2 PACKING AND COVERING

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INTRODUCTION

The basic problems in the classical theory of packings and coverings, the development of which was strongly influenced by the geometry of numbers and by crystallography, are the determination of the densest packing and the thinnest covering with congruent copies of a given body K. Roughly speaking, the density of an arrangement is the ratio between the total volume of the members of the arrangement and the volume of the whole space. In Section 2.1 we define this notion rigorously and give an account of the known density bounds.

In Section 2.2 we consider packings in, and coverings of, bounded domains. Section 2.3 is devoted to multiple arrangements and their decomposability. In Section 2.4 we make a detour to spherical and hyperbolic spaces. In Section 2.5 we discuss problems concerning the number of neighbors in a packing, while in Section 2.6 we investigate some selected problems concerning lattice arrangements. We close in Section 2.7 with problems concerning packing and covering with sequences of convex sets.

2.1 DENSITY BOUNDS FOR ARRANGEMENTS IN E^d

GLOSSARY

Convex body: A compact convex set with nonempty interior. A convex body in the plane is called a ***convex disk***. The collection of all convex bodies in d-dimensional Euclidean space $\mathbb{E}^d$ is denoted by $\mathcal{K}(\mathbb{E}^d)$. The subfamily of $\mathcal{K}(\mathbb{E}^d)$ consisting of centrally symmetric bodies is denoted by $\mathcal{K}^*(\mathbb{E}^d)$.

Operations on $\mathcal{K}(\mathbb{E}^d)$: For a set A and a real number λ we set $\lambda A = \{x \mid x = \lambda a,\ a \in A\}$. λA is called a ***homothetic copy*** of A. The ***Minkowski sum*** $A + B$ of the sets A and B consists of all points $a + b$, $a \in A$, $b \in B$. The set $A - A = A + (-A)$ is called the ***difference body*** of A. B^d denotes the unit ball centered at the origin, and $A + rB^d$ is called the ***parallel body*** of A at distance r ($r > 0$). If $A \subset \mathbb{E}^d$ is a convex body with the origin in its interior, then the ***polar body*** A^* of A is $\{x \in \mathbb{E}^d \mid \langle x, a\rangle \leq 1 \text{ for all } a \in A\}$.

The ***Hausdorff distance*** between the sets A and B is defined by

$$d(A, B) = \inf\{\varrho \mid A \subset B + \varrho B^d, B \subset A + \varrho B^d\}.$$

Lattice: The set of all integer linear combinations of a particular basis of $\mathbb{E}^d$.

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Lattice arrangement: The set of translates of a given set in $\mathbb{E}^d$ by all vectors of a lattice.

Packing: A family of sets whose interiors are mutually disjoint.

Covering: A family of sets whose union is the whole space.

The ***volume*** (Lebesgue measure) of a measurable set A is denoted by $V(A)$. In the case of the plane we use the term ***area*** and the notation $a(A)$.

Density of an arrangement relative to a set: Let $\mathcal{A}$ be an arrangement (a family of sets each having finite volume) and D a set with finite volume. The ***inner density*** $d_{\text{inn}}(\mathcal{A}|D)$, ***outer density*** $d_{\text{out}}(\mathcal{A}|D)$, and ***density*** $d(\mathcal{A}|D)$ of $\mathcal{A}$ relative to D are defined by

$$d_{\text{inn}}(\mathcal{A}|D) = \frac{1}{V(D)} \sum_{A\in\mathcal{A}, A\subset D} V(A),$$

$$d_{\text{out}}(\mathcal{A}|D) = \frac{1}{V(D)} \sum_{A\in\mathcal{A},\, A\cap D\neq\emptyset} V(A),$$

and

$$d(\mathcal{A}|D) = \frac{1}{V(D)} \sum_{A\in\mathcal{A}} V(A\cap D).$$

(If one of the sums on the right side is divergent, then the corresponding density is infinite.)

The ***lower density*** and ***upper density*** of an arrangement $\mathcal{A}$ are given by the limits $d_-(\mathcal{A}) = \liminf_{\lambda\to\infty} d_{\text{inn}}(\mathcal{A}|\lambda B^d)$, $d_+(\mathcal{A}) = \limsup_{\lambda\to\infty} d_{\text{out}}(\mathcal{A}|\lambda B^d)$. If $d_-(\mathcal{A}) = d_+(\mathcal{A})$, then we call the common value the ***density*** of $\mathcal{A}$ and denote it by $d(\mathcal{A})$. It is easily seen that these quantities are independent of the choice of the origin.

The ***packing density*** $\delta(K)$ and ***covering density*** $\vartheta(K)$ of a convex body (or more generally of a measurable set) K are defined by

$$\delta(K) = \sup\{d_+(\mathcal{P}) \mid \mathcal{P} \text{ is a packing of } \mathbb{E}^d \text{ with congruent copies of } K\}$$

and

$$\vartheta(K) = \inf\{d_-(\mathcal{C}) \mid \mathcal{C} \text{ is a covering of } \mathbb{E}^d \text{ with congruent copies of } K\}.$$

The ***translational packing density*** $\delta_T(K)$, ***lattice packing density*** $\delta_L(K)$, ***translational covering density*** $\vartheta_T(K)$, and ***lattice covering density*** $\vartheta_L(K)$ are defined analogously, by taking the supremum and infimum over arrangements consisting of translates of K and over lattice arrangements of K, respectively. It is obvious that in the definitions of $\delta_L(K)$ and $\vartheta_L(K)$ we can take maximum and minimum instead of supremum and infimum. By a theorem of Groemer, the same holds for the translational and for the general packing and covering densities.

Dirichlet cell: Given a set S of points in $\mathbb{E}^d$ such that the distances between the points of S have a positive lower bound, the Dirichlet cell, also known as the ***Voronoi cell***, associated to an element s of S consists of those points of $\mathbb{E}^d$ that are closer to s than to any other element of S.

KNOWN VALUES OF PACKING AND COVERING DENSITIES

Apart from the obvious examples of space fillers, there are only a few specific bodies for which the packing or covering densities have been determined. The bodies for which the packing density is known are given in Table 2.1.1.

TABLE 2.1.1 Bodies K for which $\delta(K)$ is known.

BODY	SOURCE
Circular disk in $\mathbb{E}^2$	[Thu10]
Parallel body of a rectangle	[Fej67]
Intersection of two congruent circular disks	[Fej71]
Centrally symmetric n-gon (algorithm in $O(n)$ time)	[MS90]
Ball in $\mathbb{E}^3$	[Hal05]
Ball in $\mathbb{E}^8$	[Via17]
Ball in $\mathbb{E}^{24}$	[CKM17]
Truncated rhombic dodecahedron in $\mathbb{E}^3$	[Bez94]

We have $\delta(B^2) = \pi/\sqrt{12}$. The longstanding conjecture that $\delta(B^3) = \pi/\sqrt{18}$ has been confirmed by Hales. A packing of balls reaching this density is obtained by placing the centers at the vertices and face-centers of a cubic lattice. We discuss the sphere packing problem in the next section.

For the rest of the bodies in Table 2.1.1, the packing density can be given only by rather complicated formulas. We note that, with appropriate modification of the definition, the packing density of a set with infinite volume can also be defined. A. Bezdek and W. Kuperberg (see [BK91]) showed that the packing density of an infinite circular cylinder is $\pi/\sqrt{12}$, that is, infinite circular cylinders cannot be packed more densely than their base. It is conjectured that the same statement holds for circular cylinders of any finite height.

A theorem of L. Fejes Tóth [Fej50] states that

$$\delta(K) \leq \frac{a(K)}{H(K)} \qquad \text{for } K \in \mathcal{K}(\mathbb{E}^2), \tag{2.1.1}$$

where $H(K)$ denotes the minimum area of a hexagon containing K. This bound is best possible for centrally symmetric disks, and it implies that

$$\delta(K) = \delta_T(K) = \delta_L(K) = \frac{a(K)}{H(K)} \qquad \text{for } K \in \mathcal{K}^*(\mathbb{E}^2).$$

The packing densities of the convex disks in Table 2.1.1 have been determined utilizing this relation.

It is conjectured that an inequality analogous to (2.1.1) holds for coverings, and this is supported by the following weaker result [Fej64]:

Let $h(K)$ denote the maximum area of a hexagon contained in a convex disk K. Let $\mathcal{C}$ be a covering of the plane with congruent copies of K such that no two copies of K cross. Then

$$d_-(\mathcal{C}) \geq \frac{a(K)}{h(K)}.$$

The convex disks A and B ***cross*** if both $A \setminus B$ and $B \setminus A$ are disconnected. As translates of a convex disk do not cross, it follows that

$$\vartheta_T(K) \geq \frac{a(K)}{h(K)} \qquad \text{for } K \in \mathcal{K}(\mathbb{E}^2).$$

Again, this bound is best possible for centrally symmetric disks, and it implies that

$$\vartheta_T(K) = \vartheta_L(K) = \frac{a(K)}{h(K)} \qquad \text{for } K \in \mathcal{K}^*(\mathbb{E}^2). \tag{2.1.2}$$

Based on this, Mount and Silverman gave an algorithm that determines $\vartheta_T(K)$ for a centrally symmetric n-gon in $O(n)$ time. Also the classical result $\vartheta(B^2) = 2\pi/\sqrt{27}$ of Kershner [Ker39] follows from this relation.

The bound $\vartheta_T(K) \leq \frac{a(K)}{h(K)}$ holds without the restriction to non-crossing disks for "fat" disks. A convex disk is r***-fat*** if it is contained in a unit circle and contains a concentric circle of radius r. G. Fejes Tóth [Fej05a] proved the inequality $\vartheta_T(K) \leq \frac{a(K)}{h(K)}$ for 0.933-fat convex disks and, sharpening an earlier result of Heppes [Hep03] also for 0.741-fat ellipses. The algorithm of Mount and Silverman enables us to determine the covering density of centrally symmetric 0.933-fat convex polygons. We note that all regular polygons with at least 10 sides are 0.933-fat, and with a modification of the proof in [Fej05a] it can be shown that the bound $\vartheta_T(K) \leq \frac{a(K)}{h(K)}$ holds also in the case when K is a regular octagon. It follows that if P_n denotes a regular n-gon, then

$$\vartheta(P_{6k}) = \frac{k \sin \frac{\pi}{3k}}{\sin \frac{\pi}{3}}$$

and

$$\vartheta(P_{6k\pm 2}) = \frac{(3k \pm 1) \sin \frac{\pi}{3k\pm 1}}{2 \sin \frac{k\pi}{3k\pm 1} + \sin \frac{(k\pm 1)\pi}{3k\pm 1}}$$

for all $k \geq 1$. The covering density is not known for any convex body other than the space fillers and the examples mentioned above.

The true nature of difficulty in removing the non-crossing condition is shown by an ingenious example by A Bezdek and W. Kuperberg [BK10]. Modifying a pentagonal tile, they constructed convex disks K with the property that in any thinnest covering of the plane with congruent copies of K, crossing pairs occur. The thinnest covering in their construction contains rotated copies of K, so it is not a counterexample for the conjectures that for every convex disk K we have $\vartheta_T(K) \leq \frac{a(K)}{h(K)}$ and $\vartheta_T(K) = \vartheta_L(K)$. The equality $\vartheta_T(K) = \vartheta_L(K)$ was first proved by Januszewski [Jan10] for triangles. Januszewski's result was extended by Sriamorn and Xue [SX15] to a wider class of convex disks containing, besides triangles, all convex quadrilaterals. A *quarter-convex disk* is the affine image of a set of the form $\{(x,y) : 0 \leq x \leq 1, 0 \leq y \leq f(x)\}$ for some positive concave function $f(x)$ defined for $0 \leq x \leq 1$. Sriamorn and Xue proved that $\vartheta_T(K) = \vartheta_L(T)$ for every quarter-convex disk.

