33 VISIBILITY

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INTRODUCTION

In a geometric context, two objects are "visible" to each other if there is a line segment connecting them that does not cross any obstacles. Over 500 papers have been published on aspects of visibility in computational geometry in the last 40 years. The research can be broadly classified as primarily focused on combinatorial issues, or primarily focused on algorithms. We partition the combinatorial work into "art gallery theorems" (Section 33.1) and illumination of convex sets (33.2), and research on visibility graphs (33.3) and the algorithmic work into that concerned with polygons (33.4), more general planar environments (33.5) paths (33.6), and mirror reflections (33.7). All of this work concerns visibility in two dimensions. Investigations in three dimensions, both combinatorial and algorithmic, are discussed in Section 33.8, and the final section (33.9) touches on visibility in $\mathbb{R}^d$.

33.1 ART GALLERY THEOREMS

A typical "art gallery theorem" provides combinatorial bounds on the number of guards needed to visually cover a polygonal region P (the art gallery) defined by n vertices. Equivalently, one can imagine light bulbs instead of guards and require full direct-light illumination.

GLOSSARY

Guard: A point, a line segment, or a line—a source of visibility or illumination.

Vertex guard: A guard at a polygon vertex.

Point guard: A guard at an arbitrary point.

Interior visibility: A guard $x \in P$ can see a point $y \in P$ if the segment xy is nowhere exterior to P: $xy \subseteq P$.

Exterior visibility: A guard x can see a point y outside of P if the segment xy is nowhere interior to P; xy may intersect ∂P, the boundary of P.

Star polygon: A polygon visible from a single interior point.

Diagonal: A segment inside a polygon whose endpoints are vertices, and which otherwise does not touch ∂P.

Floodlight: A light that illuminates from the apex of a cone with aperture α.

Vertex floodlight: One whose apex is at a vertex (at most one per vertex).

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MAIN RESULTS

The most general combinatorial results obtained to date are summarized in Table 33.1.1. Tight bounds, and ranges between lower and upper bounds are listed for the minimum number of guards sufficient for all polygons with n vertices (and possibly h holes).

TABLE 33.1.1 Number of guards needed.

PROBLEM NAME	POLYGONS	INT/EXT	GUARD	NUMBER
Art gallery theorem	simple	interior	vertex	$\lfloor n/3 \rfloor$
Fortress problem	simple	exterior	point	$\lceil n/3 \rceil$
Prison yard problem	simple	int & ext	vertex	$\lceil n/2 \rceil$
Prison yard problem	orthogonal	int & ext	vertex	$[\lceil 5n/16 \rceil, \lfloor 5n/12 \rfloor + 1]$
Orthogonal polygons	simple orthogonal	interior	vertex	$\lfloor n/4 \rfloor$
Orthogonal with holes	orthogonal with h holes	interior	vertex	$[\lfloor 2n/7 \rfloor, \lfloor (17n-8)/52 \rfloor]$
Orthogonal with holes	orthogonal with h holes	interior	vertex	$[\lfloor (n+h)/4 \rfloor, \lfloor (n+2h)/4 \rfloor]$
Polygons with holes	polygons with h holes	interior	point	$\lfloor (n+h)/3 \rfloor$

Of special note is the difficult ***orthogonal prison yard problem:*** How many vertex guards are needed to cover both the interior and the exterior of an orthogonal polygon? See Figure 33.1.1. An upper bound of $\lfloor 5n/12 \rfloor + 2$ was obtained by [HK96] via the following graph-coloring theorem: Every plane, bipartite, 2-connected graph has an ***even triangulation*** (all nodes have even degree) and therefore the resulting graph is 3-colorable. This bound was subsequently improved to $\lfloor 5n/12 \rfloor + 1$ in [MP12].

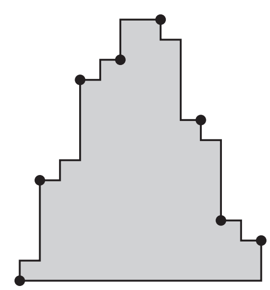

FIGURE 33.1.1
A pyramid polygon with $n = 24$ vertices whose interior and exterior are covered by 8 guards. Repeating the pattern establishes a lower bound of $5n/16 + c$ on the orthogonal prison yard problem [HK93].

COVERS AND PARTITIONS

Each art gallery theorem above implies a cover result, a cover by star polygons. Many of the theorem proofs rely on particular partitions. For example, the orthogonal polygon result depends on the theorem that every orthogonal polygon may be partitioned via diagonals into convex quadrilaterals.

Most cover problems are NP-hard, and finding a minimum guard set for a simple polygon is NP-complete. Approximation algorithms have only achieved $O(\log n)$ times the fewest guards [Gho10]. See Section 30.2 for more on covers and partitions.

