# 20 POLYHEDRAL MAPS

Ulrich Brehm and Egon Schulte

## INTRODUCTION

Historically, polyhedral maps on surfaces made their first appearance as convex polyhedra. The famous Kepler-Poinsot (star) polyhedra marked the first occurrence of maps on orientable surfaces of higher genus (namely 4), and started the branch of topology dealing with regular maps. Further impetus to the subject came from the theory of automorphic functions and from the Four-Color-Problem (Coxeter and Moser [CM80], Barnette [Bar83]).

A more systematic investigation of general polyhedral maps and nonconvex polyhedra began only around 1970, and was inspired by Grünbaum's book "Convex Polytopes" [Grü67]. Since then, the subject has grown into an active field of research on the interfaces of convex and discrete geometry, graph theory, and combinatorial topology. The underlying topology is mainly elementary, and many basic concepts and constructions are inspired by convex polytope theory.

## 20.1 POLYHEDRA

Tessellations on surfaces are natural objects of study in topology that generalize convex polyhedra and plane tessellations. For general properties of convex polyhedra, polytopes, and tessellations, see Grünbaum [Grü67], Coxeter [Cox73], Grünbaum and Shephard [GS87], and Ziegler [Zie95], or Chapters 3, 15, 16, 17, 18, and 19 of this Handbook. For a survey on polyhedral manifolds see Brehm and Wills [BW93], which also has an extensive list of references. The long list of definitions that follows places polyhedral maps in the general context of topological and geometric complexes. For an account of 2- and 3-dimensional geometric topology, see Moise [Moi77].

### GLOSSARY

***Polyhedral complex:*** A finite set $\Gamma$ of convex polytopes, the ***faces*** of $\Gamma$, in real $n$-space $\mathbb{R}^n$, such that two conditions are satisfied. First, if $Q \in \Gamma$ and $F$ is a face of $Q$, then $F \in \Gamma$. Second, if $Q_1, Q_2 \in \Gamma$, then $Q_1 \cap Q_2$ is a face of $Q_1$ and $Q_2$ (possibly the empty face $\emptyset$). The subset $||\Gamma|| := \bigcup_{Q\in\Gamma} Q$ of $\mathbb{R}^n$, equipped with the induced topology, is called the ***underlying space*** of $\Gamma$. The ***dimension*** $d := \dim \Gamma$ of $\Gamma$ is the maximum of the dimensions (of the affine hulls) of the elements in $\Gamma$. We also call $\Gamma$ a ***polyhedral d-complex***. A face of $\Gamma$ of dimension 0, 1, or $i$ is a ***vertex***, an ***edge***, or an ***i-face*** of $\Gamma$. A face that is maximal (with respect to inclusion) is called a ***facet*** of $\Gamma$. (In our applications, the facets are just the $d$-faces of $\Gamma$.)



Preliminary version (July 16, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

***Face poset:*** The set $P(\Gamma)$ of all faces of $\Gamma$, partially ordered by inclusion. As a partially ordered set, $P(\Gamma) \cup \{||\Gamma||\}$ is a ranked lattice.

***(Geometric) simplicial complex:*** A polyhedral complex $\Gamma$ all of whose nonempty faces are simplices. An ***abstract simplicial complex*** $\Delta$ is a family of subsets of a finite set $V$, the ***vertex set*** of $\Delta$, such that $\{x\} \in \Delta$ for all $x \in V$, and such that $F \subseteq G \in \Delta$ implies $F \in \Delta$. Each abstract simplicial complex $\Delta$ is isomorphic (as a poset ordered by inclusion) to the face poset of a geometric simplicial complex $\Gamma$. Once such an isomorphism is fixed, we set $||\Delta|| := ||\Gamma||$, and the terminology introduced for $\Gamma$ carries over to $\Delta$. (One often omits the qualifications "geometric" or "abstract.")

***Link:*** The link of a vertex $x$ in a simplicial complex $\Gamma$ is the subcomplex consisting of the faces that do not contain $x$ of all the faces of $\Gamma$ containing $x$.

***Polyhedron:*** A subset $P$ of $\mathbb{R}^n$ such that $P = ||\Gamma||$ for some polyhedral complex $\Gamma$. In general, given $P$, there is no canonical way to associate with it the complex $\Gamma$. However, once $\Gamma$ is specified, the terminology for $\Gamma$ regarding $P(\Gamma)$ is also carried over to $P$. (For other meanings of the term "polyhedron" see also Chapter 18.)

***Subdivision:*** If $\Gamma_1$ and $\Gamma_2$ are polyhedral complexes, $\Gamma_1$ is a subdivision of $\Gamma_2$ if $||\Gamma_1|| = ||\Gamma_2||$ and each face of $\Gamma_1$ is a subset of a face of $\Gamma_2$. If $\Gamma_1$ is a simplicial complex, this is a ***simplicial subdivision***.

***Combinatorial d-manifold:*** For $d = 1$, this is a simplicial 1-complex $\Delta$ such that $||\Delta||$ is a 1-sphere. Inductively, if $d \geq 2$, it is a simplicial $d$-complex $\Delta$ such that $||\Delta||$ is a topological $d$-manifold (without boundary) and each vertex link is a ***combinatorial*** $(\boldsymbol{d}\mathbf{-1})$***-sphere*** (that is, a combinatorial $(d-1)$-manifold whose underlying space is a $(d-1)$-sphere).

***Polyhedral d-manifold:*** A polyhedral $d$-complex $\Gamma$ admitting a simplicial subdivision that is a combinatorial $d$-manifold. If $d = 2$, this is simply a polyhedral 2-complex $\Gamma$ for which $||\Gamma||$ is a compact 2-manifold (without boundary).

***Triangulation:*** A triangulation (simplicial decomposition) of a topological space $X$ is a simplicial complex $\Gamma$ such that $X$ and $||\Gamma||$ are homeomorphic.

***Ball complex:*** A finite family $\mathcal{C}$ of topological balls (homeomorphic images of Euclidean unit balls) in a Hausdorff space, the ***underlying space*** $||\mathcal{C}||$ of $\mathcal{C}$, whose relative interiors partition $||\mathcal{C}||$ in such a way that the boundary of each ball in $\mathcal{C}$ is the union of other balls in $\mathcal{C}$. The ***dimension*** of $\mathcal{C}$ is the maximum of the dimensions of the balls in $\mathcal{C}$.

***Embedding:*** For a ball complex $\mathcal{C}$, a continuous mapping $\gamma : ||\mathcal{C}|| \mapsto \mathbb{R}^n$ that is a homeomorphism of $||\mathcal{C}||$ onto its image. $\mathcal{C}$ is said to be ***embedded*** in $\mathbb{R}^n$.