One could expect that the restriction to arrangements of translates of a set means a considerable simplification. However, this apparent advantage has not been exploited so far in dimensions greater than 2. On the other hand, the lattice packing density of some special convex bodies in $\mathbb{E}^3$ has been determined; see Table 2.1.2.

TABLE 2.1.2 Bodies $K \in \mathbb{E}^3$ for which $\delta_L(K)$ is known.

BODY	$\delta_L(K)$	SOURCE
$\{x \mid \lvert x\rvert \le 1, \lvert x_3\rvert \le \lambda\} \quad (\lambda \le 1)$	$\pi(3-\lambda^2)^{1/2}/6$	[Cha50]
$\{x \mid \lvert x_i\rvert \le 1, \lvert x_1+x_2+x_3\rvert \le \lambda\}$	$\begin{cases} \dfrac{9-\lambda^2}{9} & \text{for } 0<\lambda\le\frac{1}{2} \\ \dfrac{9\lambda(9-\lambda^2)}{4(-\lambda^3-3\lambda^2+24\lambda-1)} & \text{for } \frac{1}{2}\le\lambda\le 1 \\ \dfrac{9(\lambda^3-9\lambda^2+27\lambda-3)}{8\lambda(\lambda^2-9\lambda+27)} & \text{for } 1\le\lambda\le 3 \end{cases}$	[Whi51]
$\{x \mid \sqrt{(x_1)^2+(x_2)^2}+\lvert x_3\rvert \le 1\}$	$\pi\sqrt{6}/9 = 0.8550332\ldots$	[Whi48]
Tetrahedron	$18/49 = 0.3673469\ldots$	[Hoy70]
Octahedron	$18/19 = 0.9473684\ldots$	[Min04]
Dodecahedron	$(5+\sqrt{5})/8 = 0.9045084\ldots$	[BH00]
Icosahedron	$0.8363574\ldots$	[BH00]
Cuboctahedron	$45/49 = 0.9183633\ldots$	[BH00]
Icosidodecahedron	$(45+17\sqrt{5})/96 = 0.8647203\ldots$	[BH00]
Rhombic Cuboctahedron	$(16\sqrt{2}-20)/3 = 0.8758056\ldots$	[BH00]
Rhombic Icosidodecahedron	$(768\sqrt{5}-1290)/531 = 0.8047084\ldots$	[BH00]
Truncated Cube	$9(5-3\sqrt{2})/7 = 0.9737476\ldots$	[BH00]
Truncated Dodecahedron	$(25+37\sqrt{5})/120 = 0.8977876\ldots$	[BH00]
Truncated Icosahedron	$0.78498777\ldots$	[BH00]
Truncated Cuboctahedron	$0.8493732\ldots$	[BH00]
Truncated Icosidodecahedron	$(19+10\sqrt{5})/50 = 0.8272135\ldots$	[BH00]
Truncated Tetrahedron	$207/304 = 0.6809210\ldots$	[BH00]
Snub Cube	$0.787699\ldots$	[BH00]
Snub Dodecahedron	$0.7886401\ldots$	[BH00]

All results given in Table 2.1.2 are based on Minkowski's work [Min04] on critical lattices of convex bodies. We emphasize the following special case: Gauss's result that $\delta_L(B^3) = \pi/\sqrt{18}$ is the special case $\lambda = 1$ of Chalk's theorem concerning the frustum of the ball. In [BH00] Betke and Henk gave an efficient algorithm for computing $\delta_L(K)$ for an arbitrary 3-polytope. As an application they calculated the lattice packing densities of all regular and Archimedean polytopes.

Additional bodies can be added to Table 2.1.2 using the following observations.

It has been noticed by Chalk and Rogers [CR48] that the relation $\delta_T(K) = \delta_L(K)$ $(K \in \mathcal{K}(\mathbb{E}^2))$ readily implies that for a cylinder C in $\mathbb{E}^3$ based on a convex disk K we have $\delta_L(C) = \delta_L(K)$. Thus, $\delta_L(C)$ is determined by the lattice packing density of its base.

Next, we recall the observation of Minkowski (see [Rog64, p. 69]) that an arrangement $\mathcal{A}$ of translates of a convex body K is a packing if and only if the arrangement of translates of the body $\frac{1}{2}(K-K)$ by the same vectors is a packing. This implies that, for $K \in \mathcal{K}(\mathbb{E}^d)$,

$$\delta_T(K) = 2^d \delta_T(K-K)\frac{V(K)}{V(K-K)} \quad \text{and} \quad \delta_L(K) = 2^d \delta_L(K-K)\frac{V(K)}{V(K-K)}. \tag{2.1.3}$$

Generally, K is not uniquely determined by $K-K$; e.g., we have $K-K = B^d$ for every $K \subset \mathbb{E}^d$ that is a body of constant width 1, and the determination of $\delta_L(K)$ for such a body is reduced to the determination of $\delta_L(B^d)$, which is established for $d \le 8$ and $d = 24$. We give the known values of $\delta_L(B^d)$ and $\vartheta(B^d)$, in Table 2.1.3.

TABLE 2.1.3 Known values of $\delta_L(B^d)$ and $\vartheta_L(B^d)$.

d	$\delta_L(B^d)$	SOURCE	$\vartheta_L(B^d)$	SOURCE
2	$\frac{\pi}{2\sqrt{3}}$	[Lag73]	$\frac{2\pi}{3\sqrt{3}}$	[Ker39]
3	$\frac{\pi}{\sqrt{18}}$	[Gau]	$\frac{5\sqrt{5}\pi}{24}$	[Bam54]
4	$\frac{\pi^2}{16}$	[KZ72]	$\frac{2\pi^2}{5\sqrt{5}}$	[DR63]
5	$\frac{\pi^2}{15\sqrt{2}}$	[KZ77]	$\frac{245\sqrt{35}\pi^2}{3888\sqrt{3}}$	[RB75]
6	$\frac{\pi^3}{48\sqrt{3}}$	[Bli35]		
7	$\frac{\pi^3}{105}$	[Bli35]		
8	$\frac{\pi^4}{384}$	[Bli35]		
24	$\frac{\pi^{12}}{12!}$	[CK04]		

THE KEPLER CONJECTURE

In 1611 Kepler [Kep87] described the face-centered cubic lattice packing of congruent balls consisting of hexagonal layers. He observed that the packing is built at the same time of square layers. Then he proclaimed that this arrangement is "the tightest possible, so that in no other arrangement could more pallets be stuffed into the same container." This sentence of Kepler has been interpreted by some authors as the conjecture that the density of a packing of congruent balls cannot exceed the density of the face-centered cubic lattice packing, that is, $\pi/\sqrt{18}$. It is doubtful whether Kepler meant this, but attributing the conjecture to him certainly helped in advertising it outside the mathematical community. Today the term "Kepler conjecture" is widely accepted despite the fact that Kepler's statement as quoted above is certainly false if the container is smooth. Schürmann [Sch06] proved that if K is a smooth convex body in d-dimensional space ($d \geq 2$), then there exists a natural number n_0, depending on K, such that the densest packing of $n \geq n_0$ congruent balls in K cannot be part of a lattice arrangement.

Early research concerning Kepler's conjecture concentrated on two easier problems: proving the conjecture for special arrangements and giving upper bounds for $\delta(B^3)$.

We mentioned Gauss's result that $\delta_L(B^3) = \pi/\sqrt{18}$. A stronger result establishing Kepler's conjecture for a restricted class of packings is due to A. Bezdek, W. Kuperberg and Makai [BKM91]. They proved that the conjecture holds for packings consisting of parallel strings of balls. A string of balls is a collection of congruent balls whose centers are collinear and such that each of them touches two others. Before the confirmation of the Kepler conjecture, the best upper bound for $\delta(B^3)$ was given by Muder [Mud93], who proved that $\delta(B^3) \leq 0.773055$.

The first step toward the solution of Kepler's conjecture in its full generality was made in the early 1950s by L. Fejes Tóth (see [Fej72] pp. 174–181). He

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considered weighted averages of the volumes of Dirichlet cells of a finite collection of balls in a packing. He showed that the Kepler conjecture holds if a particular weighted average of volumes involving not more than 13 cells is greater than or equal the volume of the rhombic dodecahedron circumscribed around a ball (this being the Dirichlet cell of a ball in the face-centered cubic lattice). His argument constitutes a program that, if realizable in principle, reduces Kepler's conjecture to an optimization problem in a finite number of variables. Later, in [Fej64] p. 300, he suggested that with the use of computers $\delta(B^3)$ could be "approximated with great exactitude."

In 1990 W.-Y. Hsiang announced the solution of the Kepler conjecture. His approach is very similar to the program proposed by L. Fejes Tóth. Unfortunately, Hsiang's paper [Hsi93] contains significant gaps, so it cannot be accepted as a proof. Hsiang maintains his claim of having a proof. He gave more detail in [Hsi01]. The mathematical community lost interest in checking those details, however.

About the same time as Hsiang, Tom Hales also attacked the Kepler conjecture. His first attempt [Hal92] was a program based on the Delone subdivision of space, which is dual to the subdivision by Dirichlet cells. He modified his approach in several steps [Hal93, Hal97, Hal98]. His final version, worked out in collaboration with his graduate student Ferguson in [HF06], uses a subdivision that is a hybrid of certain Delone-type tetrahedra and Dirichlet cells. With each ball B in a saturated packing of unit balls, an object, called a *decomposition star*, is associated, consisting of certain tetrahedra having the center of B as a common vertex together with parts of a modified Dirichlet cell of B. A complicated scoring rule is introduced that takes into account the volumes of the different parts of the decomposition star with appropriate weights. The score of a decomposition star in the face-centered cubic lattice is a certain number, which Hales takes to be 8. The key property of the decomposition stars and the scoring rule is that the decomposition star of a ball B, as well as its score, depends only on balls lying in a certain neighborhood of B. From the mathematical point of view, the main step of the proof is the theorem that

> *the Kepler conjecture holds, provided the score of each decomposition star in a saturated packing of unit balls is at most 8.*

The task of proving this, which is an optimization problem in finitely many variables, has been carried out with the aid of computers. As Hales points out, there is hope that in the future such a problem "might eventually become an instance of a general family of optimization problems for which general optimization techniques exist." In the absence of such general techniques, manual procedures had to be used to guide the work of computers.

Computers are used in the proof in several ways. The topological structure of the decomposition stars is described by planar maps. A computer program enumerates around 5000 planar maps that have to be examined as potential counterexamples to the conjecture. Interval arithmetic is used to prove various inequalities. Nonlinear optimization problems are replaced by linear problems that dominate the original ones in order to apply linear programming methods. Even the organization of the few gigabytes of data is a difficult task.