EDGE GUARDS

A variation permits guards (***mobile guards***) to patrol segments, diagonals, or edges; equivalent is illumination by line segment/diagonal/edge light sources (fluorescent light bulbs). Here there are fewer results; see Table 33.1.2. Toussaint conjectures that the last line of this table should be $\lfloor n/4 \rfloor$ for sufficiently large n.

TABLE 33.1.2 Edge guards.

POLYGONS	GUARD	BOUNDS	SOURCE
Polygon	diagonal	$\lfloor n/4 \rfloor$	[O'R83]
Orthogonal polygons	segment	$\lfloor (3n+4)/16 \rfloor$	[Agg84, O'R87]
Orthogonal polygons with h holes	segment	$\lfloor (3n+4h+4)/16 \rfloor$	[GHKS96]
Polygon ($n > 11$)	closed edge	$[\lfloor n/4 \rfloor, \lfloor 3n/10 \rfloor + 1]$	[She94]
Polygon	open edge	$[\lfloor n/3 \rfloor, \lfloor n/2 \rfloor]$	[TTW12]

33.1.1 1.5D TERRAIN GUARDING

A 1.5D terrain is an x-monotone chain of edges, guarded by points on the chain. Guarding such a terrain has application to placing communication devices to cover the terrain. Although known to be NP-hard [KK11], it required further work to find a polynomial discretization, thereby establishing its NP-completeness [FHKS16].

TABLE 33.1.3 Floodlights.

APEX	ALPHA	BOUNDS	SOURCE
Any point	$[180°, 360°]$	$\lfloor n/3 \rfloor$	[Tót00]
Any point	$[90°, 180°]$	$2\lfloor n/3 \rfloor$	[Tót00]
Any point	$[45°, 60°)$	$[n-2, n-1]$	[Tót03d]
Vertex	$< 180°$	not always possible	[ECOUX95]
Vertex	$180°$	$[9n/14 - c, \lfloor 2n/3 \rfloor - 1]$	[ST05]

33.1.2 FLOODLIGHT ILLUMINATION

Urrutia introduced a class of questions involving guards with restricted vision, or, equivalently, illumination by floodlights: How many floodlights, each with aperture α, and with their apexes at distinct nonexterior points, are sufficient to cover any polygon of n vertices? One surprise is that $\lfloor n/3 \rfloor$ half-guards/π-floodlights suffice, although not when restricted to vertices. A second surprise is that, for any $\alpha < \pi$, there is a polygon that cannot be illuminated by an α floodlight at every vertex. See Table 33.1.3. A third surprise is that the best result on vertex π-floodlights employs pointed pseudotriangulations (cf. Chapter 5) in an essential way.

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33.2 ILLUMINATION OF PLANAR CONVEX SETS

A natural extension of exterior visibility is illumination of the plane in the presence of obstacles. Here it is natural to use "illumination" in the same sense as "visibility." Under this model, results depend on whether light sources are permitted to lie on obstacle boundaries: $\lfloor 2n/3 \rfloor$ lights are necessary and sufficient (for $n > 5$) if they may [O'R87], and $\lfloor 2(n+1)/3 \rfloor$ if they may not [Tót02]. More work has been done on illuminating the boundary of the obstacles, under a stronger notion of illumination, corresponding to "clear visibility."

GLOSSARY

Illuminate: x illuminates y if xy does not include a point strictly interior to an obstacle, and does not cross a segment obstacle.

Cross: xy crosses segment s if they have exactly one point p in common, and p is in the relative interior of both xy and s.

Clearly illuminate: x clearly illuminates y if the open segment (x, y) does not include any point of an obstacle.

Compact: Closed and bounded in $\mathbb{R}^d$.

Homothetic: Obtained by dilation and translation.

Isothetic: Sides parallel to orthogonal coordinate axes.

MAIN RESULTS

A third, even stronger notion of illumination is considered in Section 33.9 below. The main question that has been investigated is: How many point lights strictly exterior to a collection of n pairwise disjoint compact, convex objects in the plane are needed to clearly illuminate every object boundary point? Answers for a variety of restricted sets are shown in Table 33.2.1.

TABLE 33.2.1 Illuminating convex sets in plane.

FAMILY	BOUNDS	SOURCE
Convex sets	$4n - 7$	[Fej77]
Circular disks	$2n - 2$	[Fej77]
Isothetic rectangles	$[n - 1, n + 1]$	[Urr00]
Homothetic triangles	$[n, n + 1]$	[CRCU93]
Triangles	$[n, \lfloor (5n + 1)/4 \rfloor]$	[Tót03b]
Segments (one side)	$[4n/9 - 2, \lfloor (n + 1)/2 \rfloor]$	[CRC$^+$95, Tót03c]
Segments (both sides)	$\lfloor 4(n + 1)/5 \rfloor$	[Tót01]

The most interesting open problem here is to close the gap for triangles. Urrutia conjectures [Urr00] that $n + c$ lights suffice for some constant c.

