J. Nonparametr. Stat.*, 22:477–497, 2010.

[GRB12] A. Gillette, A. Rand, and C. Bajaj. Error estimates for generalized barycentric coordinates. *Adv. Comput. Math.*, 37:417–439, 2012.

[GRB16] A. Gillette, A. Rand, and C. Bajaj. Construction of scalar and vector finite element families on polygonal and polyhedral meshes. *Comput. Methods Appl. Math.*, 2016.

[Gre83] J.A. Gregory. C^1 rectangular and non-rectangular surface patches. In R.E. Barnhill and W. Boehm, editors, *Surfaces in Computer Aided Geometric Design*, pages 25–33, North–Holland, Amsterdam, 1983.

[Gre94] G. Greiner. Variational design and fairing of spline surface. *Comput. Graph. Forum*, 13:143–154, 1994.

[Guo91] B. Guo. Surface generation using implicit cubics. In N.M. Patrikalakis, editor, *Scientific Visualization of Physical Phenomena*, pages 485–530, Springer, Tokyo, 1991.

[Guo93] B. Guo. Non-splitting macro patches for implicit cubic spline surfaces. *Comput. Graph. Forum*, 12:434–445, 1993.

[GW91] T. Garrity and J. Warren. Geometric continuity. *Comput. Aided Geom. Design*, 8:51–65, 1991.

[Hau87] D. Haumann. Modeling the physical behavior of flexible objects. ACM SIGGRAPH Course Notes #17, 1987.

[HF84] M. Hosaka and K. Fumihiko. Non-four-sided patch expressions with control points. *Comput. Aided Geom. Design*, 1:75–86, 1984.

[HG00] A. Hubeli and M. Gross. Fairing of non-manifolds for visualization. In *Proc. IEEE Visualization*, pages 407–414, 2000.

[Hol82] K. Hollig. Multivariate splines. *SIAM J. Numer. Anal.*, 19:1013–1031, 1982.

[JLS+02] T. Ju, F. Losasso, S. Schaefer, and J. Warren. Dual contouring of Hermite data. *ACM Trans. Graph.*, 21:339–346, 2002.

[JMD+07] P. Joshi, M. Meyer, T. DeRose, B. Green, and T. Sanocki. Harmonic coordinates for character articulation. *ACM Trans. Graph.*, 26:71, 2007.

[JSW+05] T. Ju, S. Schaefer, J, Warren, and M. Desbrun. A geometric construction of coordinates for convex polyhedra using polar duals. In *Proc. 3rd Eurographics Sympos. Geom. Process.*, pages 181–186, 2005.

[KBS+01] L.P. Kobbelt, M. Botsch, U. Schwanecke, and H.-P. Seidel. Feature sensitive surface extraction from volume data. In *Proc. 28th Conf. Comput. Graph. Interactive Tech.*, pages 57–66, ACM Press, 2001.

[KCV+98] L. Kobbelt, S. Campagna, J. Vorsatz, and H.-P. Seidel. Interactive muti-resolution modeling on arbitrary meshes. In *Proc. 25th Conf. Comput. Graph. Interactive Tech.*, pages 105–114, ACM Press, 1998.

[KHP+97] L. Kobbelt, T. Hesse, H. Prautzsch, and K. Schweizerhof. Iterative mesh generation for FE-computation on free form surfaces. *Engng. Comput.*, 14:806–820, 1997.

[Kob96] L. Kobbelt. Discrete fairing. In T. Goodman and R. Martin, editors, *The Mathematics of Surfaces VII*, pages 101–129, Information Geometers, 1996.

[LD89] C.T. Loop and T.D. DeRose. A multisided generalization of Bézier surfaces. *ACM Trans. Graph.*, 8:205–234, 1989.

[LD90] C. Loop and T.D. DeRose. Generalized B-spline surfaces of arbitrary topology. *Comput. Graph.*, 24:347–356, 1990.

[Lod92] S. Lodha. *Surface Approximation by Low Degree Patches with Multiple Representations.* Ph.D. thesis, Purdue Univ., West Lafayette, 1992.

[LS07] M.-J. Lai and L.L. Schumaker. *Spline Functions on Triangulations*. Vol. 110 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, 2007.

[Mal92] J.L. Mallet. Discrete smooth interpolation in geometric modelling. *Comput. Aided Design*, 24:178–191, 1992.

[MBW+02] K. Museth, D. Breen, R. Whitaker, and A.H. Barr. Level set surface editing operators. *ACM Trans. Graph.*, 21:330–338, 2002.

[MKB+08] S. Martin, P. Kaufmann, M. Botsch, M. Wicke, and M. Gross. Polyhedral finite elements using harmonic basis functions. In *Proc. Eurographics Sympos. Geom. Process.*, pages 1521–1529, 2008.

[MLL+92] S. Mann, C. Loop, M. Lounsbery, D. Meyers, J. Painter, T.D. DeRose, and K. Sloan. A survey of parametric scattered data fitting using triangular interpolants. In H. Hagen, editor, *Curve and Surface Modeling*, pages 145–172, SIAM, Philadelphia, 1992.