After examining the proof for over two years the team of a dozen referees came to the conclusion that the general framework of the proof is sound, they did not find any error, but they cannot say for certain that everything is correct. In particular they could not check the work done by the computer. The theoretical foundation of the proof was published in the *Annals of Mathematics* [Hal05], and the details, along with historical notes, were given in a series of articles in a special issue of *Discrete and Computational Geometry* ([Hal06a, Hal06b, Hal06c, Hal06d, HF06, HF11].

It is safe to say that the 300-page proof, aided by computer calculations taking months, is one of the most complex proofs in the history of mathematics. Lagarias, who in [Lag02] extracts the common ideas of the programs of L. Fejes Tóth, W.-Y. Hsiang, and Hales and puts them into a general framework, finds that "the Hales–Ferguson proof, assumed correct, is a tour de force of nonlinear optimization."

Disappointed because the referees were unable to certify the correctness of the proof, and realizing that, besides him, probably no human being will ever check all details, Hales launched a project named Flyspeck designed for a computerized formal verification of his proof. The article [HHM10] by the Flyspeck team reorganizes the original proof into a more transparent form to provide a greater level of certification of the correctness of the computer code and other details of the proof. The final part of the paper lists errata in the original proof of the Kepler conjecture. The book [Hal12] shows in detail how geometric ideas and elements of proof are arranged and processed in preparation for the formal proof-checking scrutiny.

Marchal [Mar11] proposed an alternate subdivision associated with the packing which is simpler and provides a less complex strategy of proof than that of Hales.

On August 10, 2014 the team of the Flyspeck project announced the successful completion of the project [Fly14], where they noted that "the formal proof takes the same general approach as the original proof, with modifications in the geometric partition of space that have been suggested by Marchal."

EXISTENCE OF ECONOMICAL ARRANGEMENTS

Table 2.1.4 lists the known bounds establishing the existence of reasonably dense packings and thin coverings. When c appears in a bound without specification, it means a suitable constant characteristic of the specific bound. The proofs of most of these are nonconstructive. For constructive methods yielding slightly weaker bounds, as well as improvements for special convex bodies.

Bound 1 for the packing density of general convex bodies follows by combining Bound 6 with the relation (2.1.3) and the inequality $V(K-K) \leq \binom{2d}{d}V(K)$ of Rogers and Shephard [RS57]). For $d \geq 3$ all methods establishing the existence of dense packings rely on the theory of lattices, thus providing the same lower bounds for $\delta(K)$ and $\delta_T(K)$ as for $\delta_L(K)$.

Better bounds than for general convex bodies are known for balls. Improving earlier results by Ball [Bal92] and Vance [Van11], Venkatesh [Ven13] proved that for any constant $c > \sinh^2(\pi e)/\pi^2 e^3 = 65963.8\ldots$ there is a number $n(c)$ such that for $n > n(c)$ we have $\delta(B^n) \geq \delta_L(B^n) \geq cn2^{-n}$. Moreover, there are infinitely many dimensions n for which $\delta_L(B^n) \geq n \ln\ln n 2^{-n-1}$.

Rogers [Rog59] proved that

$$\vartheta_L(B^n) \leq cn(\log_e n)^{\frac{1}{2}\log_2 2\pi e}.$$

Gritzmann [Gri85] proved a similar bound for a larger class of convex bodies:

$$\vartheta_L(K) \leq cd(\ln d)^{1+\log_2 e}$$

holds for a suitable constant c and for every convex body K in $\mathbb{E}^d$ that has an affine image symmetric about at least $\log_2 \ln d + 4$ coordinate hyperplanes.

TABLE 2.1.4 Bounds establishing the existence of dense packings and thin coverings.

No.	BOUND	SOURCE
	Bounds for general convex bodies in $\mathbb{E}^d$	
1	$\delta_L(K) \geq cd^{3/2}/4^d$ (d large)	[Sch63a, Sch63b]
2	$\vartheta_T(K) \leq d \ln d + d \ln\ln d + 5d$	[Rog57]
3	$\vartheta_L(K) \leq d^{\log_2 \ln d + c}$	[Rog59]
	$\vartheta(K) \leq 3 \quad K \in \mathbb{E}^3$	[Smi00]
	Bounds for centrally symmetric convex bodies in $\mathbb{E}^d$	
4	$\delta_L(K) \geq \zeta(d)/2^{d-1}$	[Hla43]
5	$\delta_L(K) \geq cd/2^d$ (d large)	[Sch63a, Sch63b]
	$\delta_L(K) \geq 0:538\ldots \quad K \in \mathbb{E}^3$	[Smi05]
	Bounds for general convex bodies in $\mathbb{E}^2$	
6	$\delta(K) \geq \sqrt{3}/2 = 0.8660\ldots$	[KK90]
7	$\vartheta(K) \leq 1.2281771\ldots$	[Ism98]
8	$\delta_L(K) \geq 2/3$	[Far50]
9	$\vartheta_L(K) \leq 3/2$	[Far50]
	Bounds for centrally symmetric convex bodies in $\mathbb{E}^2$	
10	$\delta_L(K) \geq 0.892656\ldots$	[Tam70]
11	$\vartheta_L(K) \leq 2\pi/\sqrt{27}$	[Fej72, p. 103]

The determination of the densest packing of congruent regular tetrahedra is mentioned as part of Problem 18 of Hibert's famous problems [Hil00]. In recent years a series of papers was devoted to the construction of dense packing of regular tetrahedra. The presently known best arrangement was constructed by Chen, Engel and Glotzer [CEG10]. It has density $4000/4671 = 0.856347\ldots$. A nice survey on packing regular tetrahedra was written by Lagarias and Zong [LZ02].

UPPER BOUNDS FOR $\delta(K)$ AND LOWER BOUNDS FOR $\vartheta(K)$

The packing density of B^d is not known, except for the cases mentioned in Table 2.1.1. Asymptotically, the best upper bound known for $\delta(B^d)$ is

$$\delta(B^d) \leq 2^{-0.599d+o(d)} \qquad (\text{as } d \to \infty), \tag{2.1.4}$$

given by Kabatiansky and Levenshtein [KL78]. This bound is not obtained directly by the investigation of packings in $\mathbb{E}^d$ but rather through studying the analogous problem in spherical geometry, where the powerful technique of linear programming can be used (see Section 2.4). For low dimensions, Rogers's simplex bound

$$\delta(B^d) \leq \sigma_d \tag{2.1.5}$$

gives a better estimate (see [Rog58]). Here, σ_d is the ratio between the total volume of the sectors of $d+1$ unit balls centered at the vertices of a regular simplex of edge length 2 and the volume of the simplex.

Recently, Rogers's bound has been improved in low dimensions as well. On one hand, K. Bezdek [Bez02] extended the method of Rogers by investigating the

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surface area of the Voronoi regions, rather than their volume; on the other hand, Cohn and Elkies [CE03, Coh02] developed linear programming bounds that apply directly to sphere packings in $\mathbb{E}^d$. This latter method is very powerful. Using the approach of [CE03] Cohn and Kumar [CK04, CK09] reconfirmed Blichfeldt's result concerning the value of $\delta_L(B^8)$ and determined $\delta_L(B^{24})$. Finally, Viazovska [Via17] succeeded in proving that $\delta(B^8) = \delta_L(B^8)$ and Cohn, Kumar, Miller, Radchenko and Viazovska [CKM17] showed that $\delta(B^)24 = \delta_L(B^{24})$.

Coxeter, Few, and Rogers [CFR59] proved a dual counterpart to Rogers's simplex bound:

$$\vartheta(B^d) \geq \tau_d,$$

where τ_d is the ratio between the total volume of the intersections of $d+1$ unit balls with the regular simplex of edge $\sqrt{2(d+1)/d}$ if their centers lie at the vertices of the simplex, and the volume of the simplex. Asymptotically,

$$\tau_d \sim d/e^{3/2}.$$

In contrast to packings, where there is a sizable gap between the upper bound (2.1.4) and the lower bound (Bound 6 in Table 2.1.4), this bound compares quite favorably with the corresponding Bound 2 in Table 2.1.4.

It is known that there is no tiling of space by regular tetrahedra or octahedra. However, until recently no nontrivial upper bound was known for the packing density of these solids. Gravel, Elser, and Kallus [GEK11] proved upper bounds of $1 - 2.6\ldots \times 10^{-25}$ and $1 - 1.4\ldots \times 10^{-12}$, respectively, for the packing density of the regular tetrahedron and octahedron. According to a result of W. Schmidt (see [Sch61]), we have $\delta(K) < 1$ and $\vartheta(K) > 1$ for every smooth convex body; but the method of proof does not allow one to derive any explicit bound. There is a general upper bound for $\delta(K)$ that is nontrivial (smaller than 1) for a wide class of convex bodies [FK93a]. It is quite reasonable for "longish" bodies. For cylinders in $\mathbb{E}^d$, the bound is asymptotically equal to the Kabatiansky-Levenshtein bound for B^d (as $d \to \infty$). The method yields nontrivial upper bounds also for the packing density of the regular cross-polytope for all dimensions greater than 6 (see [FFV15]).

Zong [Zon14] proved the bound $\delta_T(T) \leq 0.384061$ for the translational packing density of a tetrahedron. De Oliveira Filho and Vallentin [OV] extended the method of Cohn and Elkies to obtain upper bounds for the packing density of convex bodies. In [DGOV17] this extension is used to obtain upper bounds for the translational packing density of superballs and certain Platonic and Archimedean solids in three dimensions. In particular, Zong's upper bound for the translational packing density of a tetrahedron is improved to 0.3745, getting closer to the density $18/49 = 0.3673\ldots$ of the densest lattice packing of tetrahedra.

We note that no nontrivial lower bound is known for $\vartheta(K)$ for any $K \in \mathbb{E}^d$, $d \geq 3$, other than a ball.

REGULARITY OF OPTIMAL ARRANGEMENTS

The packings and coverings attaining the packing and covering densities of a set are, of course, not uniquely determined, but it is a natural question whether there exist among the optimal arrangements some that satisfy certain regularity properties. Of particular interest are those bodies for which the densest packing and/or thinnest covering with congruent copies can be realized by a lattice arrangement.

As mentioned above, $\delta(K) = \delta_L(K)$ for $K \in \mathcal{K}^*(\mathbb{E}^2)$. A plausible interpretation of this result is that the assumption of maximum density creates from a chaotic structure a regular one. Unfortunately, certain results indicate that such bodies are rather exceptional.

Let $\mathcal{L}_p$ and $\mathcal{L}_c$ be the classes of those convex disks $K \in \mathcal{K}(\mathbb{E}^2)$ for which $\delta(K) = \delta_L(K)$ and $\vartheta(K) = \vartheta_L(K)$, respectively. Then, in the topology induced by the Hausdorff metric on $\mathcal{K}(\mathbb{E}^2)$, the sets $\mathcal{L}_p$ and $\mathcal{L}_c$ are nowhere dense [FZ94, Fej95]. It is conjectured that an analogous statement holds also in higher dimensions.