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33.3 VISIBILITY GRAPHS

Whereas art gallery theorems seek to encapsulate an environment's visibility into one function of n, the study of visibility graphs endeavors to uncover the more fine-grained structure of visibility. The original impetus for their investigation came from pattern recognition, and its connection to shape continues to be one of its primary sources of motivation; see Chapter 54. Another application is graphics (Chapter 52): illumination and radiosity depend on 3D visibility relations (Section 33.8.)

GLOSSARY

Visibility graph: A graph with a node for each object, and arcs between objects that can see one another.

Vertex visibility graph: The objects are the vertices of a simple polygon.

Endpoint visibility graph: The objects are the endpoints of line segments in the plane. See Figure 33.3.1b.

Segment visibility graph: The objects are whole line segments in the plane, either open or closed.

Object visibility: Two objects A and B are visible to one another if there are points $x \in A$ and $y \in B$ such that x sees y.

Point visibility: Two points x and y can see one another if the segment xy is not "obstructed," where the meaning of "obstruction" depends on the problem.

ϵ-visibility: Lines of sight are finite-width beams of visibility.

Hamiltonian: A graph is Hamiltonian if there is a simple cycle that includes every node.

OBSTRUCTIONS TO VISIBILITY

For polygon vertices, x sees y if xy is nowhere exterior to the polygon, just as in art gallery visibility; this implies that polygon edges are part of the visibility graph. For segment endpoints, x sees y if the closed segment xy intersects the union of all the segments either in just the two endpoints, or in the entire closed segment. This disallows grazing contact with a segment, but includes the segments themselves in the graph.

GOALS

Four goals can be discerned in research on visibility graphs:

1. Characterization: asks for a precise delimiting of the class of graphs realizable by a certain class of geometric objects.

2. Recognition: asks for an algorithm to recognize when a graph is a visibility graph.

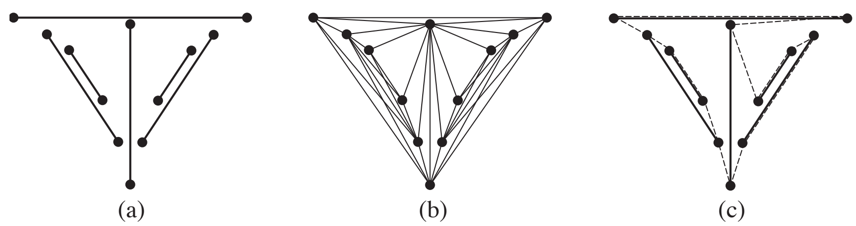

FIGURE 33.3.1
(a) *A set of 6 pairwise disjoint line segments.* (b) *Their endpoint visibility graph G.* (c) *A Hamiltonian cycle in G.*

3. Reconstruction: asks for an algorithm that will take a visibility graph as input, and output a geometric realization.

4. Counting: concerned with the number of visibility graphs under various restrictions [HN01].

POINT VISIBILITY GRAPHS

Given a set P of n points in the plane, visibility between $x, y \in P$ may be blocked by a third point in P. The recognition of point visibility graph is NP-hard [Roy16], in fact it is complete for the existential theory of the reals [CH17]. However, for planar graphs, there is complete characterization, and an $O(n)$-time recognition algorithm [GR15]. Pfender [Pfe08] constructed point visibility graphs of clique number 6 and arbitrary high chromatic number.

For example, it was established in [PPVW12] that every visibility graph with minimum degree δ has vertex connectivity of at least $\delta/2+1$, and if the number of collinear points is no more than 4, then G has connectivity of at least $2\delta/3+1$. This later quantity is conjectured to hold without the collinearity restriction. Related Ramsey-type problems and results are surveyed in [PW10].

VERTEX VISIBILITY GRAPHS

A complete characterization of vertex visibility graphs of polygons has remained elusive, but progress has been made by:

1. Restricting the class of polygons: polynomial-time recognition and reconstruction algorithms for orthogonal staircase polygons have been obtained. See Figure 33.3.2.

2. Restricting the class of graphs: every 3-connected vertex visibility graph has a 3-clique ordering, i.e., an ordering of the vertices so that each vertex is part of a triangle composed of preceding vertices.

3. Adding information: assuming knowledge of the boundary Hamiltonian circuit, four necessary conditions have been established by Ghosh and others [Gho97], and conjectured to be sufficient.

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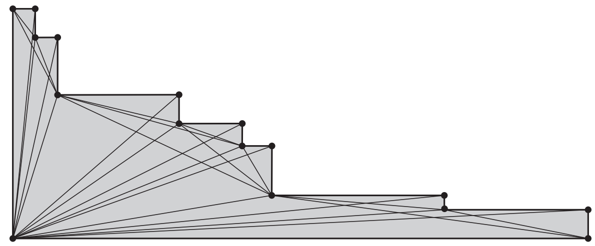

FIGURE 33.3.2
A staircase polygon and its vertex visibility graph.

