17 FACE NUMBERS OF POLYTOPES AND COMPLEXES

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INTRODUCTION

Geometric objects are often put together from simple pieces according to certain combinatorial rules. As such, they can be described as *complexes* with their constituent *cells*, which are usually polytopes and often simplices. Many constraints of a combinatorial and topological nature govern the incidence structure of cell complexes and are therefore relevant in the analysis of geometric objects. Since these incidence structures are in most cases too complicated to be well understood, it is worthwhile to focus on simpler invariants that still say something nontrivial about their combinatorial structure. The invariants to be discussed in this chapter are the *f-vectors* $f = (f_0, f_1, \ldots)$, where f_i is the number of i-dimensional cells in the complex.

The theory of f-vectors can be discussed at two levels: (1) the numerical relations satisfied by the f_i numbers, and (2) the algebraic, combinatorial, and topological facts and constructions that give rise to and explain these relations. This chapter will summarize the main facts in the numerology of f-vectors (i.e., at level 1), with emphasis on cases of geometric interest.

The chapter is organized as follows. To begin with, we treat simplicial complexes, first the general case (Section 17.1), then complexes with various Betti number constraints (Section 17.2), and finally triangulations of spheres, polytope boundaries, and manifolds (Section 17.3). Then we move on to nonsimplicial complexes, discussing first the general case (Section 17.4) and then polytopes and spheres (Section 17.5).

17.1 SIMPLICIAL COMPLEXES

GLOSSARY

The convex hull of any set of $j+1$ affinely independent points in $\mathbb{R}^n$ is called a ***j-simplex***. See Chapter 15 for more about this definition, and for the notions of *faces* and *vertices* of a simplex.

A ***geometric simplicial complex*** Γ is a finite nonempty family of simplices in $\mathbb{R}^n$ such that (i) $\sigma \in \Gamma$ implies that $\tau \in \Gamma$ for every face τ of σ, and (ii) if $\sigma, \tau \in \Gamma$ and $\sigma \cap \tau \neq \emptyset$ then $\sigma \cap \tau$ is a face of both σ and τ.

An ***abstract simplicial complex*** Δ is a finite nonempty family of subsets of some ground set V (the ***vertex set***) such that if $A \in \Delta$ and $B \subseteq A$ then

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$B \in \Delta$. (Note that always $\emptyset \in \Delta$.) The elements $A \in \Delta$ are called ***faces***. Define the ***dimension*** of a face A and of Δ itself by $\dim A = |A| - 1$; $\dim \Delta = \max_{A \in \Delta} \dim A$. By a ***d-complex*** we mean a d-dimensional complex.

With every geometric simplicial complex Γ we associate an abstract simplicial complex by taking the family of vertex sets of its simplices. Conversely, every d-dimensional abstract simplicial complex Δ can be realized in $\mathbb{R}^n$ for $n \geq 2d+1$ (and sometimes less) by some geometric simplicial complex. The latter is unique up to homeomorphism, so it is correct to think of the realization map as a one-to-one correspondence between abstract and geometric simplicial complexes. We will therefore drop the adjectives "abstract" and "geometric" and speak only of a ***simplicial complex***.

For a simplicial complex Δ, let $\Delta^i = \{i\text{-dimensional faces}\}$ and let $f_i = |\Delta^i|$. The integer sequence $f(\Delta) = (f_0, f_1, \ldots)$ is called the ***f-vector*** of Δ. (The entry $f_{-1} = 1$ is usually suppressed.) The subcomplex $\Delta^{\leq i} = \bigcup_{j \leq i} \Delta^j$ is called the ***i-skeleton*** of Δ.

A simplicial complex Δ is called ***pure*** if all maximal faces are of equal dimension. It is called ***r-colorable*** if there exists a partition of the vertex set $V = V_1 \cup \ldots \cup V_r$ such that $|A \cap V_i| \leq 1$ for all $A \in \Delta$ and $1 \leq i \leq r$. Equivalently, Δ is r-colorable if and only if its 1-skeleton $\Delta^{\leq 1}$ is r-colorable in the standard sense of graph theory. An $(r-1)$-complex that is both pure and r-colorable is sometimes called ***balanced***.