Rogers [Rog64, p. 15] conjectures that for sufficiently large d we have $\delta(B^d) > \delta_L(B^d)$. The following result of A. Bezdek and W. Kuperberg (see [BK91]) supports this conjecture: For $d \geq 3$ there are ellipsoids E in $\mathbb{E}^d$ for which $\delta(E) > \delta_L(E)$. An even more surprising result holds for coverings [FK95]: For $d \geq 3$ every strictly convex body K in $\mathbb{E}^d$ has an affine image K' such that $\vartheta(K') < \vartheta_L(K')$. In particular, there is an ellipsoid E in $\mathbb{E}^3$ for which

$$\vartheta(E) < 1.394 < \frac{3\sqrt{3}}{2}(3 \operatorname{arcsec} 3 - \pi) = \tau_3 \leq \vartheta_T(E) \leq \vartheta_L(E).$$

We note that no example of a convex body K is known for which $\delta_L(K) < \delta_T(K)$ or $\vartheta_L(K) > \vartheta_T(K)$.

Schmitt [Sch88a] constructed a star-shaped prototile for a monohedral tiling in $\mathbb{E}^3$ such that no tiling with its replicas is periodic. It is not known whether a convex body with this property exists; however, with a slight modification of Schmitt's construction, Conway produced a convex prototile that admits only nonperiodic tilings if no mirror-image is allowed (see Section 3.4). Another result of Schmitt's [Sch91] is that there are star-shaped sets in the plane whose densest packing cannot be realized in a periodic arrangement.

2.2 FINITE ARRANGEMENTS

PACKING IN AND COVERING OF A BODY WITH GIVEN SHAPE

What is the size of the smallest square tray that can hold n given glasses? Thue's result gives a bound that is asymptotically sharp as $n \to \infty$; however, for practical reasons, small values of n are of interest.

Generally, for given sets K and C and a positive integer n one can ask for the quantities

$$M_p(K, C, n) = \inf\{\lambda \mid n \text{ congruent copies of } C \text{ can be packed in } \lambda K\}$$

and

$$M_c(K, C, n) = \sup\{\lambda \mid n \text{ congruent copies of } C \text{ can cover } \lambda K\}.$$

Tables 2.2.1–2.2.2 contain the known results of $M_p(K, B^2, n)$ in the cases when K is a circle, square, or regular triangle.

Most of these results were obtained by ad hoc methods. An exception is the case of packing circles in a square. In [GMP94, Pei94] a heuristic algorithm for the determination of $M_p(K, B^2, n)$ and the corresponding optimal arrangements is given in the case where K is the unit square. The algorithm consists of the following steps:

TABLE 2.2.1 Packing of congruent circles in unit squares.

n	$M_p(K,B^2,n)$	SOURCE	n	$M_p(K,B^2,n)$	SOURCE
2	3.414213562...	(elementary)	17	8.532660354...	[GMP94]
3	3.931851653...	(elementary)	18	8.656402355...	[GMP94]
4	4	(elementary)	19	8.907460939...	[GMP94]
5	4.828427125...	(elementary)	20	8.978083353...	[GMP94]
6	5.328201177...	[Gra63, Mel94c]	21	9.358035345...	[NO97]
7	5.732050807...	Schaer (unpublished)	22	9.463845078...	[NO97]
8	5.863703305...	[SM65]	23	9.727406613...	[NO97]
9	6	[Sch95]	24	9.863703306...	[NO97]
10	6.747441523...	[GPW90]	25	10	[Wen87b]
11	7.022509506...	[GMP94]	26	10.37749821...	[NO97]
12	7.144957554...	[GMP94]	27	10.47998305...	[NO97]
13	7.463047839...	[GMP94]	28	10.67548744...	[Mar04]
14	7.732050808...	[Wen87a]	29	10.81512001...	[Mar07]
15	7.863703305...	[GMP94]	30	10.90856381...	[Mar07]
16	8	[Wen83]	36	12	[KW87]

Step 1. Find a good upper bound m for $M_p(K,B^2,n)$. This requires the construction of a reasonably good arrangement, which can be established, e.g., by the Monte Carlo method.

Step 2. Iterate an elimination process on a successively refined grid to restrict possible locations for the centers of a packing of unit circles in mK.

Step 3. Based on the result of Step 2, guess the nerve graph of the packing, then determine the optimal packing with the given graph.

Step 4. Verify that the arrangement obtained in Step 3 is indeed optimal.

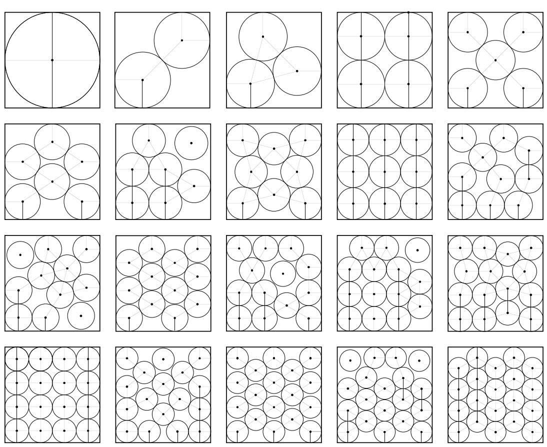

FIGURE 2.2.1
Densest packing of $n \le 20$ equal circles in a square.

We do not know whether these steps always provide the optimal arrangement in finite time, but in [GMP94, GPW90] the method was implemented successfully for $n \le 20$. The best arrangements are shown in Figure 2.2.1. Observe that quite often an optimal arrangement can contain a freely movable circle. Using more refined numerical technics Nurmela and Östergård [NO97] and Markót [Mar04, Mar07] solved the cases $21 \le n \le 30$ as well.

The sequence $M_p(K,B^2,n)$ seems to be strictly increasing when K is a square

TABLE 2.2.2 Packing of congruent circles in circles (left), and in equilateral triangles T of unit side length (right)

n	$M_p(B^2, B^2, n)$	SORCE	n	$M_p(T, B^2, n)$	SORCE
2	2	(elementary)	2	5.464101615...	(elementary)
3	2.154700538 ...	(elementary)	3	5.464101615 ...	(elementary)
4	2.414213562 ...	(elementary)	4	6.92820323 ...	[Mel93]
5	2.701301617 ...	(elementary)	5	7.464101615 ...	[Mel93]
6	3	(elementary)	6	7.464101615 ...	[Ole61, Gro66]
7	3	(elementary)	7	8.92820323 ...	[Mel93]
8	3.304764871 ...	[Pir69]	8	9.293810046 ...	[Mel93]
9	3.61312593 ...	[Pir69]	9	9.464101615 ...	[Mel93]
10	3.813898249 ...	[Pir69]	10	9.464101615 ...	[Ole61, Gro66]
11	3.9238044 ...	[Mel94a]	11	10.73008794 ...	[Mel93]
12	4.02960193 ...	[Fod00]	12	10.92820323 ...	[Mel94b]
13	4.23606797 ...	[Fod03a]	$\frac{k(k+1)}{2}$	$2(k+\sqrt{3}-1)$	[Ole61, Gro66]
14	4.32842855 ...	[Fod03b]			
19	4.86370330 ...	[Fod99]			

or when K is a circle and $n \geq 7$. In contrast to this, it is conjectured that in the case where K is a regular triangle, we have $M_p(K, B^2, n) = M_p(K, B^2, n-1)$ for all triangular numbers $n = k(k+1)/2$ $(k > 1)$.

The problem of finding the densest packing of n congruent circles in a circle has been considered also in the Minkowski plane. In terms of Euclidean geometry, this is the same as asking for the smallest number $\varrho(n, K)$ such that n mutually disjoint translates of the centrally symmetric convex disk K (the unit circle in the Minkowski metric) can be contained in $\varrho(n, K)K$. Doyle, Lagarias, and Randell [DLR92] solved the problem for all $K \in \mathcal{K}^*(\mathbb{E}^2)$ and $n \leq 7$. There is an n-gon inscribed in K having equal sides in the Minkowski metric (generated by K) and having a vertex at an arbitrary boundary point of K. Let $\alpha(n, K)$ be the maximum Minkowski side-length of such an n-gon. Then we have $\varrho(n, K) = 1 + 2/\alpha(n, K)$ for $2 \leq n \leq 6$ and $\varrho(7, K) = \varrho(6, K) = 3$.

The densest packing of n congruent balls in a cube in $\mathbb{E}^3$ was determined for $n \leq 10$ by Schaer [Sch66a, Sch66b, Sch66c, Sch94] and for $n = 14$ by Joós [Joo09a]). The problem of finding the densest packing of n congruent balls in the regular cross-polytope was solved for $n \leq 7$ by Golser [Gol77]. Böröczky Jr. and Wintsche [BW00] generalized Golser's result to higher dimensions. K. Bezdek [Bez87] solved the problem of packing n congruent balls in a regular tetrahedron, for $n = 5, 8, 9$ and 10. The known values of $M_c(K, B^2, n)$ in the cases when K is a circle, square, or regular triangle are shown in Table 2.2.3.

G. Kuperberg and W. Kuperberg solved the problem of thinnest covering of the cube with k congruent balls for $k = 2$, 3, 4 and 8. Joós [Joo14a, Joo14b] settled the cases $k = 5$ and 6. Joós [Joo08, Joo09b] also proved that the minimum radius of 8 congruent balls that can cover the unit 4-dimensional cube is $\sqrt{5/12}$. The problem of covering the n-dimensional cross-polytope with k congruent balls of minimum radius was studied by Böröczky, Jr., Fábián and Wintsche [BFW06] who found the solution for $k = 2$, n, and $2n$. Remarkably, the case $k = n = 3$ is exceptional, breaking the pattern holding for $k = n \neq 3$.

TABLE 2.2.3 Covering circles, squares, and equilateral triangles with congruent circles.

K	n	$M_c(K, B^2, n)$	SOURCE
B^2	2	2	(elementary)
	3	$2/\sqrt{3}$	(elementary)
	4	$\sqrt{2}$	(elementary)
	5	$1.64100446\ldots$	[Bez84]
	6	$1.7988\ldots$	[Bez84]
	7	2	(elementary)
	8–10	$1 + 2\cos\frac{2\pi}{n-1}$	[Fej05b]
Unit square	2	$4\sqrt{5}/5$	(elementary)
	3	$\frac{16}{\sqrt{65}}$	[BB85, HM97]
	4	$2\sqrt{2}$	[BB85, HM97]
	5	$3.065975\ldots$	[BB85, HM97]
	6	$\frac{12}{\sqrt{13}}$	[BB85]
	7	$1 + \sqrt{7}$	[BB85, HM97]
Regular triangle of side 1	2	2	(elementary)
	3	$2\sqrt{3}$	[BB85, Mel97]
	4	$2 + \sqrt{3}$	[BB85, Mel97]
	5	4	[BB85, Mel97]
	6	$3\sqrt{3}$	[BB85, Mel97]
	9	6	[BB85]
	10	$4\sqrt{3}$	[BB85]

SAUSAGE CONJECTURES

Intensive research on another type of finite packing and covering problem has been generated by the sausage conjectures of L. Fejes Tóth and Wills (see [GW93]):

What is the convex body of minimum volume in $\mathbb{E}^d$ that can accommodate k nonoverlapping unit balls? What is the convex body of maximum volume in $\mathbb{E}^d$ that can be covered by k unit balls?

According to the conjectures mentioned above, for $d \geq 5$ the extreme bodies are "sausages" and in the optimal arrangements the centers of the balls are equally spaced on a line segment (Figure 2.2.2).