ENDPOINT VISIBILITY GRAPHS

A set of n pairwise disjoint line segments forms a noncrossing perfect matching on the $2n$ endpoints in the plane. For segment endpoint visibility graphs, there have been three foci:

1. Are the graphs Hamiltonian? See Figure 33.3.1c. Posed by Mirzaian, this was settled in the affirmative [HT03]: YES, there is always a Hamiltonian polygon (i.e., a noncrossing circuit) for pairwise disjoint line segments, not all lying on a line.

2. In the quest for generating a *random* noncrossing perfect matching, Aichholzer et al. [ABD$^+$09] conjecture that any two such matchings are connected by sequence of noncrossing perfect matchings in which consecutive matching are *compatible* (the union of the two matchings is also noncrossing). Every matching on $4n$ vertices is known to have a compatible matching [IST13].

3. Size questions: there must be at least $5n - 4$ edges [SE87], and at least $6n-6$ when no segment is a "chord" splitting the convex hull [GOH$^+$02]; the smallest clique cover has size $\Omega(n^2/\log^2 n)$ [AAAS94].

SEGMENT VISIBILITY GRAPHS

Whole segment visibility graphs have been investigated most thoroughly under the restriction that the segments are all (say) vertical and visibility is horizontal. Such segments are often called ***bars***. The visibility is usually required to be ϵ-visibility. Endpoints on the same horizontal line often play an important role here, as does the distinction between closed segments and intervals (which may or may not include their endpoints). There are several characterizations:

1. G is representable by segments, with no two endpoints on the same horizontal line, iff there is a planar embedding of G such that, for every interior k-face F, the induced subgraph of F has exactly $2k - 3$ edges.

2. G is representable by segments, with endpoints on the same horizontal permitted, iff there is a planar embedding of G with all cutpoints on the exterior face.

3. Every 3-connected planar graph is representable by intervals.

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OBSTACLE NUMBER

An interesting variant of visibility graphs has drawn considerable attention. Given a graph G, an *obstacle representation* of G is a mapping of its nodes to the plane such that edge (x, y) is in G if and only if the segment xy does not intersect any "obstacle." An obstacle is any connected subset of $\mathbb{R}^2$. The obstacle number of G is the minimum number of obstacles in an obstacle representation of G. At least one obstacle is needed to represent any graph other than the complete graph. There are graphs with obstacle number $\Omega(n/(\log\log n)^2)$ [DM13]. No upper bound better than $O(n^2)$ is known.

When the obstacles are points and G is the empty graph on n vertices, this quantity is known as the *blocking number* $b(n)$; see [Mat09, PW10]. It is conjectured that $\lim_{n\to\infty} b(n)/n = \infty$, but the best bound is $b(n) \geq (25/8 - o(1))n$ [DPT09].

INVISIBILITY GRAPHS

For a set $X \subseteq \mathbb{R}^d$, its *invisibility graph* $\mathcal{I}(\mathcal{X})$ has a vertex for each point in X, and an edge between two vertices u and v if the segment uv is not completely contained in X. The chromatic number $\chi(X)$ and clique number $\omega(X)$ of $\mathcal{I}(\mathcal{X})$ have been studied, primarily in the context of the covering number, the fewest convex sets whose union is X. It is clear that $\omega(X) \leq \chi(X)$, and it was conjectured in [MV99] that for planar sets X, there is no upper bound on χ as a function of ω. This conjecture was settled positively in [CKM$^+$10].

OTHER VISIBILITY GRAPHS

The notion of a visibility graph can be extended to objects such as disjoint disks: each disk is a node, with an arc if there is a segment connecting them that avoids touching any other disk. Rappaport proved that the visibility graph of disjoint congruent disks is Hamiltonian [Rap03]. ***Rectangle visibility graphs***, which restrict visibility to vertical or horizontal lines of sight between disjoint rectangles, have been studied for their role in graph drawing (Chapter 55). A typical result is that any graph with a maximum vertex degree 4 can be realized as a rectangle visibility graph [BDHS97].

OPEN PROBLEMS

Ghosh and Goswami list 44 open problems on visibility graphs in their survey [GG13]. Below we list just three.

1. Given a visibility graph G and a Hamiltonian circuit C, construct in polynomial time a simple polygon such that its vertex visibility graph is G, with C corresponding to the polygon's boundary.
2. Given a visibility graph G of a simple polygon P, find the Hamiltonian cycle that corresponds to the boundary of P.
3. Develop an algorithm to recognize whether a polygon vertex visibility graph is planar. Necessary and sufficient conditions are known [LC94].