***Polyhedral embedding:*** For a ball complex $\mathcal{C}$, an embedding $\gamma$ that maps each ball in $\mathcal{C}$ onto a convex polytope.

***Immersion:*** For a ball complex $\mathcal{C}$, a continuous mapping $\gamma : ||\mathcal{C}|| \mapsto \mathbb{R}^n$ that is locally injective (hence the image may have self-intersections). $\mathcal{C}$ is said to be ***immersed*** in $\mathbb{R}^n$.

***Polyhedral immersion:*** For a ball complex $\mathcal{C}$, an immersion $\gamma$ that maps each ball in $\mathcal{C}$ onto a convex polytope.

***Map on a surface:*** An embedded finite graph $M$ (without loops or multiple edges) on a compact 2-manifold (surface) $S$ such that two conditions are satisfied: The closures of the connected components of $S \setminus M$, the ***faces*** of $M$, are closed

2-cells (closed topological disks), and each vertex of $M$ has valency at least 3. (Note that some authors use a broader definition of maps; e.g., see [CM80].)

***Polyhedral map:*** A map $M$ on $S$ such that the intersection of any two distinct faces is either empty, a common vertex, or a common edge.

Figure 20.1.1 shows a polyhedral map on a surface of genus 3, known as ***Dyck's regular map***. We will further discuss this map in Sections 20.4 and 20.5.

***Type:*** A map $M$ on $S$ is of type $\{p, q\}$ if all its faces are topological $p$-gons such that $q$ meet at each vertex. The symbol $\{p, q\}$ is the ***Schläfli symbol*** for $M$.

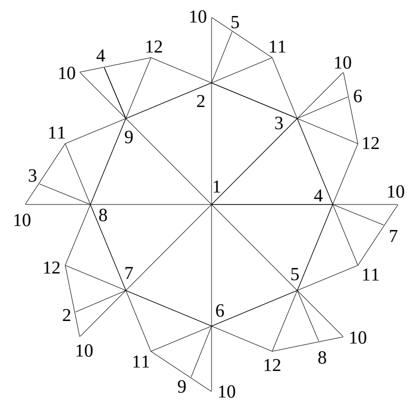


FIGURE 20.1.1
*Dyck's regular map, of type $\{3, 8\}$.*
*Vertices with the same label are identified.*

## BASIC RESULTS

Simplicial complexes are important in topology, geometry, and combinatorics. Each abstract simplicial $d$-complex $\Delta$ with $n$ vertices is isomorphic to the face poset of a geometric simplicial complex $\Gamma$ in $\mathbb{R}^{2d+1}$ that is obtained as the image under a projection (Schlegel diagram—see Chapter 15) of a simplicial $d$-subcomplex in the boundary complex of the cyclic convex $(2d+2)$-polytope $C(n, 2d+2)$ with $n$ vertices; see [Grü67] or Chapter 15 of this Handbook.

Let $\mathcal{C}$ be a ball complex and $P(\mathcal{C})$ the associated poset (i.e., $\mathcal{C}$ ordered by inclusion). Let $\Delta(P(\mathcal{C}))$ denote the ***order complex*** of $P(\mathcal{C})$; that is, the simplicial complex whose vertex set is $\mathcal{C}$ and whose $k$-faces are the $k$-chains $x_0 < x_1 < \ldots < x_k$ in $P(\mathcal{C})$. Then $||\mathcal{C}||$ and $||\Delta(P(\mathcal{C}))||$ are homeomorphic. This means that the poset $P(\mathcal{C})$ already carries complete topological information about $||\mathcal{C}||$. See [Bjö95] or [BW93], as well as Chapter 17, for further information.

Each polyhedral $d$-complex is a $d$-dimensional ball complex. The set $\mathcal{C}$ of vertices, edges, and faces of a map $M$ on a 2-manifold $S$ is a 2-dimensional ball complex. In particular, a map $M$ is a polyhedral map if and only if the intersection of any two elements of $\mathcal{C}$ is empty or an element of $\mathcal{C}$. A map is usually identified with its poset of vertices, edges, and faces, ordered by inclusion. If $M$ is a polyhedral map, then this poset is a lattice when augmented by $\emptyset$ and $S$ as smallest and largest elements. The dual lattice (obtained by reversing the order) again gives a polyhedral map, the ***dual map***, on the same 2-manifold $S$. Note that in the context of polyhedral maps, the qualification "polyhedral" does not mean that it can be realized as a polyhedral complex. However, a polyhedral 2-manifold can always be regarded as a polyhedral map.

An important problem is the following:

**PROBLEM 20.1.1** General Embeddability Problem

*When is a given finite poset isomorphic to the face poset of some polyhedral complex in a given space $\mathbb{R}^n$? When can a ball complex be polyhedrally embedded or polyhedrally immersed in $\mathbb{R}^n$?*

These questions are different from the embeddability problems that are discussed in piecewise-linear topology, because simplicial subdivisions are excluded. A complete answer is available only for the face posets of spherical maps:

**THEOREM 20.1.2** Steinitz's Theorem

*Each polyhedral map $M$ on the 2-sphere is isomorphic to the boundary complex of a convex 3-polytope. Equivalently, a finite graph is the edge graph of a convex 3-polytope if and only if it is planar and 3-connected (it has at least 4 vertices and the removal of any 2 vertices leaves a connected graph).*

Very little is known about polyhedral embeddings of orientable polyhedral maps of positive genus $g$. There are some general necessary combinatorial conditions for the existence of polyhedral embeddings in $n$-space $\mathbb{R}^n$ [BGH91]. Given a simplicial polyhedral map of genus $g$ it is generally difficult to decide whether or not it admits a polyhedral embedding in 3-space $\mathbb{R}^3$. For each $g \geq 6$, there are examples of simplicial polyhedral maps that cannot be embedded in $\mathbb{R}^3$ [BO 00]. Each nonorientable closed surface can be immersed but not embedded in $\mathbb{R}^3$. However, the Möbius strip and therefore each nonorientable surface can be triangulated in such a way that the resulting simplicial polyhedral map cannot be polyhedrally immersed in $\mathbb{R}^3$ [Bre83]. On the other hand, each triangulation of the torus can be polyhedrally embedded in $\mathbb{R}^3$ [ABM08], and each triangulation of the real projective plane $\mathbb{R}\mathrm{P}^2$ can be polyhedrally embedded in $\mathbb{R}^4$ [BS95].