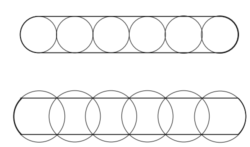

FIGURE 2.2.2
Sausage-like arrangements of circles.

After several partial results supporting these conjectures (see [GW93]) the breakthrough concerning the sausage conjecture for ball packings was achieved by Betke, Henk, and Wills [BHW94]: they proved that the conjecture holds for dimensions $d \geq 13387$. Betke and Henk [BH98] improved the bound on d to $d \geq 42$.

Several generalizations of the problems mentioned above have been considered. Connections of these types of problems to the classical theory of packing and coverings, as well as to crystallography, have been observed. For details we refer to [Bor04].

THE COVERING PROBLEMS OF BORSUK, HADWIGER, and LEVI

In 1933, Borsuk [Bor33] formulated the conjecture that any bounded set in $\mathbb{E}^d$ can be partitioned into $d+1$ subsets of smaller diameter. Borsuk verified the conjecture for $d = 2$, and the three-dimensional case was settled independently by Eggleston, Grünbaum, and Heppes. The Borsuk conjecture was proved for many special cases: for smooth convex bodies by Hadwiger [Had46] and Melzak [Mel67], for sets having the symmetry group of the regular simplex by Rogers [Rog71], and for sets of revolution by Kołodziejczyk [Kol88]. Hadwiger and Melzak's result was generalized by Dekster [Dek93] who proved that the conjecture holds for every convex body for which there exists a direction in which every line tangent to the body contains at least one point of the body's boundary at which the tangent hyperplane is unique.

However, Kahn and Kalai [KK93] showed that Borsuk's conjecture is false in the following very strong sense: Let $b(d)$ denote the smallest integer such that every bounded set in $\mathbb{E}^d$ can be partitioned into $b(d)$ subsets of smaller diameter. Then $b(d) \geq (1.2)^{\sqrt{d}}$ for every sufficiently large value of d. The best known upper bound for $b(d)$ is $b(d) \leq (\sqrt{3/2} + o(1))^n$ due to Schramm [Sch88b]. The lowest dimension known in which the Borsuk conjecture fails is 64 [JB14]. The papers [Rai07] and [Rai08] are excellent surveys on Borsuk's problem.

In the 1950's, Hadwiger and Levi, independently of each other, asked for the smallest integer $h(K)$ such that the convex body K can be covered by $h(K)$ smaller positive homothetic copies of K. Boltjanskiĭ observed that the Hadwiger-Levi covering problem for convex bodies is equivalent to an illumination problem. We say that a boundary point x of the convex body K is ***illuminated from the direction u*** if the ray emanating from x in the direction u intersects the interior of K. Let $i(K)$ be the minimum number of directions from which the boundary of K can be illuminated. Then $h(K) = i(K)$ for every convex body.

It is conjectured that $h(K) \leq 2^d$ for all $K \in \mathcal{K}(\mathbb{E}^d)$ and that equality holds only for parallelotopes. Levi verified the conjecture for the plane, but it is open for $d \geq 3$. Lassak proved Hadwiger's conjecture for centrally symmetric convex bodies in $\mathbb{E}^3$, and K. Bezdek extended Lassak's result to convex polytopes with any affine symmetry. There is a great variety of results confirming the conjecture for special classes of bodies in $\mathbb{E}^n$ by establishing upper bounds for $h(K)$ or $i(K)$ smaller than 2^n. The difficulty of the problem is exposed by the following example of Naszódi [Nas16b]. Clearly, we have $h(B^d) = d + 1$. On the other hand, for any $\varepsilon > 0$ there is a centrally symmetric convex body K and a positive constant $c = c(\varepsilon)$ such that K is ε close to B^d and $h(K) \geq c^d$. For literature and further results concerning the Hadwiger-Levi problem, we refer to [Bez93, BMS97, MS99].

2.3 MULTIPLE ARRANGEMENTS

GLOSSARY

k-Fold packing: An arrangement $\mathcal{A}$ such that each point of the space belongs to the interior of at most k members of $\mathcal{A}$.

***k*-Fold covering:** An arrangement $\mathcal{A}$ such that each point of the space belongs to at least k members of $\mathcal{A}$.

Densities: In analogy to the packing and covering densities of a body K, we define the quantities $\delta^k(K)$, $\delta^k_T(K)$, $\delta^k_L(K)$, $\vartheta^k(K)$, $\vartheta^k_T(K)$, and $\vartheta^k_L(K)$ as the suprema of the densities of all k-fold packings and the infima of the densities of all k-fold coverings with congruent copies, translates, and lattice translates of K, respectively.

TABLE 2.3.1 Bounds for k-fold packing and covering densities.

BOUND	SOURCE
$\delta^k_T(K) \ge ck \qquad K \in \mathcal{K}(\mathbb{E}^d)$	[ER62]
$\vartheta^k_L(K) \le ((k+1)^{1/d} + 8d)^d \qquad K \in \mathcal{K}(\mathbb{E}^d)$	[Coh76]
$\delta^k_L(K) \ge k - ck^{2/5} \qquad K \in \mathcal{K}(\mathbb{E}^2)$	[Bol89]
$\vartheta^k_L(K) \le k + ck^{2/5} \qquad K \in \mathcal{K}(\mathbb{E}^2)$	[Bol89]
$\delta^k_L(P) \ge k - ck^{1/3} \qquad P$ convex polygon	[Bol89]
$\vartheta^k_L(P) \le k + ck^{1/3} \qquad P$ convex polygon	[Bol89]
$\delta^k_L(B^d) \le k - ck^{\frac{d-1}{2d}} \qquad d \not\equiv 1\ (4)$	[Bol79, Bol82]
$\delta^k_L(B^d) \le k - ck^{\frac{d-3}{2d}} \qquad d \equiv 1\ (4)$	[Bol79, Bol82]
$\vartheta^k_L(B^2) \ge k + ck^{\frac{d-1}{2d}} \qquad d \not\equiv 1\ (4)$	[Bol79, Bol82]
$\vartheta^k_L(B^2) \ge k + ck^{\frac{d-3}{2d}} \qquad d \equiv 1\ (4)$	[Bol79, Bol82]
$\delta^k(B^d) \ge (2k/(k+1))^{d/2}\delta(B^d)$	[Few64]
$\delta^k_L(B^d) \ge (2k/(k+1))^{d/2}\delta_L(B^d)$	[Few64]
$\delta^k(B^d) \le (1+d^{-1})((d+1)^k - 1)(k/(k+1))^{d/2}$	[Few64]
$\delta^2(B^d) \le \frac{4}{3}(d+2)(\frac{2}{3})^{d/2}$	[Few68]
$\vartheta^k(B^d) \ge ck \qquad c = c_d > 1$	[Fej79]
$\delta^k(B^2) \le \dfrac{\pi}{6}\cot\dfrac{\pi}{6k}$	[Fej76]
$\vartheta^k(B^2) \ge \dfrac{\pi}{3}\csc\dfrac{\pi}{3k}$	[Fej76]

The information known about the asymptotic behavior of k-fold packing and covering densities is summarized in Table 2.3.1. There, in the various bounds, different constants appear, all of which are denoted by c. The known values of $\delta^k_L(B^d)$ and $\vartheta^k_L(B^d)$ (for $k \ge 2$) are given in Table 2.3.2.

The k-fold lattice packing density and the k-fold lattice covering density of a triangle T was determined for all k by Sriamorn [Sri16]. We have $\delta^k_L(T) = \frac{2k^2}{2k+1}$ and $\vartheta^k_L(T) = \frac{2k+1}{2}$. Moreover, Sriamorn [Sri14] showed tat $\delta^k_T(T) = \delta^k_L(T)$ and Sriamorn and Wetayawanich [SW15] showed that $\vartheta^k_T(T) = \vartheta^k_L(T)$ for all k.

General methods for the determination of the densest k-fold lattice packings and the thinnest k-fold lattice coverings with circles have been developed by Horváth, Temesvári, and Yakovlev [THY87] and by Temesvári [Tem88], respectively.

These methods reduce both problems to the determination of the optima of finitely many well-defined functions of one variable. The proofs readily provide algorithms for finding the optimal arrangements; however, the authors did not try to implement them. Only the values of $\delta^9_L(B^2)$ and $\vartheta^8_L(B^2)$ have been added in this

TABLE 2.3.2 Known values of $\delta_L^k(B^d)$ and $\vartheta_L^k(B^d)$.

RESULT	SOURCE
$\delta_L^2(B^2) = \frac{\pi}{\sqrt{3}}$	[Hep59]
$\delta_L^3(B^2) = \frac{\sqrt{3}\pi}{2}$	[Hep59]
$\delta_L^4(B^2) = \frac{2\pi}{\sqrt{3}}$	[Hep59]
$\delta_L^5(B^2) = \frac{4\pi}{\sqrt{7}}$	[Blu63]
$\delta_L^6(B^2) = \frac{35\pi}{8\sqrt{6}}$	[Blu63]
$\delta_L^7(B^2) = \frac{8\pi}{\sqrt{15}}$	[Bol76]
$\delta_L^8(B^2) = \frac{3969\pi}{4\sqrt{220-2\sqrt{193}}\sqrt{449+32\sqrt{193}}}$	[Yak83]
$\delta_L^9(B^2) = \frac{25\pi}{2\sqrt{21}}$	[Tem94]
$\delta_L^2(B^3) = \frac{8\pi}{9\sqrt{3}}$	[FK69]
$\vartheta_L^2(B^2) = \frac{4\pi}{3\sqrt{3}}$	[Blu57]
$\vartheta_L^3(B^2) = \frac{\pi\sqrt{27138+2910\sqrt{97}}}{216}$	[Blu57]
$\vartheta_L^4(B^2) = \frac{25\pi}{18}$	[Blu57]
$\vartheta_L^5(B^2) = \frac{32\pi}{7\sqrt{7}}$	[Tem84]
$\vartheta_L^6(B^2) = \frac{98\pi}{27\sqrt{3}}$	[Tem92a]
$\vartheta_L^7(B^2) = 7.672\ldots$	[Tem92b]
$\vartheta_L^8(B^2) = \frac{32\pi}{3\sqrt{15}}$	[Tem94]
$\vartheta_L^2(B^3) = \frac{8\pi}{\sqrt{3}\sqrt{76\sqrt{6}-159}}$	[Few67]

way to the list of values of $\delta_L^k(B^2)$ and $\vartheta_L^k(B^2)$ that had been determined previously by ad hoc methods.

We note that we have $\delta_L^k(B^2) = k\delta_L(B^2)$ for $k \le 4$ and $\vartheta_L^2(B^2) = 2\vartheta_L(B^2)$. These are the only cases where the extreme multiple arrangements of circles are not better than repeated simple arrangements. These relations have been extended to arbitrary centrally symmetric convex disks by Dumir and Hans-Gill [DH72a, DH72b] and by G. Fejes Tóth (see [Fej84a]). There is a simple reason for the relations $\delta_L^3(K) = 3\delta_L(K)$ and $\delta_L^4(K) = 4\delta_L(K)$ ($K \in \mathcal{K}^*(\mathbb{E}^2)$): Every 3-fold lattice packing of the plane with a centrally symmetric disk is the union of 3 simple lattice packings and every 4-fold packing is the union of two 2-fold packings.