The ***clique complex*** of a graph is the collection of vertex sets of all its cliques (complete induced subgraphs). These are also known as ***flag complexes***.

For integers $k, n \geq 1$, there is a unique way of writing

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_i}{i}$$

so that $a_k > a_{k-1} > \ldots > a_i \geq i \geq 1$. Then define

$$\partial_k(n) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_i}{i-1},$$

and

$$\partial^k(n) = \binom{a_k - 1}{k-1} + \binom{a_{k-1} - 1}{k-2} + \ldots + \binom{a_i - 1}{i-1}.$$

Also let $\partial_k(0) = \partial^k(0) = 0$.

Let $\mathbb{N}^\infty$ denote the set of sequences $(n_0, n_1, \ldots)$ of nonnegative integers, and $\mathbb{N}^{(\infty)}$ the subset of sequences such that $n_k = 0$ for all sufficiently large k. We call $n \in \mathbb{N}^{(\infty)}$ a ***K-sequence*** if

$$\partial_{k+1}(n_k) \leq n_{k-1} \qquad \text{for all } k \geq 1.$$

We call $n \in \mathbb{N}^\infty$ an ***M-sequence*** if

$$n_0 = 1 \ \text{ and } \ \partial^k(n_k) \leq n_{k-1} \qquad \text{for all } k \geq 2.$$

THE KRUSKAL-KATONA THEOREM AND SOME RELATIVES

The following basic result characterizes the f-vectors of simplicial complexes.

THEOREM 17.1.1 *Kruskal-Katona Theorem*

For $f = (f_0, f_1, \ldots) \in \mathbb{N}^{(\infty)}$ the following are equivalent:

(i) *f is the f-vector of a simplicial complex;*
(ii) *f is a K-sequence.*

Theorem 17.1.1 has a generalization to colored complexes, whose statement will require some additional definitions. Fix an integer $r > 0$. Then define $\binom{n}{k}_r$ as follows: partition $\{1, \ldots, n\}$ into r subsets $V_1, \ldots, V_r$ as evenly as possible (so every subset V_i will have $\lfloor \frac{n}{r} \rfloor$ or $\lfloor \frac{n}{r} \rfloor + 1$ elements), and let $\binom{n}{k}_r$ be the number of k-subsets $F \subseteq \{1, \ldots, n\}$ such that $|F \cap V_i| \leq 1$ for $1 \leq i \leq r$. For $k \leq r$ every positive integer n can be uniquely written

$$n = \binom{a_k}{k}_r + \binom{a_{k-1}}{k-1}_{r-1} + \ldots + \binom{a_i}{i}_{r-k+i},$$

where $\frac{a_j}{a_{j-1}} > \frac{r-k+j}{r-k+j-1}$ for $j = k, k-1, \ldots, i+1$, and $a_i \geq i \geq 1$. Then define

$$\partial_k^{(r)}(n) = \binom{a_k}{k-1}_r + \binom{a_{k-1}}{k-2}_{r-1} + \ldots + \binom{a_i}{i-1}_{r-k+i},$$

and let $\partial_k^{(r)}(0) = 0$.

THEOREM 17.1.2

For $f = (f_0, \ldots, f_{d-1})$, $d \leq r$, the following are equivalent:

(i) *f is the f-vector of an r-colorable simplicial complex;*
(ii) *$\partial_{k+1}^{(r)}(f_k) \leq f_{k-1}$, for all $1 \leq k \leq d-1$.*

Note that for r sufficiently large Theorem 17.1.2 specializes to Theorem 17.1.1.

THEOREM 17.1.3

The f-vector of any $(r-1)$-dimensional clique complex is the f-vector of some r-colorable complex.