Another important type of problem asks for topological properties of the space of all polyhedral embeddings, or of all convex $d$-polytopes, with a given face lattice. This is the ***realization space*** for this face lattice. Every convex 3-polytope has an open ball as its realization space. However, the realization spaces of convex 4-polytopes can be arbitrarily complicated; see the "Universality Theorem" by Richter-Gebert [Ric96] in Chapter 15 of this Handbook.

For further embeddability results in higher dimensions, as well as for a discussion of some related problems such as the polytopality problems and isotopy problems, see [Zie95, BLS$^+$93, BW93]. For a computational approach to the embeddability problem in terms of oriented matroids, see Bokowski and Sturmfels [BS89], as well as Chapter 6 of this Handbook. We shall revisit the embeddability problem in Sections 20.2 and 20.5 for interesting special classes of polyhedral maps.

Many interesting maps $M$ on compact surfaces $S$ have a Schläfli symbol $\{p, q\}$; for examples, see Section 20.4. These maps can then be obtained from the regular tessellation $\{p, q\}$ of the 2-sphere, the Euclidean plane, or the hyperbolic plane by making identifications. Trivially, $qf_0 = 2f_1 = pf_2$, where $f_0, f_1, f_2$ are the numbers of vertices, edges, and faces of $M$, respectively. Also, if the Euler characteristic $\chi$ of $S$ is negative and $m$ denotes the number of flags (incident triples consisting of a vertex, an edge, and a face) of $M$, then

$$\chi = f_0 - f_1 + f_2 = \frac{m}{2}\left(\frac{1}{q} - \frac{1}{2} + \frac{1}{p}\right) \leq -\frac{m}{84}\,, \qquad (20.1.1)$$

and equality holds on the right-hand side if and only if $M$ is of type $\{3, 7\}$ or $\{7, 3\}$.

## 20.2 EXTREMAL PROPERTIES

There is a natural interest in polyhedral maps and polyhedra defined by certain minimality properties. For relations with the famous Map Color Theorem, which gives the minimum genus of a surface on which the complete graph $K_n$ can be embedded, see Ringel [Rin74], Barnette [Bar83], Gross and Tucker [GT87], and Mohar and Thomassen [MT01]. See also Brehm and Wills [BW93].

### GLOSSARY

***f-vector:*** For a map $M$, the vector $f(M) = (f_0, f_1, f_2)$, where $f_0, f_1, f_2$ are the numbers of vertices, edges, and faces of $M$, respectively.

***Weakly neighborly:*** A polyhedral map is weakly neighborly (a ***wnp map***) if any two vertices lie in a common face.

***Neighborly:*** A map is neighborly if any two vertices are joined by an edge.

***Nonconvex vertex:*** A vertex $x$ of a polyhedral 2-manifold $M$ in $\mathbb{R}^3$ is a ***convex vertex*** if at least one of the two components into which $M$ divides a small convex neighborhood of $x$ in $\mathbb{R}^3$ is convex; otherwise, $x$ is nonconvex.

***Tight polyhedral 2-manifold:*** A polyhedral 2-manifold $M$ embedded in $\mathbb{R}^3$ such that every hyperplane strictly supporting $M$ locally at a point supports $M$ globally.

### BASIC RESULTS

#### THEOREM 20.2.1

*Let $M$ be a polyhedral map of Euler characteristic $\chi$ with $f$-vector $(f_0, f_1, f_2)$. Then*

$$f_0 \geq \lceil (7 + \sqrt{49 - 24\chi})/2 \rceil. \tag{20.2.1}$$

Here, $\lceil t \rceil$ denotes the smallest integer greater than or equal to $t$. This lower bound is known as the ***Heawood bound*** and is an easy consequence of Euler's formula $f_0 - f_1 + f_2 = \chi$ ($= 2 - 2g$ if $M$ is orientable of genus $g$).

#### THEOREM 20.2.2

*Except for the nonorientable 2-manifolds with $\chi = 0$ (Klein bottle) or $\chi = -1$ and the orientable 2-manifold of genus $g = 2$ ($\chi = -2$), each 2-manifold admits a triangulation for which the lower bound* (20.2.1) *is attained.*

This is closely related to the Map Color Theorem. The same lower bound (20.2.1) holds for the number $f_2$ of faces of $M$, since the dual of $M$ is a polyhedral map with the same Euler characteristic and with $f$-vector $(f_2, f_1, f_0)$.

The exact minimum for the number $f_1$ of edges of a polyhedral map is known for only some manifolds. Let $E_+(\chi)$ or $E_-(\chi)$, respectively, denote the smallest

Preliminary version (July 16, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

number $f_1$ such that there is a polyhedral map with $f_1$ edges on the orientable 2-manifold, or on the nonorientable 2-manifold, respectively, of Euler characteristic $\chi$. The known values of $E_+(\chi)$ and $E_-(\chi)$ are listed in Table 20.2.1; undecided cases are left blank. The polyhedral maps that attain the minimal values $E_+(2)$, $E_+(-8)$, $E_-(0)$, and $E_-(-6)$ are uniquely determined.

TABLE 20.2.1 The known values of $E_+(\chi)$ and $E_-(\chi)$.

| $\chi$ | 2 | 1 | 0 | $-1$ | $-2$ | $-3$ | $-4$ | $-5$ | $-6$ | $-7$ | $-8$ | $-26$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $E_+(\chi)$ | 6 | – | 18 | – | 27 | – | 33 | – | 38 | – | 40 | 78 |
| $E_-(\chi)$ | – | 15 | 18 | 23 | 26 | 30 | 33 | 35 | 36 | 40 | 42 | |

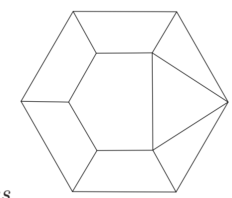

FIGURE 20.2.1
*A self-dual polyhedral map on* $\mathbb{R}\mathrm{P}^2$ *with the minimum number* (15) *of edges.*

For a map on $\mathbb{R}\mathrm{P}^2$ with 15 edges, see Figure 20.2.1. For the unique polyhedral map with 40 edges on the orientable 2-manifold of genus 5 ($\chi = -8$), see Figure 20.2.2 (and [Bre90a]). This map is weakly neighborly and self-dual, and has a cyclic group of automorphisms acting regularly on the set of vertices and on the set of faces.