[Mou10] B. Mourrain. On the dimension of spline spaces on planar t-meshes. *Math. Comput.*, 83:847–871, 2010.

[MP08] P. Milbradt and T. Pick. Polytope finite elements. *Int. J. Numer. Methods Eng.*, 73:1811–1835, 2008.

[MS92] H.P. Moreton and C.H. Séquin. Functional optimization for fair surface design. *ACM Comput. Graph.*, 26:167–176 , 1992.

[MW91] D. Moore and J. Warren. Approximation of dense scattered data using algebraic surfaces. In *Proc. 24th Hawaii Int. Conf. System Sci.*, pages 681–690, IEEE, 1991.

[Néd80] J.-C. Nédélec. Mixed finite elements in $\mathbf{R}^3$. *Numer. Math.*, 35:315–341, 1980.

[Néd86] J.-C. Nédélec. A new family of mixed finite elements in $\mathbb{R}^3$. *Numer. Math.*, 50:57–81, 1986.

[OBB00] Y. Ohtake, A.G. Belyaev, and I.A. Bogaevski. Polyhedral surface smoothing with simultaneous mesh regularization. In *Proc. Geom. Modeling Process.*, pages 229–237, 2000.

[OF03] S.J. Osher and R. Fedkiw. *Level Set Methods and Dynamic Implicit Surfaces.* Vol. 153 of *Applied Mathematical Sciences*, Springer-Verlag, New York, 2003.

[PB88] J. Platt and A.H. Barr. Constraint methods for flexible models. *SIGGRAPH Comput. Graph.*, 22:279–288, 1988.

[Pet90a] J. Peters. Local cubic and bicubic C^1 surface interpolation with linearly varying boundary normal. *Comput. Aided Geom. Design*, 7:499–516, 1990.

[Pet90b] J. Peters. Smooth mesh interpolation with cubic patches. *Comput. Aided Design*, 22:109–120, 1990.

[Pet91] J. Peters. Smooth interpolation of a mesh of curves. *Constr. Approx.*, 7:221–246, 1991.

[Pet93] J. Peters. Smooth free-form surfaces over irregular meshes generalizing quadratic splines. *Comput. Aided Geom. Design*, 10:347–361, 1993.

[Pet02] J. Peters. C^2 free-form surfaces of degree (3,5). *Comput. Aided Geom. Design*, 19:113–126, 2002.

[PK89] N.M. Patrikalakis and G.A. Kriezis. Representation of piecewise continuous algebraic surfaces in terms of b-splines. *Visual Comput.*, 5:360–374, 1989.

[PM87] P. Perona and J. Malik. Scale space and edge detection using anisotropic diffusion. In *IEEE Comput. Soc. Workshop Comput. Vision*, 1987.

[PR99] T. Preußer and M. Rumpf. An adaptive finite element method for large scale image processing. In *Scale-Space Theories in Computer Vision*, pages 232–234, 1999.

[PR08] J. Peters and U. Reif. *Subdivision Surfaces.* Springer, Berlin, 2008.

[Rei92] U. Reif. A unified approach to subdivision algorithms. Tech. Rep. 92-16, Mathematisches Institut A, Universität Stuttgart, 1992.

[RGB13] A. Rand, A. Gillette, and C. Bajaj. Interpolation error estimates for mean value coordinates. *Adv. Comput. Math.*, 39:327–347, 2013.

[RGB14] A. Rand, A. Gillette, and C. Bajaj. Quadratic serendipity finite elements on polygons using generalized barycentric coordinates. *Math. Comput.*, 83:2691–2716, 2014.

[RS06] M. Rashid and M. Selimotic. A three-dimensional finite element method with arbitrary polyhedral elements. *Int. J. Numer. Methods Eng.*, 67:226–252, 2006.

[RT77] P.-A. Raviart and J.M. Thomas. A mixed finite element method for 2nd order elliptic problems. In *Mathematical Aspects of Finite Element Methods*, pages 292–315, vol. 606 of *Lecture Notes in Math.*, Springer, Berlin, 1977.

[Sap94] N. Sapidis. *Designing Fair Curves and Surfaces.* SIAM, Philadelphia, 1994.

[Sar87] R.F. Sarraga. G^1 interpolation of generally unrestricted cubic Bézier curves. *Comput. Aided Geom. Design*, 4:23–39, 1987.

[Sch94] L.L. Schumaker. Applications of multivariate splines. In *Math. Comput., 1943–1993: A Half–century of Computations Mathematics*, vol. 48 of *Proc. Symposia Appl. Math.*, AMS, Providence, 1994.

[SDS96] E.J. Stollnitz, T.D. DeRose, and D. Salesin. *Wavelets for Computer Graphics: Theory and Applications.* Morgan Kaufmann, San Francisco, 1996.

[Sed85] T.W. Sederberg. Piecewise algebraic patches. *Comput. Aided Geom. Design*, 2:53–59, 1985.

[Sei89] H.-P. Seidel. A new multiaffine approach to B-splines. *Comput. Aided Geom. Design*, 6:23–32, 1989.