33.4 ALGORITHMS FOR VISIBILITY IN A POLYGON

Designing algorithms to compute aspects of visibility in a polygon P was a major focus of the computational geometry community in the 1980s. For most of the basic problems, optimal algorithms were found, several depending on Chazelle's linear-time triangulation algorithm [Cha91]. See [Gho07] for a book-length survey.

GLOSSARY

Throughout, P is a simple polygon.

Kernel: The set of points in P that can see all of P. See Figure 33.4.4.

Point visibility polygon: The region visible from a point in P.

Segment visibility polygon: The region visible from a segment in P.

MAIN RESULTS

The main algorithms are listed in Table 33.4.1. We discuss two of these algorithms below to illustrate their flavor.

TABLE 33.4.1 Polygon visibility algorithms.

ALGORITHM TO COMPUTE	TIME COMPLEXITY	SOURCE
Kernel	$O(n)$	[LP79]
Point visibility polygon	$O(n)$	[JS87]
Segment visibility polygon	$O(n)$	[GHL+87]
Shortest illuminating segment	$O(n)$	[DN94]
Vertex visibility graph	$O(E)$	[Her89]

VISIBILITY POLYGON ALGORITHM

Let $x \in P$ be the visibility source. Lee's linear-time algorithm [JS87] processes the vertices of P in a single counterclockwise boundary traversal. At each step, a vertex is either pushed on or popped off a stack, or a *wait* event is processed. The latter occurs when the boundary at that point is invisible from x. At any stage, the stack represents the visible portion of the boundary processed so far.

Although this algorithm is elementary in its tools, it has proved delicate to implement correctly.

VISIBILITY GRAPH ALGORITHM

In contrast, Hershberger's vertex visibility algorithm [Her89] uses sophisticated tools to achieve output-size sensitive time complexity $O(E)$, where E is the number of edges of the graph. His algorithm exploits the intimate connection between shortest paths and visibility in polygons. It first computes the *shortest path map* (Chapter 31) in $O(n)$ time for a vertex, and then systematically transforms this into the map of an adjacent vertex in time proportional to the number of changes. Repeating this achieves $O(E)$ time overall.

Most of the above algorithms have been parallelized; see, for example, [GSG92].

33.5 ALGORITHMS FOR VISIBILITY AMONG OBSTACLES

The shortest path between two points in an environment of polygonal obstacles follows lines of sight between obstacle vertices. This has provided an impetus for developing efficient algorithms for constructing visibility regions and graphs in such settings. The obstacles most studied are noncrossing line segments, which can be joined end-to-end to form polygonal obstacles. Many of the questions mentioned in the previous section can be revisited for this environment.

The major results are shown in Table 33.5.1; the first three are described in [O'R87]; the fourth is discussed below.

TABLE 33.5.1 Algorithms for visibility among obstacles.

ALGORITHM TO COMPUTE	TIME COMPLEXITY
Point visibility region	$O(n \log n)$
Segment visibility region	$\Theta(n^4)$
Endpoint visibility graph	$O(n^2)$
Endpoint visibility graph	$O(n \log n + E)$

ENDPOINT VISIBILITY GRAPH

The largest effort has concentrated on constructing the endpoint visibility graph. Worst-case optimal algorithms were first discovered by constructing the line arrangement dual to the endpoints in $O(n^2)$ time. Since many visibility graphs have less than a quadratic number of edges, an output-size sensitive algorithm was a significant improvement: $O(n \log n + E)$ where E is the number of edges of the graph [GM91]. This was further improved to $O(n + h \log h + E)$ for h polygonal obstacles with a total of n vertices [DW15].

A polygon with n vertices and h holes can be preprocessed into a data structure with space complexity $O(\min(|E|, nh) + n)$ in time $O(E + n \log n)$, and can report the visibility polygon $V(q)$ of a query point q in time $O(|V(q)| \log n + h)$ [IK09].

33.6 VISIBILITY PATHS

A fruitful idea was introduced to visibility research in the mid-1980s: the notion of "link distance" between two points, which represents the smallest number of mutually visible relay stations needed to communicate from one point to another; see Section 31.3. A related notion called "watchman tours" was introduced a bit later, mixing shortest paths and visibility problems, and employing many of the concepts developed for link-path problems (Section 31.4).

33.7 MIRROR REFLECTIONS

GLOSSARY

Light ray reflection: A light ray reflects from an interior point of a mirror with reflected angle equal to incident angle; a ray that hits a mirror endpoint is absorbed.

Mirror polygon: A polygon all of whose edges are mirrors reflecting light rays.

Periodic light ray: A ray that reflects from a collection of mirrors and, after a finite number of reflections, rejoins its path (and thenceforth repeats that path).

Trapped light ray: One that reflects forever, and so never "reaches" infinity.