This last observation brings us to the topic of decompositions of multiple arrangements. Our goal here is to find insight into the structure of multiple arrangements by decomposing them into possibly few simple ones. Pach [Pac85] showed that any double packing with positive homothetic copies of a convex disk can be decomposed into 4 simple packings. Further, if $\mathcal{P}$ is a k-fold packing with convex disks such that for some integer L the inradius $r(K)$ and the area $a(K)$ of each member K of $\mathcal{P}$ satisfy the inequality $9\pi^2 kr^2(K)/a(K) \le L$, then $\mathcal{P}$ can be decomposed into L simple packings.

A set S is *cover-decomposable* if there is an integer $k = k(S)$ such that every locally finite k-fold covering with translates of S can be decomposed into two coverings. Pach [Pac86] proved that centrally symmetric convex polygons are cover-decomposable and conjectured that all convex disks are cover-decomposable. His proof heavily uses the property of central symmetry, and more than two decades elapsed until the cover-decomposability of a non-symmetric polygon, namely of the triangle was proved by Tardos and Tóth [TT07]. Soon after, Pálvölgyi and Tóth [PT10] proved that all convex polygons are cover-decomposable. On the other hand, it turned out [Pál13] that the circle is not cover-decomposable. Moreover, convex sets that have two parallel tangents which both touch it in a point where the boundary has positive finite curvature, are not cover-decomposable. Pálvölgyi [Pál10] showed that concave polygons with no parallel sides are not cover-decomposable. He also proved that polyhedra in 3 and higher dimensions are not cover-decomposable. For details we refer to the survey paper [PPT13] and the web-site `http://www.cs.elte.hu/~dom/covdec/`.

2.4 PROBLEMS IN NONEUCLIDEAN SPACES

Research on packing and covering in spherical and hyperbolic spaces has been concentrated on arrangements of balls. In contrast to spherical geometry, where the finite, combinatorial nature of the problems, as well as applications, have inspired research, investigations in hyperbolic geometry have been hampered by the lack of a reasonable notion of density relative to the whole hyperbolic space.

SPHERICAL SPACE

Let $M(d,\varphi)$ be the maximum number of caps of spherical diameter φ forming a packing on the d-dimensional spherical space $\mathbb{S}^d$, that is, on the boundary of B^{d+1}, and let $m(d,\varphi)$ be the minimum number of caps of spherical diameter φ covering $\mathbb{S}^d$. An upper bound for $M(d,\varphi)$, which is sharp for certain values of d and φ and yields the best estimate known as $d \to \infty$, is the so-called ***linear programming bound*** (see [CS93, pp. 257-266]). It establishes a surprising connection between $M(d,\varphi)$ and the expansion of real polynomials in terms of certain Jacobi polynomials. The Jacobi polynomials, $P_i^{(\alpha,\beta)}(x)$, $i = 0, 1 \dots, \alpha > -1$, $\beta > -1$, form a complete system of orthogonal polynomials on $[-1,1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. Set $\alpha = \beta = (d-1)/2$ and let

$$f(t) = \sum_{i=0}^{k} f_i P_i^{(\alpha,\alpha)}(t)$$

be a real polynomial such that $f_0 > 0$, $f_i \ge 0$ $(i = 1, 2, \dots, k)$, and $f(t) \le 0$ for $-1 \le t \le \cos\varphi$. Then

$$M(d,\varphi) \le f(1)/f_0.$$

With the use of appropriate polynomials Kabatiansky and Levenshtein [KL78] obtained the asymptotic bound:

$$\frac{1}{d}\ln M(d,\varphi) \le \frac{1+\sin\varphi}{2\sin\varphi}\ln\frac{1+\sin\varphi}{2\sin\varphi} - \frac{1-\sin\varphi}{2\sin\varphi}\ln\frac{1-\sin\varphi}{2\sin\varphi} + o(1).$$

This implies the simpler bound

$$M(d,\varphi) \le (1-\cos\varphi)^{-d/2}2^{-0.099d+o(d)} \qquad (\text{as } d\to\infty,\ \varphi\le\varphi^* = 62.9974\ldots).$$

Bound (2.1.4) for $\delta(B^d)$ follows in the limiting case when $\varphi \to 0$.

The following is a list of values of d and φ for which the linear programming bound turns out to be exact (see [Lev79]).

$M(2,\arccos 1/\sqrt{5}) = 12$	$M(3,\arccos 1/5) = 120$	$M(4,\arccos 1/5) = 16$
$M(5,\arccos 1/4) = 27$	$M(6,\arccos 1/3) = 56$	$M(7,\pi/3) = 240$
$M(20,\arccos 1/7) = 162$	$M(21,\arccos 1/11) = 100$	$M(21,\arccos 1/6) = 275$
$M(21,\arccos 1/4) = 891$	$M(22,\arccos 1/5) = 552$	$M(22,\arccos 1/3) = 4600$
	$M(23,\pi/3) = 196560$	

For small values of d and specific values of φ the linear programming bound is superseded by the "simplex bound" of Böröczky [Bor78], which is the generalization of Rogers's bound (2.1.5) for ball packings in $\mathbb{S}^d$. The value of $M(d,\varphi)$ has been determined for all d and $\varphi \ge \pi/2$ (see [DH51, Ran55]). We have

$$M(d,\varphi) = i+1 \qquad \text{for } \frac{1}{2}\pi + \arcsin\frac{1}{i+1} < \varphi \le \frac{1}{2}\pi + \arcsin\frac{1}{i}, \qquad i = 1,\ldots,d,$$

$$M(d,\varphi) = d+2 \qquad \text{for } \frac{1}{2}\pi < \varphi \le \frac{1}{2}\pi + \arcsin\frac{1}{d+1},$$

and

$$M(d,\frac{1}{2}\pi) = 2(d+1).$$

Consider a decreasing continuous positive-valued potential function f defined on $(0,4]$. How should N distinct points $\{x_1,x_2,\ldots,x_N\}$ be placed on the unit sphere in n-dimensional space to minimize the potential energy $\sum_{i\neq j} f(\|x_i - x_j\|^2)$. Of special interest are ***completely monotonic*** potential functions f which are infinitely differentiable and satisfy the inequalities $(-1)^k f^{(k)}(x) \ge 0$ for every $k \ge 0$ and every $x \in (0,4]$. Cohn and Kumar [CK07] introduced the notion of universally optimal arrangement. An arrangement of points is ***universally optimal*** if its potential energy is minimal under every completely monotonic potential function.

Cohn and Kumar [CK07] proved universal optimality of the sets of vertices of all regular simplicial polytopes in every dimension, as well as of several other arrangements in dimensions 2–8 and 21–24. These arrangements are listed in Table 1 of their paper and they coincide with the arrangements for which the linear programming bound for $M(d,\varphi)$ is sharp. For the potential function $f(r) = 1/r^s$, the leading term of the potential energy for large s comes from the minimal distance. It follows that for universally optimal arrangements of points the minimal distance is maximal. For, if there were an arrangement with the same number of points but a larger minimal distance, then it would have lower potential energy when s is sufficiently large. With the special choice of $f(r) = 2 - 1/r^2$ and $f(r) = \log(4/r)$, respectively, it also follows that these arrangements maximize the sum, as well as the product, of the distances between pairs of points.

Rogers [Rog63] proved the existence of thin coverings of the sphere by congruent balls. His bound was improved by Böröczky and Wintsche [BW00], Verger-Gaugry [Ver05] and Dumer [Dum07]. The latter author proved that the d-dimensional sphere can be covered by congruent balls with density $\frac{1}{2}d \ln d + \ln d + 5d$. In the limiting case when the radius of the balls approaches zero, this improves bound 2 of Table 2.1.4 by a factor of $\frac{1}{2}$. Naszódi [Nas16a] proved the existence of economic coverings of the sphere by congruent copies of a general spherically convex set.

TABLE 2.4.1 Densest packing and thinnest covering with congruent circles on a sphere.

n	a_n	Source	A_n	Source
2	$180°$	(elementary)	$180°$	(elementary)
3	$120°$	(elementary)	$180°$	(elementary)
4	$109.471\ldots°$	[Fej43a]	$141.047\ldots°$	[Fej43b]
5	$90°$	[SW51]	$126.869\ldots°$	[Sch55]
6	$90°$	[Fej43a]	$109.471\ldots°$	[Fej43b]
7	$77.866\ldots°$	[SW51]	$102.053\ldots°$	[Sch55]
8	$74.869\ldots°$	[SW51]		
9	$70.528\ldots°$	[SW51]		
10	$66.316\ldots°$	[Dan86, Har86]	$84.615\ldots°$	[Fej69a]
11	$63.435\ldots°$	[Dan86, Bor83]		
12	$63.435\ldots°$	[Fej43a]	$74.754\ldots°$	[Fej43b]
13	$57.136\ldots°$	[MT12]		
14	$55.670\ldots°$	[MT15]	$69.875\ldots°$	[Fej69a]
24	$43.667\ldots°$	[Rob61]		

Extensive research has been done on circle packings and circle coverings on $\mathbb{S}^2$. Traditionally, here the inverse functions of $M(2,\varphi)$ and $m(2,\varphi)$ are considered. Let a_n be the maximum number such that n caps of spherical diameter a_n can form a packing and let A_n be the minimum number such that n caps of spherical diameter A_n can form a covering on $\mathbb{S}^2$. The known values of a_n and A_n are given in Table 2.4.1. In addition, conjecturally best circle packings and circle coverings for $n \leq 130$, as well as good arrangements with icosahedral symmetry for $n \leq 55000$, have been constructed [HSS12]. The ad hoc methods of the earlier constructions have recently been replaced by different computer algorithms, but none of them has been shown to give the optimum.

Observe that $a_5 = a_6$ and $a_{11} = a_{12}$. Also, $A_2 = A_3$. It is conjectured that $a_n > a_{n+1}$ and $A_n > A_{n+1}$ in all other cases.

HYPERBOLIC SPACE

The density of a general arrangement of sets in d-dimensional hyperbolic space $\mathbb{H}^d$ cannot be defined by a limit as in $\mathbb{E}^d$ (see [FK93b]). The main difficulty is that in hyperbolic geometry the volume and the surface area of a ball of radius r are of the same order of magnitude as $r \to \infty$. In the absence of a reasonable definition of density with respect to the whole space, two natural problems arise:

(i) Estimate the density of an arrangement relative to a bounded domain.

(ii) Find substitutes for the notions of densest packing and thinnest covering.

Concerning the first problem, we mention the following result of K. Bezdek (see [Bez84]). Consider a packing of finitely many, but at least two, circles of radius r in the hyperbolic plane $\mathbb{H}^2$. Then the density of the circles relative to the outer parallel domain of radius r of the convex hull of their centers is at most $\pi/\sqrt{12}$.

As a corollary it follows that if at least two congruent circles are packed in a circular domain in $\mathbb{H}^2$, then the density of the packing relative to the domain is at most $\pi/\sqrt{12}$. We note that the density of such a finite packing relative to the convex hull of the circles can be arbitrarily close to 1 as $r \to \infty$. K. Böröczky Jr. [Bor05] proved a dual counterpart to the above-mentioned theorem of K. Bezdek, a corollary of which is that if at least two congruent circles cover a circular domain in $\mathbb{H}^2$, then the density of the covering relative to the domain is at most $2\pi/\sqrt{27}$.