[Sib80] R. Sibson. A vector identity for the Dirichlet tessellation. *Math. Proc. Cambridge Philos. Soc.*, 87:151–155, 1980.

[SK00] R. Schneider and L. Kobbelt. Generating fair meshes with G^1 boundary conditions. In *Geometric Modeling Processing*, pages 251–261, 2000.

[SK01] R. Schneider and L. Kobbelt. Geometric fairing of triangular meshes for free-form surface design. *Comput. Aided Geom. Design*, 18:359–379, 2001.

[SM06] N. Sukumar and E.A. Malsch. Recent advances in the construction of polygonal finite element interpolants. *Archives Comput. Methods. Eng.*, 13:129–163, 2006.

[SS95] P. Schröder and W. Sweldens. Spherical wavelets: Efficiently representing functions on the sphere. *Comput. Graphics*, 29:161–172, 1995.

[ST04] N. Sukumar and A. Tabarraei. Conforming polygonal finite elements. *Int. J. Numer. Methods Eng.*, 61:2045–2066, 2004.

[SZB+03] T.W. Sederberg, J. Zheng, A. Bakenov, and A. Nasri. T-splines and T-NURCCs. *ACM Trans. Graph.*, 22:477–484, 2003.

[Tau95] G. Taubin. A signal processing approach to fair surface design. In *Proc. 22nd Conf. Comput. Graph. Interactive Tech.*, pages 351–358, ACM Press, 1995.

[TC90] J. Thingvold and E. Cohen. Physical modeling with B-spline surfaces for interactive design and animation. *Comput. Graphics*, 24:129–137, 1990.

[TF88] D. Terzopoulos and K. Fleischer. Modeling inelastic deformation: Visoelasticity, plasticity, fracture. In *SIGGRAPH Comput. Graph.*, 22:269–278, 1988.

[TPB+87] D. Terzopoulos, J. Platt, A. Barr, and K. Fleischer. Elastically deformable models. *Comput. Graphics*, 21:205–214, 1987.

[TQ94] D. Terzopoulos and H. Qin. Dynamic nurbs with geometric constraints for interactive sculpting. *ACM Trans. Graph.*, 13:103–136, 1994.

[Wac11] E.L. Wachspress. Barycentric coordinates for polytopes. *Comput. Math. Appl.*, 61:3319–3321, 2011.

[Wah90] G. Wahba. *Spline Models for Observational Data.* CBMS-NSF in Applied Mathematics, SIAM, Philadelphia, 1990.

[War92] J. Warren. Creating multisided rational Bézier surfaces using base points. *ACM Trans. Graph.*, 11:127–139, 1992.

[War96] J. Warren. Barycentric coordinates for convex polytopes. *Adv. Comput. Math.*, 6:97–108, 1996.

[WB98] R. Whitaker and D. Breen. Level set models for the deformation of solid objects. In *Proc. 3rd Int. Eurographics Workshop Implicit Surfaces*, pages 19–35, 1998.

[WBG07] M. Wicke, M. Botsch, and M. Gross. A finite element method on convex polyhedra. *Comput. Graphics Forum*, 26:355–364, 2007.

[Wei98] J. Weickert. *Anisotropic Diffusion in Image Processing.* B.G. Teubner, Stuttgart, 1998.

[WFB87] A. Witkin, K. Fleischer, and A.H. Barr. Energy constraints on parameterized models. In *SIGGRAPH Comput. Graph.*, 21:225–232, 1987.

[WJE00] R. Westermann, C. Johnson, and T. Ertl. A level-set method for flow visualization. In *Proc. IEEE Visualization*, pages 147–154, 2000.

[WSH$^+$07] J. Warren, S. Schaefer, A.N. Hirani, and M. Desbrun. Barycentric coordinates for convex sets. *Adv. Comput. Math.*, 27:319–338, 2007.

[WW92] W. Welch and A. Witkin. Variational surface modeling. *Comput. Graph.*, 26:157–166, 1992.

[WW94] W. Welch and A. Witkin. Free-form shape design using triangulated surfaces. In *Proc. 21st Conf. Comput. Graph. Interactive Tech.*, pages 247–256, 1994.

[WW02] J. Warren and H. Weimer. *Subdivision Methods for Geometric Design: A Constructive Approach.* Morgan Kaufmann, San Francisco, 2002.

[XB00] G. Xu and C.L. Bajaj. Regular algebraic curve segments (I)—Definitions and characteristics. *Comput. Aided Geom. Des.*, 17:485–501, 2000.

[XBE01] G. Xu, C.L. Bajaj, and S. Evans. C^1 modeling with hybrid multiple-sided A-patches. *Internat. J. Found. Comput. Sci.*, 13:261–284, 2001.

[ZOF01] H.-K. Zhao, S. Osher, and R. Fedkiw. Fast surface reconstruction using the level set method. In *Proc. IEEE Workshop Variational Level Set Methods Comput. Vis.*, 2001.

[ZOM$^+$00] H.K. Zhao, S.J. Osher, B. Merriman, and M. Kang. Implicit and non-parametric shape reconstruction from unorganized points using variational level set method. *Comput. Vision Graph. Image Understanding*, 80:295–319, 2000.

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