MULTICOMPLEXES AND MACAULAY'S THEOREM

A ***multicomplex*** $\mathcal{M}$ is a nonempty collection of monomials in finitely many variables such that if m is in $\mathcal{M}$ then so is every divisor of m. Let $f_i(\mathcal{M})$ be the number of degree i monomials in $\mathcal{M}$; $f(\mathcal{M}) = (f_0, f_1, \ldots)$ is called the ***f-vector*** of $\mathcal{M}$.

THEOREM 17.1.4 *Macaulay's Theorem*

For $f \in \mathbb{N}^\infty$ the following are equivalent:

(i) *f is the f-vector of a multicomplex;*

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(ii) *f is an M-sequence;*
(iii) *$f_i = \dim_{\boldsymbol{k}} R_i$, $i \geq 0$, for some finitely generated commutative graded $\boldsymbol{k}$-algebra $R = \oplus_{i\geq 0} R_i$ such that $R_0 \cong \boldsymbol{k}$ (a field) and R_1 generates R.*

A simplicial complex can be viewed as a multicomplex of squarefree monomials. Hence, a K-sequence is (except for a shift in the indexing) an M-sequence: If $(f_0, \ldots, f_{d-1})$ is a K-sequence then $(1, f_0, \ldots, f_{d-1})$ is an M-sequence. For this reason (and others, see, e.g., Theorem 17.2.3), properties of M-sequences are of interest also if one cares mainly about the special case of simplicial complexes.

A multicomplex is ***pure*** if all its maximal (under divisibility) monomials have the same degree.

THEOREM 17.1.5

Let $(f_0, \ldots, f_r)$ be the f-vector of a pure multicomplex, $f_r \neq 0$. Then $f_i \leq f_j$ for all $i < j \leq r - i$.

COMMENTS

Simplicial complexes (abstract and geometric) are treated in most books on algebraic topology; see, e.g., [Mun84, Spa66]. The Kruskal-Katona theorem (independently discovered by M.-P. Schützenberger, J.B. Kruskal, G.O.H. Katona, L.H. Harper, and B. Lindström during the years 1959-1966) is discussed in many places and several proofs have appeared; see, e.g., [And87, Zie95].

A Kruskal-Katona type theorem for simplicial complexes with vertex-transitive symmetry group appears in [FK96].

Theorem 17.1.2 is from [FFK88]. (*Remark:* The definition of the $\partial_k^{(r)}(\cdot)$ operator is incorrectly stated in [FFK88], in particular the uniqueness claim in [FFK88, Lemma 1.1] is incorrect. The version stated here was suggested to us by J. Eckhoff.)

Theorem 17.1.3 was conjectured by J. Eckhoff and G. Kalai and proved by Frohmader [Fro08].

For Macaulay's theorem we refer to [And87, Sta96]. There is a common generalization of Macaulay's theorem and the Kruskal-Katona theorem due to Clements and Lindström; see [And87]. Theorem 17.1.5 is from [Hib89].

17.2 BETTI NUMBER CONSTRAINTS

GLOSSARY

The ***Euler characteristic*** $\chi(\Delta)$ of a simplicial complex Δ with f-vector $(f_0, \ldots, f_{d-1})$ is $\chi(\Delta) = \sum_{i=0}^{d-1} (-1)^i f_i$.

The ***h-vector*** $(h_0, \ldots, h_d)$ of a $(d-1)$-dimensional simplicial complex is defined by

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1} (x-1)^{d-i}.$$

The corresponding ***g-vector*** $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ is defined by $g_0 = 1$ and $g_i = h_i - h_{i-1}$, for $i \geq 1$.

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The ***Betti number*** $\beta_i(\Delta)$ is the dimension (as a $\mathbb{Q}$-vector space) of the ith reduced simplicial homology group $\widetilde{H}_i(\Delta, \mathbb{Q})$; see any textbook on algebraic topology (e.g., [Mun84]) for the definition. We call $(\beta_0, \ldots, \beta_{\dim \Delta})$ the ***Betti sequence*** of Δ.

The ***link*** $\ell k_\Delta(F)$ of a face F is the subcomplex of Δ defined by $\ell k_\Delta(F) = \{A \in \Delta \mid A \cap F = \emptyset, A \cup F \in \Delta\}$. Note that $\ell k_\Delta(\emptyset) = \Delta$.