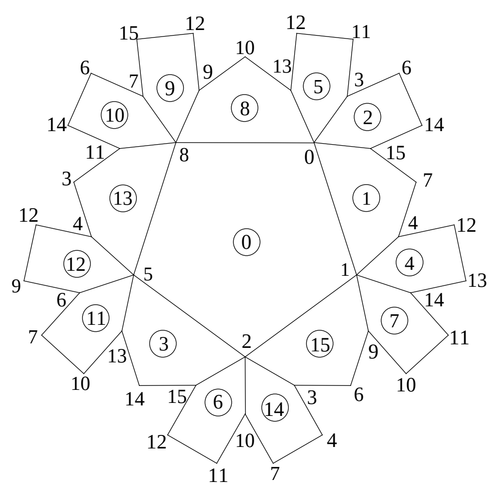

FIGURE 20.2.2
*The unique polyhedral map of genus* 5 *with the minimum number* (40) *of edges.*

A general bound for the number $f_1$ of edges is given by

**THEOREM 20.2.3** [Bre90a]

$$f_1 \geq -\chi + \min\{y \in \mathbb{N} \mid y(\sqrt{2y} - 6) \geq -8\chi \text{ and } y \geq 8\}\,,$$

*where* $\mathbb{N}$ *is the set of natural numbers.*

Preliminary version (July 16, 2017). To appear in the Handbook of Discrete and Computational Geometry,
J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

If $M$ is a polyhedral map on a surface $S$, then a new polyhedral map $M'$ on $S$ can be obtained from $M$ by the following operation, called ***face splitting***. A new edge $xy$ is added across a face of $M$, where $x$ and $y$ are points on edges of $M$ that are not contained in a common edge. The new vertices $x$ and $y$ of $M'$ may be vertices of $M$, or one or both may be relative interior points of edges of $M$. The dual operation is called ***vertex splitting***. On the sphere $S^2$, the (boundary complex of the) tetrahedron is the only polyhedral map that is minimal with respect to face splitting. On the real projective plane $\mathbb{RP}^2$, there are exactly 16 polyhedral maps that are minimal with respect to face splitting [Bar91], and exactly 7 that are minimal with respect to both face splitting and vertex splitting. These are exactly the polyhedral maps on $\mathbb{RP}^2$ with 15 edges, which is the minimum number of edges for $\mathbb{RP}^2$. For an example, see Figure 20.2.1.

For neighborly polyhedral maps we always have equality in (20.2.1). Weakly neighborly polyhedral maps (wnp maps) are a generalization of neighborly polyhedral maps. On the 2-sphere, the only wnp maps are the (boundary complexes of the) pyramids and the triangular prism. Every other 2-manifold admits only finitely many combinatorially distinct wnp maps. Moreover,

$$\limsup_{\chi\to\infty} V_{\max}(\chi)\cdot|2\chi|^{-2/3}\le 1\ ,$$

where $V_{\max}(\chi)$ denotes the maximum number of vertices of a wnp map of Euler characteristic $\chi$; see [BA86], which also discusses further equalities and inequalities for general polyhedral maps. For several 2-manifolds, all wnp maps have been determined. For example, on the torus there are exactly five wnp maps, and three of them are geometrically realizable as polyhedra in $\mathbb{R}^3$.

In some instances, the combinatorial lower bound (20.2.1) can also be attained geometrically by (necessarily orientable) polyhedra in $\mathbb{R}^3$. Trivially, the tetrahedron minimizes $f_0\,(=4)$ for $g=0$. For $g=1$ there is a polyhedron with $f_0=7$ known as the ***Császár torus***; see Figure 20.2.3. A pair of congruent copies of the torus shown in Figure 20.2.3b can be linked (if the coordinates orthogonal to the plane of projection are sufficiently small). Polyhedra that have the minimum number of vertices have also been found for $g=2$ (the exceptional case), 3, 4, or 5, with 10, 10, 11, or 12 vertices, respectively. For $g\le 4$ each triangulation with the minimum number of vertices admits a realization as a polyhedron in $\mathbb{R}^3$, but for $g=5$ there are also minimal triangulations which do not admit such a realization. For $g=6$, none of the 59 combinatorially different triangulations with 12 vertices admits a geometric realization as a polyhedron in $\mathbb{R}^3$ [Sch10].

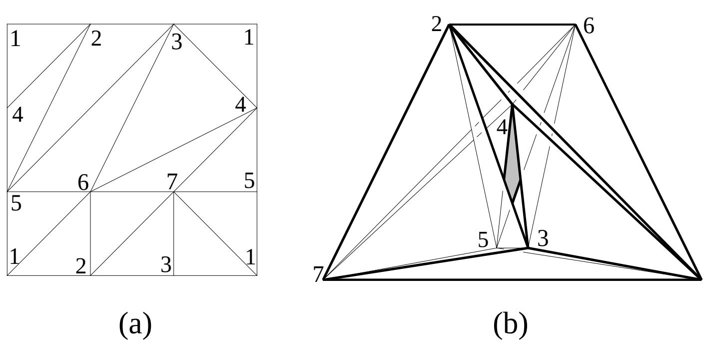


FIGURE 20.2.3
(a) *The unique 7-vertex triangulation of the torus and* (b) *a symmetric realization as a polyhedron.*

The minimum number of vertices for polyhedral maps that admit polyhedral immersions in $\mathbb{R}^3$ is 9 for the real projective plane $\mathbb{R}\mathrm{P}^2$ [Bre90b], the Klein bottle, and the surface of Euler characteristic $\chi = -1$. The minimum number is 10 for the surface with $\chi = -3$ [Leo09]. The lower bound for $\mathbb{R}\mathrm{P}^2$ follows directly from the fact that each immersion of $\mathbb{R}\mathrm{P}^2$ in $\mathbb{R}^3$ has (generically) a triple point (like the classical Boy surface).

There are also some surprising results for higher genus. For example, for each $q \geq 3$ there exists a polyhedral map $M_q$ of type $\{4, q\}$ with $f_0 = 2^q$ and $g = 2^{q-3}(q-4)+1$ such that $M_q$ and its dual have polyhedral embeddings in $\mathbb{R}^3$ [MSW83]. These polyhedra are combinatorially regular in the sense of Section 20.5. Note that $f_0 = O(g/\log g)$. Thus for sufficiently large genus, $M_q$ has more handles than vertices, and its dual has more handles than faces.

Every polyhedral 2-manifold in $\mathbb{R}^3$ of genus $g \geq 1$ contains at least 5 nonconvex vertices. This bound is attained for each $g \geq 1$. For tight polyhedral 2-manifolds, the lower bound for the number of nonconvex vertices is larger and depends on $g$. For a survey on tight polyhedral submanifolds see [Küh95].

## 20.3 EBERHARD'S THEOREM AND RELATED RESULTS

Eberhard's theorem is one of the oldest nontrivial results about convex polyhedra. The standard reference is Grünbaum [Grü67, Grü70]. For recent developments see also Jendrol [Jen93].