Rogers's simplex bound (2.1.5) for ball packings in $\mathbb{E}^d$ has been extended by Böröczky [Bor78] to $\mathbb{H}^d$ as follows. If balls of radius r are packed in $\mathbb{H}^d$ then the density of each ball relative to its Dirichlet cell is less than or equal to the density of $d+1$ balls of radius r centered at the vertices of a regular simplex of side-length $2r$ relative to this simplex. Of course, we should not interpret this result as a global density bound. The impossibility of such an interpretation is shown by an ingenious example of Böröczky (see [FK93b]). He constructed a packing $\mathcal{P}$ of congruent circles in $\mathbb{H}^2$ and two tilings, $\mathcal{T}_1$ and $\mathcal{T}_2$, both consisting of congruent tiles, such that each tile of $\mathcal{T}_1$, as well as each tile of $\mathcal{T}_2$, contains exactly one circle from $\mathcal{P}$, but such that the tiles of $\mathcal{T}_1$ and $\mathcal{T}_2$ have different areas.

The first notion that has been suggested as a substitute for densest packing and thinnest covering is "solidity." $\mathcal{P}$ is a ***solid packing*** if no finite subset of $\mathcal{P}$ can be rearranged so as to form, together with the rest of $\mathcal{P}$, a packing not congruent to $\mathcal{P}$. Analogously, $\mathcal{C}$ is a ***solid covering*** if no finite subset of $\mathcal{C}$ can be rearranged so as to form, together with the rest of $\mathcal{C}$, a covering not congruent to $\mathcal{C}$. Obviously, in $\mathbb{E}^d$ a solid packing with congruent copies of a body K has density $\delta(K)$, and a solid covering with congruent copies of K has density $\vartheta(K)$. This justifies the use of solidity as a natural substitute for "densest packing" and "thinnest covering" in hyperbolic space.

The tiling with Schläfli symbol $\{p,3\}$ (see Chapters 18 or 20 of this Handbook) has regular p-gonal faces such that at each vertex of the tiling three faces meet. There exists such a tiling for each $p \geq 2$: for $p \leq 5$ on the sphere, for $p \geq 7$ on the hyperbolic plane, while for $p = 6$ we have the well-known hexagonal tiling in Euclidean plane. The incircles of such a tiling form a solid packing and the circumcircles form a solid covering. In addition, several packings and coverings by incongruent circles, including the incircles and the circumcircles of certain trihedral Archimedean tilings have been confirmed to be solid (see [Fej68, Fej74, Hep92, Flo00, Flo01, FH00]).

Other substitutes for the notion of densest packing and thinnest covering have been proposed in [FKK98] and [Kup00]. A packing $\mathcal{P}$ with congruent copies of a body K is ***completely saturated*** if no finite subset of $\mathcal{P}$ can be replaced by a greater number of congruent copies of K that, together with the rest of $\mathcal{P}$, form a packing. Analogously, a covering $\mathcal{C}$ with congruent copies of K is ***completely reduced*** if no finite subset of $\mathcal{C}$ can be replaced by a smaller number of congruent copies of K that, together with the rest of $\mathcal{C}$, form a covering. While there are convex bodies that do not admit a solid packing or solid covering, it has been shown [Bow03, FKK98] that each body in $\mathbb{E}^d$ or $\mathbb{H}^d$ admits a completely saturated packing and a completely reduced covering. However, the following rather counterintuitive result of Bowen makes it doubtful whether complete saturatedness and complete reducedness are good substitutes for the notions of densest packing and thinnest covering in hyperbolic space. For any $\varepsilon > 0$ there is a body K in $\mathbb{H}^d$ that admits a tiling and at the same time a completely saturated packing $\mathcal{P}$ with the following property. For every point p in $\mathbb{H}^d$, the limit

$$\lim_{\lambda\to\infty} \frac{1}{V(B_\lambda(p))} \sum_{P\in\mathcal{P}} V(P \cap (B_\lambda(p))$$

exists, is independent of p, and is less than ε. Here $V(\cdot)$ denotes the volume in $\mathbb{H}^d$ and $B_\lambda(p)$ denotes the ball of radius λ centered at p.

Bowen and Radin [BR03, BR04] proposed a probabilistic approach to analyze the efficiency of packings in hyperbolic geometry. Their approach can be sketched as follows.

Instead of studying individual arrangements, one considers the space Σ_K consisting of all saturated packings of $\mathbb{H}^d$ by congruent copies of K. A suitable metric on Σ_K is introduced that makes Σ_K compact and makes the natural action of the group $\mathcal{G}^d$ of rigid motions of $\mathbb{H}^d$ on Σ_K continuous. We consider Borel probability measures on Σ_K that are invariant under $\mathcal{G}^d$. For such an invariant measure μ the ***density*** $d(\mu)$ of μ is defined as $d(\mu) = \mu(A)$, where A is the set of packings $\mathcal{P} \in \Sigma_K$ for which the origin of $\mathbb{H}^d$ is contained in some member of $\mathcal{P}$. It follows easily from the invariance of μ that this definition is independent of the choice of the origin. The connection of density of measures to density of packings is established by the following theorem.

If μ is an ergodic invariant Borel probability measure on Σ_K, then—with the exception of a set of μ-measure zero—for every packing $\mathcal{P} \in \Sigma_K$, and for all $p \in \mathbb{H}^d$,

$$\lim_{\lambda\to\infty} \frac{1}{V(B_\lambda(p))} \sum_{P\in\mathcal{P}} V(P \cap (B_\lambda(p)) = d(\mu). \tag{2.4.1}$$

(A measure μ is ergodic if it cannot be expressed as the positive linear combination of two invariant measures.)

The ***packing density*** $\delta(K)$ of K can now be defined as the supremum of $d(\mu)$ for all ergodic invariant measures on Σ_K. A packing $\mathcal{P} \in \Sigma_K$ is ***optimally dense*** if there is an ergodic invariant measure μ such that the orbit of $\mathcal{P}$ under $\mathcal{G}^d$ is dense in the support of μ and, for all $p \in \mathbb{H}^d$, (2.4.1) holds.

It is shown in [BR03] and [BR04] that there exists an ergodic invariant measure μ with $d(\mu) = \delta(K)$ and a subset of the support of μ of full μ-measure of optimally dense packings. Bowen and Radin prove several results justifying that this is a workable notion of optimal density and optimally dense packings. In particular, the definitions carry over without any change to $\mathbb{E}^d$, and there they coincide with the usual notions. The advantage of this probabilistic approach is that it neglects pathological packings such as the example by Böröczky. As for packings of balls, it is shown in [BR03] that there are only countably many radii for which there exists an optimally dense packing of balls of the given radius that is periodic.

2.5 NEIGHBORS

GLOSSARY

Neighbors: Two members of a packing whose closures intersect.

Newton number $N(K)$ of a convex body K: The maximum number of neighbors of K in all packings with congruent copies of K.

Hadwiger number $H(K)$ of a convex body K: The maximum number of neighbors of K in all packings with translates of K.

n-neighbor packing: A packing in which each member has exactly n neighbors.

n^+-neighbor packing: A packing in which each member has at least n neighbors.

Table 2.5.1 contains the results known about Newton numbers and Hadwiger numbers. Cheong and Lee [CL07] showed that, surprisingly, there are non-convex Jordan regions with arbitrary large Hadwiger number.

It seems that the maximum number of neighbors of one body in a lattice packing with congruent copies of K is considerably smaller than $H(K)$. While $H(B^d)$ is of exponential order of magnitude, the highest known number of neighbors in a lattice packing with B^d occurs in the Barnes-Wall lattice and is $c^{O(\log d)}$ [CS93]. Moreover, Gruber [Gru86] showed that, in the sense of Baire categories, most convex bodies in $\mathbb{E}^d$ have no more than $2d^2$ neighbors in their densest lattice packing. Talata [Tal98b] gave examples of convex bodies in $\mathbb{E}^d$ for which the difference between the Hadwiger number and the maximum number of neighbors in a lattice packing is 2^{d-1}. Alon [Alo97] constructed a finite ball packing in $\mathbb{E}^d$ in which each ball has $c^{O(\sqrt{d})}$ neighbors.

A problem related to the determination of the Hadwiger number concerns the maximum number $C(K)$ of mutually nonoverlapping translates of a set K that have a common point. No more than four nonoverlapping translates of a topological disk in the plane can share a point [BKK95], while for $d \geq 3$ there are starlike bodies in $\mathbb{E}^d$ for which $C(K)$ is arbitrarily large.

For a given convex body K, let $M(K)$ denote the maximum natural number with the property that an $M(K)$-neighbor packing with finitely many congruent copies of K exists. For $n \leq M(K)$, let $L(n, K)$ denote the minimum cardinality, and, for $n > M(K)$, let $\lambda(n, K)$ denote the minimum density, of an n-neighbor packing with congruent copies of K. The quantities $M_T(K)$, $M^+(K)$, $M_T^+(K)$, $L_T(n, K)$, $L^+(n, K)$, $L_T^+(n, K)$, $\lambda_T(n, K)$, $\lambda^+(n, K)$, and $\lambda_T^+(n, K)$ are defined analogously.

Österreicher and Linhart [OL82] showed that for a smooth convex disk K we have $L(2, K) \geq 3$, $L(3, K) \geq 6$, $L(4, K) \geq 8$, and $L(5, K) \geq 16$. All of these inequalities are sharp. We have $M_T^+(K) = 3$ for all convex disks, and there exists a 4-neighbor packing of density 0 with translates of any convex disk. There exists a 5-neighbor packing of density 0 with translates of a parallelogram, but Makai [Mak85] proved that $\lambda_T^+(5, K) \geq 3/7$ and $\lambda_T^+(6, K) \geq 1/2$ for every $K \in \mathcal{K}(\mathbb{E}^2)$ that is not a parallelogram, and that $\lambda_T^+(5, K) \geq 9/14$ and $\lambda_T^+(6, K) \geq 3/4$ for every $K \in \mathcal{K}^*(\mathbb{E}^2)$ that is not a parallelogram. The case of equality characterizes triangles and affinely regular hexagons, respectively. According to a result of Chvátal [Chv75], $\lambda_T^+(6, P) = 11/15$ for a parallelogram P.

TABLE 2.5.1 Newton and Hadwiger numbers.