A simplicial complex Δ is ***acyclic*** if $\beta_i(\Delta) = 0$ for all i.

A simplicial complex Δ is ***Cohen-Macaulay*** if $\beta_i(\ell k_\Delta(F)) = 0$ for all $F \in \Delta$ and all $i < \dim \ell k_\Delta(F)$.

A simplicial complex Δ is ***m-Leray*** if $\beta_i(\ell k_\Delta(F)) = 0$ for all $F \in \Delta$ and all $i \geq m$.

FIXED BETTI NUMBERS

A simplicial complex is ***connected*** if its 1-skeleton is connected in the sense of graph theory. This is equivalent to demanding $\beta_0 = 0$.

THEOREM 17.2.1

For $f \in \mathbb{N}^{(\infty)}$ the following are equivalent:

(i) *f is the f-vector of a connected simplicial complex;*
(ii) *f is a K-sequence and $\partial^3(f_2) \leq f_1 - f_0 + 1$.*

The most basic relationship between f-vectors and Betti numbers is the ***Euler-Poincaré formula***:

$$\chi(\Delta) = f_0 - f_1 + f_2 - \ldots = 1 + \beta_0 - \beta_1 + \beta_2 - \ldots$$

This is in fact the only linear one in the following complete set of relations.

THEOREM 17.2.2

For $f = (f_0, f_1, \ldots) \in \mathbb{N}^{(\infty)}$ and $\beta = (\beta_0, \beta_1, \ldots) \in \mathbb{N}^{(\infty)}$ the following are equivalent:

(i) *f is the f-vector of some simplicial complex with Betti sequence β;*
(ii) *if $\chi_{k-1} = \sum_{j \geq k} (-1)^{j-k}(f_j - \beta_j)$, $k \geq 0$, then $\chi_{-1} = 1$ and $\partial_{k+1}(\chi_k + \beta_k) \leq \chi_{k-1}$ for all $k \geq 1$.*

By putting $\beta_i = 0$ for all i one gets as a special case a characterization of the f-vectors of acyclic simplicial complexes, viz., $\sum_{i \geq 0} f_{i-1} x^i = (1 + x) \sum_{i \geq 0} f'_{i-1} x^i$, where $(f'_0, f'_1, \ldots)$ is a K-sequence.

COHEN-MACAULAY COMPLEXES

Examples of Cohen-Macaulay complexes are triangulations of manifolds whose Betti numbers vanish below the top dimension, in particular triangulations of spheres and balls. Other examples are matroid complexes (the independent sets of a matroid), Tits buildings, and the order complexes (simplicial complex of totally ordered subsets) of several classes of posets, e.g., semimodular lattices (including distributive

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and geometric lattices). Shellable complexes (see Chapters 16 and 19) are Cohen-Macaulay. Cohen-Macaulay complexes are always pure.

The definition of h-vector given in the glossary shows that the h-vector and the f-vector of a complex mutually determine each other via the formulas:

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}\,, \qquad f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{i-j} h_j\,,$$

for $0 \le i \le d$. Hence, we may state f-vector results in terms of h-vectors whenever convenient.

THEOREM 17.2.3

For $h = (h_0, \ldots, h_d) \in \mathbb{Z}^{d+1}$ the following are equivalent:

(i) *h is the h-vector of a $(d-1)$-dimensional Cohen-Macaulay complex;*
(ii) *h is the h-vector of a $(d-1)$-dimensional shellable complex;*
(iii) *h is an M-sequence.*

Since there are a total of $\binom{n+k-1}{k}$ monomials of degree k in n variables, and by Theorems 17.1.5 and 17.2.3 the h-vector of a $(d-1)$-dimensional Cohen-Macaulay complex counts certain monomials in $h_1 = f_0 - d$ variables, we derive the inequalities

$$0 \le h_i \le \binom{f_0 - d + i - 1}{i}$$

for the h-vectors of Cohen-Macaulay complexes. The lower bound can be improved for complexes with fixed-point-free involutive symmetry.