### GLOSSARY

***p*-sequence:** For a polyhedral map $M$, the sequence $p(M) = (p_k(M))_{k\geq 3}$, where $p_k = p_k(M)$ is the number of $k$-gonal faces of $M$.

***v*-sequence:** For a polyhedral map $M$, the sequence $v(M) = (v_k(M))_{k\geq 3}$, where $v_k = v_k(M)$ is the number of vertices of $M$ of degree $k$.

### EBERHARD-TYPE RESULTS

Significant results are known for the general problem of determining what kind of polygons, and how many of each kind, may be combined to form the faces of a polyhedral map $M$ on an orientable surface of genus $g$. These refine results (for $d = 3$) about the boundary complex and the number of $i$-dimensional faces ($i = 0, \dots, d-1$) of a convex $d$-polytope [Grü67, Zie95]; see Chapter 17.

If $M$ is a polyhedral map of genus $g$ with $f$-vector $(f_0, f_1, f_2)$, then

$$\sum_{k\geq 3} p_k = f_2, \quad \sum_{k\geq 3} v_k = f_0, \quad \sum_{k\geq 3} k p_k = 2f_1 = \sum_{k\geq 3} k v_k\,. \tag{20.3.1}$$

Further, Euler's formula $f_0 - f_1 + f_2 = 2(1-g)$ implies the equations

$$\sum_{k\geq 3}(6-k)p_k + 2\sum_{k\geq 3}(3-k)v_k = 12(1-g) \tag{20.3.2}$$

Preliminary version (July 16, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

and

$$\sum_{k\geq 3}(4-k)(p_k+v_k) = 8(1-g)\,. \tag{20.3.3}$$

These equations contain no information about $p_6, v_3$ and $p_4, v_4$, respectively.

Eberhard-type results deal with the problem of determining which pairs $(p_k)_{k\geq 3}$ and $(v_k)_{k\geq 3}$ of sequences of nonnegative integers can occur as $p$-sequences $p(M)$ and $v$-sequences $v(M)$ of polyhedral maps $M$ of a given genus $g$. The above equations yield simple necessary conditions. As a consequence of Steinitz's theorem (Section 20.1), the problem for $g = 0$ is equivalent to a similar such problem for convex 3-polytopes [Grü67, Grü70]. The classical theorem of Eberhard says the following:

**THEOREM 20.3.1** Eberhard's Theorem

*For each sequence $(p_k \mid 3 \leq k \neq 6)$ of nonnegative integers satisfying*

$$\sum_{k\geq 3}(6-k)p_k = 12,$$

*there exists a value of $p_6$ such that the sequence $(p_k)_{k\geq 3}$ is the $p$-sequence of a spherical polyhedral map all of whose vertices have degree 3, or, equivalently, of a convex 3-polytope that is simple (has vertices only of degree 3).*

This is the case $g = 0$ and $v_3 = f_0,\ v_k = 0\ (k \geq 4)$.

More general results have been established [Jen93]. Given two sequences $p' = (p_k \mid 3 \leq k \neq 6)$ and $v' = (v_k \mid k > 3)$ of nonnegative integers such that the equation (20.3.2) is satisfied for a given genus $g$, let $E(p', v'; g)$ denote the set of integers $p_6 \geq 0$ such that $(p_k)_{k\geq 3}$ and $(v_k)_{k\geq 3}$, with $v_3 := (\sum_{k\geq 3} kp_k - \sum_{k\geq 4} kv_k)/3$ determined by (20.3.1), are the $p$-sequences and $v$-sequences, respectively, of a polyhedral map of genus $g$. For all but two admissible triples $(p', v', g)$, the set $E(p', v'; g)$ is known up to a finite number of elements. For example, for $g = 0$, the set $E(p', v'; 0)$ is nonempty if and only if $\sum_{k\not\equiv 0\,(\mathrm{mod}\,3)} v_k \neq 1$ or $p_k \neq 0$ for at least one odd $k$. In particular, for each such nonempty set, there exists a constant $c$ depending on $(p', v')$ such that $E(p', v'; 0) = \{j \mid c \leq j\}$, $\{j \mid c \leq j \equiv 0\,(\mathrm{mod}\,2)\}$, or $\{j \mid c \leq j \equiv 1\,(\mathrm{mod}\,2)\}$. Similarly, for each triple with $g \geq 2$, there is a constant $c$ depending on $(p', v', g)$ such that $E(p', v'; g) = \{j \mid c \leq j\}$. There are analogous results for sequences $(p_k \mid 3 \leq k \neq 4)$ and $(v_k \mid 3 \leq k \neq 4)$ that satisfy the equation (20.3.3) or other related equations.

For $g = 1$ there is also a more geometric Eberhard-type result available, which requires the polyhedral map $M$ to be polyhedrally embedded in $\mathbb{R}^3$:

**THEOREM 20.3.2** [Gri83]

*Let $s$, $p_k$ $(k \geq 3, k \neq 6)$ be nonnegative integers. Then there exists a toroidal polyhedral 2-manifold $M$ in $\mathbb{R}^3$ with $p_k(M) = p_k$ $(k \neq 6)$ and $\sum_{k\geq 3}(k-3)v_k(M) = s$ if and only if $\sum_{k\geq 3}(6-k)p_k = 2s$ and $s \geq 6$.*

Also, for toroidal polyhedral 2-manifolds in $\mathbb{R}^3$ (as well as for convex 3-polytopes), the exact range of possible $f$-vectors is known [Grü67, BW93].

**THEOREM 20.3.3**

*A polyhedral embedding in $\mathbb{R}^3$ of some torus with $f$-vector $(f_0, f_1, f_2)$ exists if and* *only if* $f_0 - f_1 + f_2 = 0$, $f_2(11 - f_2)/2 \leq f_0 \leq 2f_2$, $f_0(11 - f_0)/2 \leq f_2 \leq 2f_0$, *and* $2f_1 - 3f_0 \geq 6$.

For generalizations of Eberhard's theorem to tilings of the Euclidean plane, see also [GS87].

# 20.4 REGULAR MAPS

Regular maps are topological analogues of the ordinary regular polyhedra and star-polyhedra on surfaces. Historically they became important in the context of transformations of algebraic equations and representations of algebraic curves in homogeneous complex variables. There is a large body of literature on regular maps and their groups. The classical text is Coxeter and Moser [CM80]. For more recent texts see McMullen and Schulte [MS02], and Conder, Jones, Siran, and Tucker [CJST].

## GLOSSARY

***(Combinatorial) automorphism:*** An incidence-preserving bijection (of the set of vertices, edges, and faces) of a map $M$ on a surface $S$ to itself. The (***combinatorial automorphism***) ***group*** $A(M)$ of $M$ is the group of all such bijections. It can be "realized" by a group of homeomorphisms of $S$.