Klee asked whether every polygonal room whose walls are mirrors (a mirror polygon) is illuminable from every interior point [Kle69, KW91]. Tokarsky answered NO by constructing rooms that leave one point dark when the light source is located at a particular spot [Tok95]. Complementing Tokarsky's result, it is now known that if P is a *rational polygon* (angles rational multiples of π), for every x there are at most finitely many dark points y in P [LMW16]. However, a second question of Klee remains open: Is every mirror polygon illuminable from *some* interior point?

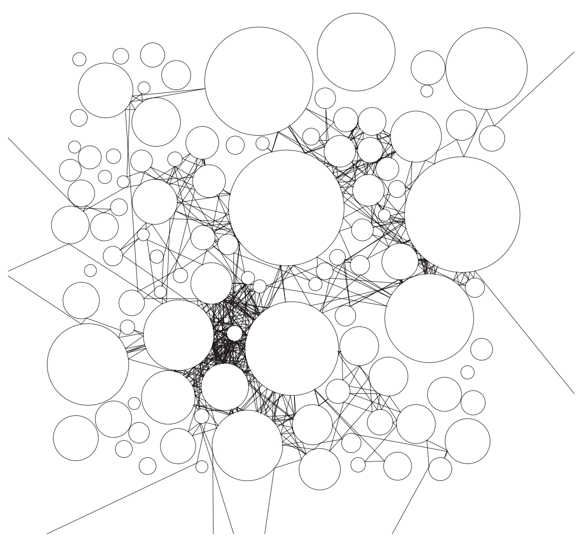

FIGURE 33.7.1
100 *mirror disks fail to trap* 10 *rays from a point source (near the center)* [OP01].

The behavior of light reflecting in a polygon is complex. Aronov et al. [ADD+98] proved that after k reflections, the boundary of the illuminated region has combinatorial complexity $O(n^{2k})$, with a matching lower bound for any fixed k. Even determining whether every triangle supports a periodic ray is unresolved; see [HH00].

Pach asked whether a finite set of disjoint circular mirrors can trap all the rays from a point light source [Ste96]. See Fig. 33.7.1. This and many other related questions [OP01] remain open.

33.8 VISIBILITY IN THREE DIMENSIONS

Research on visibility in three dimensions (3D) has concentrated on three topics: hidden surface removal, polyhedral terrains, and various 3D visibility graphs.

33.8.1 HIDDEN SURFACE REMOVAL

"Hidden surface removal" is one of the key problems in computer graphics (Chapter 52), and has been the focus of intense research for two decades. The typical problem instance is a collection of (planar) polygons in space, from which the view from $z = \infty$ must be constructed. Traditionally, hidden-surface algorithms have been classified as either *image-space* algorithms, exploiting the ultimate need to compute visible colors for image pixels, and *object-space* algorithms, which perform exact computations on object polygons. We only discuss the latter.

The complexity of the output scene can be quadratic in the number of input vertices n. A worst-case optimal $\Theta(n^2)$ algorithm can be achieved by projecting the lines containing each polygon edge to a plane and constructing the resulting arrangement of lines [Dév86, McK87]. More recent work has focused on obtaining output-size sensitive algorithms, whose time complexity depends on the number of vertices k in the output scene (the complexity of the ***visibility map***), which is often less than quadratic in n. See Table 33.8.1 for selected results. In the table, k is the complexity of the ***visibility map***, the "wire-frame" projection of the scene. A notable example is based on careful construction of "visibility maps," which leads, e.g., to a complexity of $O((n+k)\log^2 n)$ for performing hidden surface removal on nonintersecting spheres, where k is the complexity of the output map.

TABLE 33.8.1 Hidden-surface algorithm complexities.

ENVIRONMENT	COMPLEXITY	SOURCE
Isothetic rectangles	$O((n+k)\log n)$	[BO92]
Polyhedral terrain	$O((n+k)\log n \log\log n)$	[RS88]
Nonintersecting polyhedra	$O(n\sqrt{k}\log n)$	[SO92]
	$O(n^{1+\epsilon}\sqrt{k})$	[BHO+94]
	$O(n^{2/3+\epsilon}k^{2/3} + n^{1+\epsilon})$	[AM93]
Arbitrary intersecting spheres	$O(n^{2+\epsilon})$	[AS00]
Nonintersecting spheres	$O(k + n^{3/2}\log n)$	[SO92]
Restricted-intersecting spheres	$O((n+k)\log^2 n)$	[KOS92]

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33.8.2 BINARY SPACE PARTITION TREES

Binary Space Partition (BSP) trees are a popular method of implementing the basic ***painter's algorithm***, which displays objects back-to-front to obtain proper occlusion of front-most surfaces. A ***BSP*** partitions $\mathbb{R}^d$ into empty, open convex sets by hyperplanes in a recursive fashion. A BSP for a set S of n line segments in $\mathbb{R}^2$ is a partition such that all the open regions corresponding to leaf nodes of the tree are empty of points from S: all the segments in S lie along the boundaries of the regions. An example is shown in Fig. 33.8.1. In general, a BSP for S will "cut up" the segments in S, in the sense that a particular $s \in S$ will not lie in the boundary of a single leaf region. In the figure, partitions 1 and 2 both cut segments, but partition 3 does not.