BODY K	RESULT	SOURCE
B^3	$N(K) = 12$	[SW53]
B^4	$N(K) = 24$	[Mus08]
B^8	$N(K) = 240$	[Lev79, OS79]
B^{24}	$N(K) = 196560$	[Lev79, OS79]
Regular triangle	$N(K) = 12$	[Bor71]
Square	$N(K) = 8$	[You39, Bor71, KLL95]
Regular pentagon	$N(K) = 6$	[ZX02]
Regular n-gon for $n \geq 6$	$N(K) = 6$	[Bor71, Zha98]
Isosceles triangle with base angle $\pi/6$	$N(K) = 21$	[Weg92]
Convex disk of diameter d and width w	$N(K) \leq (4 + 2\pi)\frac{d}{w} + \frac{w}{d} + 2$	[Fej69b]
Parallelotope in $\mathbb{E}^d$	$H(K) = 3^d - 1$	[Had46]
Tetrahedron	$H(K) = 18$	[Tal99a]
Octahedron	$H(K) = 18$	[LZ99]
Rhombic dodecahedron	$H(K) = 18$	[LZ99]
Starlike region in $\mathbb{E}^2$	$H(K) \leq 35$	[Lan11]
Centrally symmetric starlike region in $\mathbb{E}^2$	$H(K) \leq 12$	[Lan09]
Convex body in $\mathbb{E}^d$	$H(K) \leq 3^d - 1$	[Had46]
Convex body in $\mathbb{E}^d$	$H(K) \geq 2^{cd},\ c > 0$	[Tal98a]
Simplex in $\mathbb{E}^d$	$H(K) \geq 1.13488^{d-o(d)}$	[Tal00]
Unit ball in L_p norm	$2^{n+o(1)}$ as $p \to \infty$	[Xu07]
Set in $\mathbb{E}^d$ with int $(K - K) \neq \emptyset$	$H(K) \geq d^2 + d$	[Smi75]

A construction of Wegner (see [FK93c]) shows that $M(B^3) \geq 6$ and $L(6, B^3) \leq 240$, while Kertész [Ker94] proved that $M(B^3) \leq 8$. It is an open problem whether an n-neighbor or n^+-neighbor packing of finitely many congruent balls exists for $n = 7$ and $n = 8$.

The long-standing conjecture of L. Fejes Tóth [Fej69c] that a 12-neighbor packing of congruent balls consist of parallel hexagonal layers was recently confirmed independently by Hales [Hal13] and by Böröczky and Szabó [BS15]. Both proofs heavily depend on the use of computers. Hales uses the technique he developed for the proof of the Kepler conjecture, while the proof by Böröczky and Szabó relies on the computer-aided solution of the "strong thirteen spheres problem" by Musin and Tarasov.

Harborth, Szabó and Ujváry-Menyhárt [HSU02] constructed finite n-neighbor packings of incongruent balls in $\mathbb{E}^3$ for all $n \leq 12$ except for 11. The question whether a finite 11-neighbor packing of balls exists remains open. Since the smallest ball in a finite ball-packing has at most twelve neighbors, there is no finite n-neighbor packing for $n \geq 12$. On the other hand, the average number of neighbors in a finite packing of balls can be greater than 12. G. Kuperberg and Schramm [KS94] constructed a finite packing of balls in which the average number of neighbors is $666/53 = 12.566$ and showed that the average number of neighbors in every finite packing of balls is at most $8 + 4\sqrt{3} = 14.928$.

For 6^+-neighbor packings with (not necessarily equal) circles, the following nice theorem of Bárány, Füredi, and Pach [BFP84]) holds:

In a 6^+-neighbor packing with circles, either all circles are congruent or arbitrarily small circles occur.

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2.6 SELECTED PROBLEMS ON LATTICE ARRANGEMENTS

In this section we discuss, from the vast literature on lattices, some special problems concerning arrangements of convex bodies in which the restriction to lattice arrangements is automatically imposed by the nature of the problem.

GLOSSARY

Point-trapping arrangement: An arrangement $\mathcal{A}$ such that every component of the complement of the union of the members of $\mathcal{A}$ is bounded.

Connected arrangement: An arrangement $\mathcal{A}$ such that the union of the members of $\mathcal{A}$ is connected.

j-impassable arrangement: An arrangement $\mathcal{A}$ such that every j-dimensional flat intersects the interior of a member of $\mathcal{A}$.

Obviously, a point-trapping arrangement of congruent copies of a body can be arbitrarily thin. On the other hand, Bárány, Böröczky, Makai, and Pach showed that the density of a point-trapping *lattice* arrangement of any convex body in $\mathbb{E}^d$ is greater than or equal to $1/2$. For $d \geq 3$, equality is attained only in the "checkerboard" arrangement of parallelotopes (see [BBM86]).

Bleicher [Ble75] showed that the minimum density of a point-trapping lattice of unit balls in $\mathbb{E}^3$ is equal to

$$32\sqrt{(7142 + 1802\sqrt{17})^{-1}} = 0.265\ldots.$$

The extreme lattice is generated by three vectors of length $\frac{1}{2}\sqrt{7+\sqrt{17}}$, any two of which make an angle of $\arccos \frac{\sqrt{17}-1}{8} = 67.021\ldots^\circ$.

For a convex body K, let $c(K)$ denote the minimum density of a connected lattice arrangement of congruent copies of K. According to Groemer [Gro66],

$$\frac{1}{d!} \leq c(K) \leq \frac{\pi^{d/2}}{2^d\Gamma(1+d/2)} \qquad \text{for } K \in \mathcal{K}^d.$$

The lower bound is attained when K is a simplex or cross-polytope, and the upper bound is attained for a ball.

For a given convex body K in $\mathbb{E}^d$, let $\varrho_j(K)$ denote the infimum of the densities of all j-impassable lattice arrangements of copies of K. Obviously, $\varrho_0(K) = \vartheta_L(K)$. Let $\widehat{K} = (K-K)^*$ denote the polar body of the difference body of K. Between $\varrho_{d-1}(K)$ and $\delta_L(\widehat{K})$ Makai [Mak78], and independently also Kanan and Lovász [KL88], found the following surprising connection:

$$\varrho_{d-1}(K)\delta_L(\widehat{K}) = 2^d V(K) V(\widehat{K}).$$

Little is known about $\varrho_j(K)$ for $0 < j < d-1$. The value of $\varrho_1(B^3)$ has been determined by Bambah and Woods [BW94]. We have

$$\varrho_1(B^3) = 9\pi/32 = 0.8835\ldots.$$

An extreme lattice is generated by the vectors $\frac{4}{3}(1,1,0)$, $\frac{4}{3}(0,1,1)$, and $\frac{4}{3}(1,0,1)$.

2.7 PACKING AND COVERING WITH SEQUENCES OF CONVEX BODIES

In this section we consider the following problem: Given a convex set K and a sequence $\{C_i\}$ of convex bodies in $\mathbb{E}^d$, is it possible to find rigid motions σ_i such that $\{\sigma_i C_i\}$ covers K, or forms a packing in K? If there are such motions σ_i, then we say that the sequence $\{C_i\}$ permits an ***isometric covering*** of K, or an ***isometric packing*** in K, respectively. If there are not only rigid motions but even translations τ_i so that $\{\tau_i C_i\}$ is a covering of K, or a packing in K, then we say that $\{C_i\}$ permits a ***translative covering*** of K, or a ***translative packing*** in K, respectively.

First we consider translative packings and coverings of cubes by sequences of boxes. By a ***box*** we mean an orthogonal parallelotope whose sides are parallel to the coordinate axes. We let $I^d(s)$ denote a cube of side length s in $\mathbb{E}^d$.

Groemer (see [Gro85]) proved that a sequence $\{C_i\}$ of boxes whose edge lengths are at most 1 permits a translative covering of $I^d(s)$ if

$$\sum_i V(C_i) \geq (s+1)^d - 1,$$

and that it permits a translative packing in $I^d(s)$ if

$$\sum_i V(C_i) \leq (s-1)^d - \frac{s-1}{s-2}((s-1)^{d-2} - 1).$$

Slightly stronger conditions (see [Las97]) guarantee even the existence of on-line algorithms for the determination of the translations τ_i. This means that the determination of τ_i is based only on C_i and the previously fixed sets $\tau_i C_i$.

We recall (see [Las97]) that to any convex body K in $\mathbb{E}^d$ there exist two boxes, say Q_1 and Q_2, with $V(Q_1) \geq 2d^{-d}V(K)$ and $V(Q_2) \leq d!V(K)$, such that $Q_1 \subset K \subset Q_2$. It follows immediately that if $\{C_i\}$ is a sequence of convex bodies in $\mathbb{E}^d$ whose diameters are at most 1 and

$$\sum_i V(C_i) \geq \frac{1}{2}d^d((s+1)^d - 1),$$

then $\{C_i\}$ permits an isometric covering of $I^d(s)$; and that if

$$\sum_i V(C_i) \leq \frac{1}{d!}\left((s-1)^d - \frac{s-1}{s-2}((s-1)^{d-2} - 1)\right),$$

then it permits an isometric packing in $I^d(s)$.

The sequence $\{C_i\}$ of convex bodies is ***bounded*** if the set of the diameters of the bodies is bounded. As further consequences of the results above we mention the following. If $\{C_i\}$ is a bounded sequence of convex bodies such that $\sum V(C_i) = \infty$, then it permits an isometric covering of $\mathbb{E}^d$ with density $\frac{1}{2}d^d$ and an isometric packing in $\mathbb{E}^d$ with density $\frac{1}{d!}$. Moreover, if all the sets C_i are boxes, then $\{C_i\}$ permits a translative covering of $\mathbb{E}^d$ and a translative packing in $\mathbb{E}^d$ with density 1.

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In $\mathbb{E}^2$, any bounded sequence $\{C_i\}$ of convex disks with $\sum a(C_i) = \infty$ permits even a translative packing and covering with density $\frac{1}{2}$ and 2, respectively. It is an open problem whether for $d > 2$ any bounded sequence $\{C_i\}$ of convex bodies in $\mathbb{E}^d$ with $\sum V(C_i) = \infty$ permits a translative covering. If the sequence $\{C_i\}$ is unbounded, then the condition $\sum V(C_i) = \infty$ no longer suffices for $\{C_i\}$ to permit even an isometric covering of the space. For example, if C_i is the rectangle of side lengths i and $\frac{1}{i^2}$, then $\sum a(C_i) = \infty$ but $\{C_i\}$ does not permit an isometric covering of $\mathbb{E}^2$. There is a simple reason for this, which brings us to one of the most interesting topics of this subject, namely Tarski's plank problem.

A ***plank*** is a region between two parallel hyperplanes. Tarski conjectured that if a convex body of minimum width w is covered by a collection of planks in $\mathbb{E}^d$, then the sum of the widths of the planks is at least w. Tarski's conjecture was first proved by Bang. Bang's theorem immediately implies that the sequence of rectangles above does not permit an isometric covering of $\mathbb{E}^2$, not even of $(\frac{\pi^2}{12} + \epsilon)B^2$.

In his paper, Bang asked whether his theorem can be generalized so that the width of each plank is measured relative to the width of the convex body being covered, in the direction normal to the plank. Bang's problem has been solved for centrally symmetric bodies by Ball [Bal91]. This case has a particularly appealing formulation in terms of normed spaces: If the unit ball in a Banach space is covered by a countable collection of planks, then the total width of the planks is at least 2.

The paper of Groemer [Gro85] gives a nice account on problems about packing and covering with sequences of convex bodies.

2.8 SOURCES AND RELATED MATERIAL

SURVEYS

The monographs [Bor04, Fej72, Rog64, Zon99] are devoted solely to packing and covering; also the books [CS93, CFG91, EGH89, Fej64, Gru07, GL87, BMP05, PA95] contain results relevant to this chapter. Additional material and bibliography can be found in the following surveys: [Bar69, Fej83, Fej84b, Fej99, FK93b, FK93c, FK01, Flo87, Flo02, GW93, Gro85, Gru79].

RELATED CHAPTERS

Chapter 3: Tilings
Chapter 7: Lattice points and lattice polytopes
Chapter 13: Geometric discrepancy theory and uniform distribution
Chapter 18: Symmetry of polytopes and polyhedra
Chapter 20: Polyhedral maps
Chapter 64: Crystals, periodic and aperiodic

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