***Regular map:*** A map $M$ on $S$ whose group $A(M)$ is transitive on the flags (incident triples consisting of a vertex, an edge, and a face) of $M$.

***Chiral map:*** A map $M$ on $S$ whose group $A(M)$ has two orbits on the flags such that any two adjacent flags are in distinct orbits. Here two flags are ***adjacent*** if they differ in precisely one element: a vertex, an edge, or a face. (For a chiral map the underlying surface $S$ must be orientable.)

## GENERAL RESULTS

Each regular map $M$ is of type $\{p, q\}$ for some finite $p$ and $q$. Its group $A(M)$ is transitive on the vertices, the edges, and the faces of $M$. In general, the Schläfli symbol $\{p, q\}$ does not determine $M$ uniquely. The group $A(M)$ is generated by involutions $\rho_0, \rho_1, \rho_2$ such that the ***standard relations***

$$\rho_0{}^2 = \rho_1{}^2 = \rho_2{}^2 = (\rho_0\rho_1)^p = (\rho_1\rho_2)^q = (\rho_0\rho_2)^2 = 1$$

hold, but in general there are also further independent relations. Any triangle in the "barycentric subdivision" (order complex) of $M$ is a fundamental region for $A(M)$ on the underlying surface $S$; see Section 20.1. For any fixed such triangle, we can take for $\rho_i$ the "combinatorial reflection" in its side opposite to the vertex that corresponds to an $i$-dimensional element of $M$. The set of standard relations gives a presentation for the symmetry group of the regular tessellation $\{p, q\}$ on the 2-sphere, in the Euclidean plane, or in the hyperbolic plane, whichever is the universal covering of $M$. See Figure 20.5.1 (a) for a conformal (hyperbolic) drawing of the Dyck map (shown also in Figure 20.1.1) with a fundamental region shaded. The identifications on the boundary of the drawing are indicated by letters.

Preliminary version (July 16, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

The regular maps on orientable surfaces of genus $g \leq 301$ and on non-orientable surfaces of genus $h \leq 602$ have been enumerated by computer (see [Con12]). A similar enumeration is also known for chiral maps on orientable surfaces of genus $g \leq 301$. Up to isomorphism, if $g = 0$, there are just the Platonic solids (or regular spherical tessellations) $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$, and $\{5,3\}$. When $g = 1$ there are three infinite families of torus maps of type $\{3,6\}$, $\{6,3\}$, and $\{4,4\}$, each a quotient of the corresponding Euclidean universal covering tessellations $\{3,6\}$, $\{6,3\}$, and $\{4,4\}$, respectively. For $g \geq 2$, the universal covering tessellation $\{p,q\}$ is hyperbolic and there are only finitely many regular maps on a surface of genus $g$. The latter follows from the ***Hurwitz formula*** $|A(M)| \leq 84\,|\chi|$ (or from the inequality 20.1.1), where $\chi$ is the Euler characteristic of $S$. Each regular map on a nonorientable surface is doubly covered by a regular map of the same type on an orientable surface, and this covering map is unique [Wil78].

Generally speaking, given $M$, the topology of $S$ is reflected in the relations that have to be added to the standard relations to obtain a presentation for $A(M)$. Conversely, many interesting regular maps can be constructed by adding certain kinds of extra relations for the group. Two examples are the regular maps $\{p,q\}_r$ and $\{p,q|r\}$ obtained by adding the extra relations $(\rho_0\rho_1\rho_2)^r = 1$ or $(\rho_0\rho_1\rho_2\rho_1)^r = 1$, respectively. Often these are "infinite maps" on noncompact surfaces, but there are also many (finite) maps on compact surfaces. The Dyck map $\{3,8\}_6$ and the famous ***Klein map*** $\{3,7\}_8$ (with group $PGL(2,7)$) are both of genus 3 and of the first kind, while the traditional regular skew polyhedra in Euclidean 3-space or 4-space are of the second kind. For more details and further interesting classes of regular maps, see [CM80, MS02, CJST] and Chapter 18 of this Handbook. In Section 20.5 we shall discuss polyhedral embeddings of regular maps in ordinary 3-space.

The rotation subgroup (orientation preserving subgroup) of the group of an orientable regular map (of type $\{3,7\}$ or $\{7,3\}$) that achieves equality in the Hurwitz formula is also called a ***Hurwitz group***. The Klein map is the regular map of smallest genus whose rotation subgroup is a Hurwitz group [Con90].

## 20.5 SYMMETRIC POLYHEDRA

Traditionally, much of the appeal of polyhedral 2-manifolds comes from their combinatorial or geometric symmetry properties. For surveys on symmetric polyhedra in $\mathbb{R}^3$ see Schulte and Wills [SW91, SW12], Bokowski and Wills [BW88], and Brehm and Wills [BW93].

### GLOSSARY

***Combinatorially regular:*** A polyhedral 2-manifold (or polyhedron) $P$ is combinatorially regular if its combinatorial automorphism group $A(P)$ is flag-transitive (or, equivalently, if the underlying polyhedral map is a regular map).

***Equivelar:*** A polyhedral 2-manifold (or polyhedron) $P$ is equivelar of type $\{p,q\}$ if all its 2-faces are convex $p$-gons and all its vertices are $q$-valent.

Preliminary version (July 16, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

## GENERAL RESULTS

See Section 20.4 for results about regular maps. Up to isomorphism, the Platonic solids are the only combinatorially regular polyhedra of genus 0. For the torus, each regular map that is a polyhedral map also admits an embedding in $\mathbb{R}^3$ as a combinatorially regular polyhedron. Much less is known for maps of genus $g \geq 2$. Two infinite sequences of combinatorially regular polyhedra have been discovered, one consisting of polyhedra of type $\{4, q\}$ ($q \geq 3$) and the other of their duals of type $\{q, 4\}$ (see [MSW83, RZ11]). These are polyhedral embeddings of the maps $M_q$ and their duals mentioned in Section 20.2. Several famous regular maps have also been realized as polyhedra, including Klein's $\{3, 7\}_8$, Dyck's $\{3, 8\}_6$, the map $\{3, 10\}_6$ (a relative of Dyck's map), and Coxeter's $\{4, 6|3\}$, $\{6, 4|3\}$, $\{4, 8|3\}$, and $\{8, 4|3\}$ [SW85, SW91, BS89, BL16]. It is conjectured that there are just eight regular maps in the genus range $2 \leq g \leq 6$ that can be realized as combinatorially regular polyhedra in $\mathbb{R}^3$ (see [SW12]). However, a complete classification of the regular maps of high genus that admit realizations as combinatorially regular polyhedra in $\mathbb{R}^3$ does not seem to be within reach at present. See Figure 20.5.1 for an illustration of a polyhedral realization of Dyck's regular map $\{3, 8\}_6$ shown in Figure 20.1.1. Figure 20.5.1(a) shows a conformal drawing of the Dyck map, with a fundamental region shaded, while (b) shows a maximally symmetric polyhedral realization. For the Klein map $\{7, 3\}_8$, the dual of $\{3, 7\}_8$, there exists also a non-self-intersecting polyhedral realization with non-convex heptagonal faces [McC09].