An attractive feature of BSPs is that an implementation to construct them is easy: In $\mathbb{R}^3$, select a polygon, partition all objects by the plane containing it, and recurse. Bounding the size (number of leaves) of BSP trees has been a challenge. The long-standing conjecture that $O(n)$ size in $\mathbb{R}^2$ is achievable was shown to be false. See Table 33.8.2 for selected results.

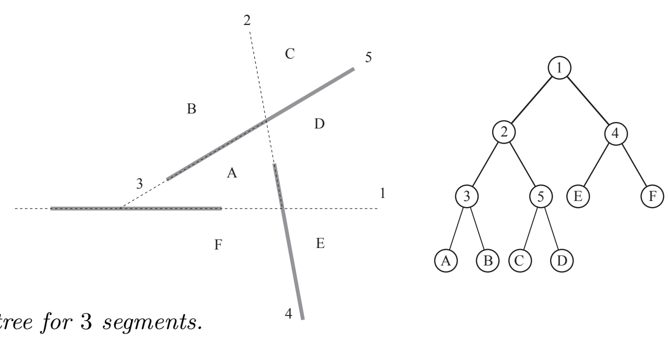


FIGURE 33.8.1
A binary space partition tree for 3 segments.

TABLE 33.8.2 BSP complexities.

DIM	CLASS	BOUND	SOURCE
2	segments	$O(n \log n)$	[PY90]
2	isothetic	$\Theta(n)$	[PY92]
2	fat	$\Theta(n)$	[BGO97]
2	segments	$\Theta(n \log n / \log \log n)$	[Tót03a, Tót11]
3	polyhedra	$O(n^2)$	[PY90]
3	polyhedra	$\Omega(n^2)$	[Cha84]
3	isothetic	$\Theta(n^{3/2})$	[PY92]
3	fat orthog. rects.	$O(n \log^8 n)$	[Tót08]

33.8.3 POLYHEDRAL TERRAINS

Polyhedral terrains are an important special class of 3D surfaces, arising in a variety of applications, most notably geographic information systems (Chapter 59).

GLOSSARY

Polyhedral terrain: A polyhedral surface that intersects every vertical line in at most a single point.

Perspective view: A view from a point.

Orthographic view: A view from infinity (parallel lines of sight).

Ray-shooting query: A query asking which terrain face is first hit by a ray shooting in a given direction from a given point. (See Chapter 41.)

$\alpha(n)$: The inverse Ackermann function (nearly a constant). See Section 28.10.

COMBINATORIAL BOUNDS

Several almost-tight bounds on the maximum number of combinatorially different views of a terrain have been obtained, as listed in Table 33.8.3.

TABLE 33.8.3 Bounds for polyhedral terrains.

VIEW TYPE	BOUND	SOURCE
Along vertical	$O(n^2 2^{\alpha(n)})$	[CS89]
Orthographic	$O(n^{5+\epsilon})$	[AS94]
Perspective	$O(n^{8+\epsilon})$	[AS94]

Bose et al. established that $\lfloor n/2 \rfloor$ vertex guards are sometimes necessary and always sufficient to guard a polyhedral terrain of n vertices [BSTZ97, BKL96].

ALGORITHMS

Algorithms seek to exploit the terrain constraints to improve on the same computations for general polyhedra:

1. To compute the orthographic view from above the terrain:
   time $O((k+n)\log n \log\log n)$, where k is the output size [RS88].

2. To preprocess for $O(\log n)$ ray-shooting queries for rays with origin on a vertical line [BDEG94].

33.8.4 3D VISIBILITY GRAPHS

GLOSSARY

Aspect graph: A graph with a node for each combinatorially distinct view of a collection of polyhedra, with two nodes connected by an arc if the views can be reached directly from one another by a continuous movement of the viewpoint.

Preliminary version (August 6, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

Isothetic: Edges parallel to Cartesian coordinate axes.

Box visibility graph: A graph realizable by disjoint isothetic boxes in 3D with orthogonal visibility.

K_n: The complete graph on n nodes.

There have been three primary motivations for studying visibility graphs of objects in three dimensions.

1. Computer graphics: Useful for accelerating interactive "walkthroughs" of complex polyhedral scenes [TS91], and for radiosity computations [TH93]. See Chapter 52.

2. Computer vision: "Aspect graphs" are used to aid image recognition. The maximum number of nodes in an aspect graph for a polyhedron of n vertices depends on both convexity and the type of view. See Table 33.8.4. Note that the nonconvex bounds are significantly larger than those for terrains.