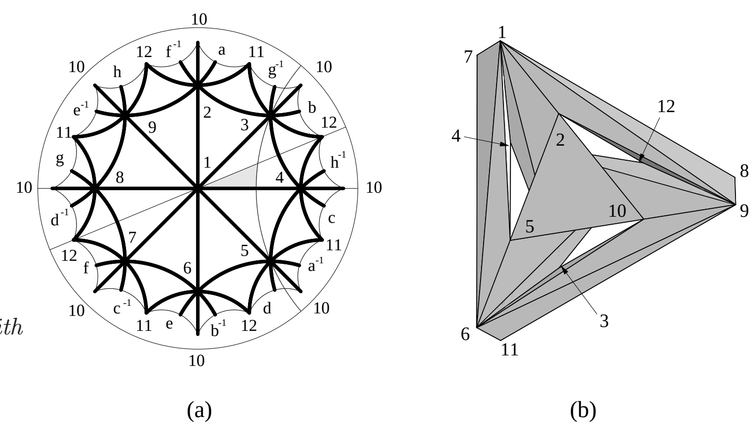


FIGURE 20.5.1
*Dyck's regular map:*
(a) *a conformal drawing, with fundamental region shaded;*
(b) *a symmetric polyhedral realization.*

For a more general concept of polyhedra in $\mathbb{R}^3$ or higher-dimensional spaces, as well as an enumeration of the corresponding regular polyhedra, see Chapter 18 of this Handbook. The latter also contains a depiction of the polyhedral realization of $\{4, 8|3\}$.

Equivelarity is a local regularity condition. Each combinatorially regular polyhedron in $\mathbb{R}^3$ is equivelar. However, there are many other equivelar polyhedra. For sufficiently large genus $g$, for example, there are equivelar polyhedra for each of the types $\{3, q\}$ with $q = 7, 8, 9$; $\{4, q\}$ with $q = 5, 6$; and $\{q, 4\}$ with $q = 5, 6$ [BW93].

The symmetry group of a polyhedron is generally much smaller than the combinatorial automorphism group of the underlying polyhedral map. In particular, the five Platonic solids are the only polyhedra in $\mathbb{R}^3$ with a flag-transitive symmetry group. Note that even for higher genus (namely, for $g = 1, 3, 5, 7, 11$, and 19) there exist polyhedra with vertex-transitive symmetry groups. Such a polyhedral torus is shown in Figure 20.5.2.

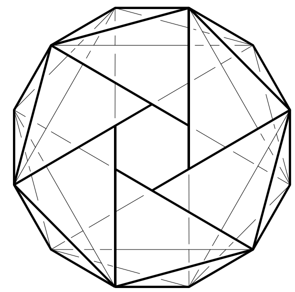

FIGURE 20.5.2
*A vertex-transitive polyhedral torus.*

Finally, if we relax the requirement that a polyhedron be free of self-intersections and allow more general "polyhedral realizations" of maps (for instance, polyhedral immersions), then there is much more flexibility in the construction of "polyhedra" with high symmetry properties. The most famous examples are the Kepler-Poinsot star-polyhedra, but there are also many others. For more details see [SW91, BW88, BW93, MS02] and Chapter 18 of this Handbook.

## 20.6 SOURCES AND RELATED MATERIAL

### SURVEYS

[Bar83]: A text about colorings of maps and polyhedra.

[Bjö95]: A survey on topological methods in combinatorics.

[BLS+93]: A monograph on oriented matroids.

[BS89]: A text about computational aspects of geometric realizability.

[BW93]: A survey on polyhedral manifolds in 2 and higher dimensions.

[Con90]: A survey on Hurwitz groups.

[CJST]: A monograph on regular maps on surfaces.

[Cox73]: A monograph on regular polytopes, regular tessellations, and reflection groups.

[CM80]: A monograph on discrete groups and their presentations.

[GT87]: A text about topological graph theory, in particular graph embeddings in surfaces.

[Grü67]: A monograph on convex polytopes.

[Grü70]: A survey on convex polytopes complementing the exposition in [Grü67].

Preliminary version (July 16, 2017). To appear in the Handbook of Discrete and Computational Geometry,
J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

[GS87]: A monograph on plane tilings and patterns.

[Küh95]: A survey on tight polyhedral manifolds.

[Moi77]: A text about geometric topology in low dimensions.

[MT01]: A text about embeddings of graphs in surfaces.

[MS02]: A monograph on abstract regular polytopes and their groups.

[Rin74]: A text about maps on surfaces and the Map Color Theorem.

[SW91]: A survey on combinatorially regular polyhedra in 3-space.

[Zie95]: A graduate textbook on convex polytopes.

## RELATED CHAPTERS

Chapter 3: Tilings
Chapter 6: Oriented matroids
Chapter 15: Basic properties of convex polytopes
Chapter 16: Subdivisions and triangulations of polytopes
Chapter 17: Face numbers of polytopes and complexes
Chapter 18: Symmetry of polytopes and polyhedra

## REFERENCES


[ABM08] D. Archdeacon, C.P. Bonnington, and J.A. Ellis-Monaghan. How to exhibit toroidal maps in space. *Discrete Comput. Geom.* 38:573–594, 2008.

[BA86] U. Brehm and A. Altshuler. On weakly neighborly polyhedral maps of arbitrary genus. *Israel J. Math.*, 53:137–157, 1986.

[Bar83] D.W. Barnette. *Map Coloring, Polyhedra, and the Four-Color Problem.* MAA, Washington, 1983.

[Bar91] D.W. Barnette. The minimal projective plane polyhedral maps. In P. Gritzmann and B. Sturmfels, editors, *Applied Geometry and Discrete Mathematics—The Victor Klee Festschrift*, vol. 4 of*DIMACS Ser. Discrete Math. Theoret. Comp. Sci.*, pages 63–70, AMS, Providence, 1991.

[BGH91] D.W. Barnette, P. Gritzmann, and R. Höhne On valences of polyhedra. *J. Combin. Theory Ser. A*, 58:279–300, 1991.