TABLE 33.8.4 Combinatorial complexity of visibility graphs.

CONVEXITY	ORTHOGRAPHIC	PERSPECTIVE	SOURCE
Convex polyhedron	$\Theta(n^2)$	$\Theta(n^3)$	[PD90]
Nonconvex polyhedron	$\Theta(n^6)$	$\Theta(n^9)$	[GCS91]

3. Combinatorics: It has been shown that K_{22} is realizable by disjoint isothetic rectangles in "$2\frac{1}{2}$D" with vertical visibility (all rectangles are parallel to the xy-plane), but that K_{56} (and therefore all larger complete graphs) cannot be so represented [BEF+93]. It is known that K_{42} is a box visibility graph [BJMO94] but that K_{184} is not [FM99].

33.9 PENETRATING ILLUMINATION OF CONVEX BODIES

A rich vein of problems was initiated by Hadwiger, Levi, Gohberg, and Markus; see [MS99] for the complex history. The problems employ a different notion of exterior illumination, which could be called ***penetrating illumination*** (or perhaps "stabbing"), and focuses on a single convex body in $\mathbb{R}^d$.

GLOSSARY

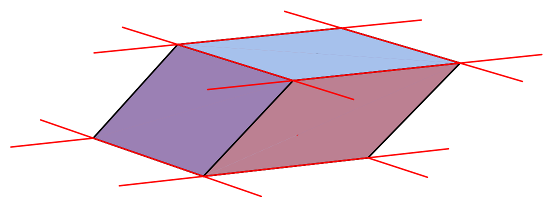
Penetrating illumination: An exterior point x penetratingly illuminates a point y on the boundary ∂K of an object K if the ray from x through y has a non-empty intersection with the interior int K of K.

Direction illumination: A point $y \in \partial K$ is illuminated from direction $\mathbf{v}$ if the ray from the exterior through y with direction $\mathbf{v}$ has a non-empty intersection with int K.

Affine symmetry: An object in $\mathbb{R}^3$ has affine symmetry if it unchanged after reflection through a point, reflection in a plane, or rotation about a line by angle $2\pi/n$, $n = 2, 3, \ldots$.

The central problem may be stated: What is the minimum number of exterior points sufficient to penetratingly illuminate any compact, convex body K in $\mathbb{R}^d$? The problem is only completely solved in 2D: 4 lights are needed for a parallelogram, and 3 for all other convex bodies. In 3D it is known that 8 lights are needed for a parallelepiped (Fig. 33.9.1), and conjectured that 7 suffice for all other convex bodies. Bezdek proved that 8 lights suffice for any 3-polytope with an affine symmetry [Bez93]. Lassak proved that no more than 20 lights are needed for any compact, convex body in 3D [Bol81].

FIGURE 33.9.1
A parallelepiped requires $2^3 = 8$ *lights for penetrating illumination of its boundary.*



One reason for the interest in this problem is its connection to other problems, particularly covering problems. Define:

$I_0(K)$: the minimum number of points sufficient to penetratingly illuminate K.

$I_\infty(K)$: the minimum number of directions sufficient to direction-illuminate K.

$H(K)$: the minimum number of smaller homothetic copies of K that cover K.

$i(K)$: the minimum number of copies of int K that cover K.

Remarkably,

$$I_0(K) = I_\infty(K) = H(K) = i(K) \ ,$$

as established by Boltjanski, Hadwiger, and Soltan; see again [MS99]. Several have conjectured that these quantities are $\leq 2^d$ for compact, convex bodies in $\mathbb{R}^d$, with equality only for the d-parallelotope. The conjecture has been established only for special classes of bodies in 3 and higher dimensions, e.g., [Bol01, Bez11].

33.10 SOURCES AND RELATED MATERIAL

SURVEYS

All results not given an explicit reference above may be traced in these surveys.

[O'R87]: A monograph devoted to art gallery theorems and visibility algorithms.

[She92]: A survey of art gallery theorems and visibility graphs, updating [O'R87].

[O'R92]: A short update to [She92].

[Urr00]: Art gallery results, updating [She92].

[GG13]: Survey of open problems on visibility graphs.

[O'R93]: Survey of visibility graph results.

[Gho07]: Survey of visibility algorithms in $\mathbb{R}^2$.

[MSD00]: Survey of link-distance algorithms.

[Mur99]: A Ph.D. thesis on hidden-surface removal algorithms.

[Tót05]: Survey of binary space partitions.

[MS99, Bez06, BK16]: Surveys of illumination of convex bodies.

RELATED CHAPTERS

Chapter 29: Triangulations and mesh generation
Chapter 30: Polygons
Chapter 31: Shortest paths and networks
Chapter 41: Ray shooting and lines in space
Chapter 42: Geometric intersection
Chapter 52: Computer graphics
Chapter 54: Pattern recognition
Chapter 59: Geographic information systems

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