[Bjö95] A. Björner. Topological methods. In R.L. Graham, M. Grötschel, and L. Lovász, editors, *Handbook of Combinatorics*, pages 1819–1872, Elsevier, Amsterdam, 1995.

[BL16] U. Brehm and U. Leopold. Symmetric realizations of the regular map $\{3, 10\}_6$ of genus 6 with 15 vertices as a polyhedron. In preparation, 2016.

[BLS+93] A. Björner, M. Las Vergnas, B. Sturmfels, N. White, and G.M. Ziegler. *Oriented Matroids.* Vol. 46 of *Encyclopedia Math. Appl.*, Cambridge University Press, 1993.

[BO 00] J. Bokowski and A.G. de Oliveira. On the generation of oriented matroids. *Discrete Comput. Geom.* 24:197–208, 2000.

[Bre83] U. Brehm. A nonpolyhedral triangulated Möbius strip. *Proc. Amer. Math. Soc.* 89:519–522, 1983.

[Bre90a] U. Brehm. Polyhedral maps with few edges. In R. Bodendiek and R. Henn, editors, *Topics in Combinatorics and Graph Theory*, pages 153–162, Physica-Verlag, Heidelberg, 1990.

[Bre90b] U. Brehm. How to build minimal polyhedral models of the Boy surface. *Math. Intelligencer* 12:51–56, 1990.

[BS89] J. Bokowski and B. Sturmfels. *Computational Synthetic Geometry.* Vol. 1355 of *Lecture Notes in Math.*, Springer-Verlag, Berlin, 1989.

[BS95] U. Brehm and G. Schild. Realizability of the torus and the projective plane in $\mathbb{R}^4$. *Israel J. Math.*, 91:249–251, 1995.

[BW88] J. Bokowki and J.M. Wills. Regular polyhedra with hidden symmetries. *Math. Intelligencer*, 10:27–32, 1988.

[BW93] U. Brehm and J.M. Wills. Polyhedral manifolds. In P.M. Gruber and J.M. Wills, editors, *Handbook of Convex Geometry*, Vol. A, pages 535–554, North-Holland, Amsterdam, 1993.

[CJST] M.D.E. Conder, G.A. Jones, J. Širáň, and T.W. Tucker. *Regular Maps.* In preparation.

[CM80] H.S.M. Coxeter and W.O.J. Moser. *Generators and Relations for Discrete Groups*, 4th edition. Springer-Verlag, Berlin, 1980.

[Con90] M. Conder. Hurwitz groups: A brief survey. *Bull. Amer. Math. Soc.*, 23:359–370, 1990.

[Con12] M.D.E. Conder. Lists of regular maps, hypermaps and polytopes, trivalent symmetric graphs, and surface actions. `www.math.auckland.ac.nz/~conder`.

[Cox73] H.S.M. Coxeter. *Regular Polytopes*, 3rd edition. Dover, New York, 1973.

[Gri83] P. Gritzmann. The toroidal analogue of Eberhard's theorem. *Mathematika*, 30:274–290, 1983.

[Grü67] B. Grünbaum. *Convex Polytopes*. Interscience, London, 1967. Revised edition (V. Kaibel, V. Klee, and G.M. Ziegler, editors), vol. 221 of *Grad. Texts in Math.*, Springer, New York, 2003.

[Grü70] B. Grünbaum. Polytopes, graphs, and complexes. *Bull. Amer. Math. Soc.*, 76:1131–1201, 1970.

[GS87] B. Grünbaum and G.C. Shephard. *Tilings and Patterns.* Freeman, New York, 1987.

[GT87] J.L. Gross and T.W. Tucker. *Topological Graph Theory*. Wiley, New York, 1987.

[Jen93] S. Jendrol. On face-vectors and vertex-vectors of polyhedral maps on orientable 2-manifolds. *Math. Slovaca*, 43:393–416, 1993.

[Küh95] W. Kühnel. *Tight Polyhedral Submanifolds and Tight Triangulations.* Vol. 1612 of *Lecture Notes in Math.*, Springer-Verlag, New York, 1995.

[Leo09] U. Leopold. *Polyhedral Embeddings and Immersions of Triangulated 2-Manifolds.* Diplomarbeit (Master's Thesis), Dresden, 2009.

[McC09] D. McCooey. A non-self-intersecting polyhedral realization of the all-heptagon Klein map. *Symmetry Cult. Sci.* 20:247–268, 2009.

[Moi77] E.E. Moise. *Geometric Topology in Dimensions 2 and 3.* Vol. 47 of *Graduate Texts in Math.*, Springer-Verlag, New York, 1977.

[MS02] P. McMullen and E. Schulte. *Abstract Regular Polytopes*. Cambridge University Press, 2002.

[MSW83] P. McMullen, C. Schulz, and J.M. Wills. Polyhedral manifolds in $E^3$ with unusually large genus. *Israel J. Math.*, 46:127–144, 1983.

[MT01] B. Mohar and C. Thomassen. *Graphs on Surfaces*. Johns Hopkins University Press, Baltimore, 2001.

[Ric96] J. Richter-Gebert. *Realization Spaces of Polytopes.* Vol. 1643 of *Lecture Notes in Math.*, Springer-Verlag, Berlin, 1996.

[Rin74] G. Ringel. *Map Color Theorem.* Springer-Verlag, Berlin, 1974.

[RZ11] T. Rörig and G.M. Ziegler. Polyhedral surfaces in wedge products. *Geom. Dedicata* 151:155–173, 2011.

[Sch10] L. Schewe. Nonrealizable minimal vertex triangulations of surfaces: showing nonrealizability using oriented matroids and satisfiability solvers. *Discrete Comput. Geom.* 43:289–302, 2010.

[SW12] E. Schulte and J.M. Wills. Convex-faced combinatorially regular polyhedra of small genus. *Symmetry* 4:1–14, 2012.

[SW85] E. Schulte and J.M. Wills. A polyhedral realization of Felix Klein's map $\{3,7\}_8$ on a Riemann surface of genus 3. *J. London Math. Soc.*, 32:539–547, 1985.

[SW91] E. Schulte and J.M. Wills. Combinatorially regular polyhedra in three-space. In K.H. Hofmann and R. Wille, editors, *Symmetry of Discrete Mathematical Structures and Their Symmetry Groups*, pages 49–88. Heldermann Verlag, Berlin, 1991.

[Wil78] S.E. Wilson. Non-orientable regular maps. *Ars Combin.*, 5:213–218, 1978.

[Zie95] G.M. Ziegler. *Lectures on Polytopes.* Vol. 152 of *Graduate Texts in Math.*, Springer-Verlag, New York, 1995.