THEOREM 17.2.4

Let $h = (h_0, \ldots, h_d)$ be the h-vector of a Cohen-Macaulay complex admitting an automorphism α of order 2, such that $\alpha(F) \neq F$ for all $F \in \Delta \setminus \{\emptyset\}$. Then

$$h_i \ge \binom{d}{i} \qquad \textit{for } 0 \le i \le d.$$

Consequently, $f_{d-1} = h_0 + \ldots + h_d \ge 2^d$.

Another condition on a Cohen-Macaulay complex that forces stricter conditions on its h-vector is being r-colorable.

THEOREM 17.2.5

For $h = (h_0, \ldots, h_d) \in \mathbb{Z}^{d+1}$ the following are equivalent:

(i) *h is the h-vector of a $(d-1)$-dimensional and d-colorable Cohen-Macaulay complex;*
(ii) *$(h_1, \ldots h_d)$ is the f-vector of a d-colorable simplicial complex.*

Hence in this case the h-vector is not only an M-sequence, but the special kind of K-sequence characterized in Theorem 17.1.2.

LERAY COMPLEXES

Examples of Leray complexes arise as follows. Let $\mathcal{K} = \{K_1, \ldots, K_t\}$ be a family of convex sets in $\mathbb{R}^m$, and let $\Delta(\mathcal{K}) = \{A \subseteq \{1, \ldots, t\} \mid \bigcap_{i \in A} K_i \neq \emptyset\}$. Then the simplicial complex $\Delta(\mathcal{K})$ is m-Leray.

Fix $m \geq 0$, and let $f = (f_0, \ldots, f_{d-1})$ be the f-vector of a simplicial complex Δ. Define

$$h_k^* = \begin{cases} f_k & \text{for } 0 \leq k \leq m-1 \\ \sum\limits_{j \geq 0} (-1)^j \binom{k+j-m}{j} f_{k+j} & \text{for } k \geq m. \end{cases}$$

The sequence $h^* = (h_0^*, \ldots, h_{d-1}^*)$ is the ***h^*-vector*** of Δ. The two vectors f and h^* mutually determine each other.

THEOREM 17.2.6

For $h^* = (h_0^*, h_1^*, \ldots) \in \mathbb{Z}^{(\infty)}$ the following are equivalent:

(i) *h^* is the h^*-vector of an m-Leray complex;*

(ii) *h^* is the h^*-vector of $\Delta(\mathcal{K})$ for some family $\mathcal{K}$ of convex sets in $\mathbb{R}^m$;*

(iii)

$$\begin{cases} h_k^* \geq 0 & \text{for } k \geq 0 \\ \partial_{k+1}(h_k^*) \leq h_{k-1}^* & \text{for } 1 \leq k \leq m-1 \\ \partial_m(h_k^*) \leq h_{k-1}^* - h_k^* & \text{for } k \geq m. \end{cases}$$

COMMENTS

The Euler-Poincaré formula (due to Poincaré, 1899) is proved in most books on algebraic topology.

A good general source on Cohen-Macaulay complexes is [Sta96]; it contains Theorems 17.2.3, 17.2.4, and 17.2.5, as well as references to the original sources. Theorem 17.2.2 is from [BK88]. A common generalization of Theorems 17.1.1, 17.2.2, and 17.2.3 is given in [Bjö96]. Theorem 17.2.1 is a special case. There are several additional results about h-vectors of Cohen-Macaulay complexes. For instance, for complexes with nontrivial automorphism groups, see [Sta96, Section III.8]; for matroid complexes, see [Sta96, Section III.3]; and for Cohen-Macaulay complexes that are r-colorable for $r < d$, see the references mentioned in [Sta96, Section III.4].

Cohen-Macaulay complexes are pure. However, there is an extension of their theory to a class of nonpure complexes, the so-called *sequentially Cohen-Macaulay* complexes, introduced in [Sta96]. In [ABG17], to which we refer for definitions and references, a numerical characterization is given of the so-called h-triangles (doubly indexed h-numbers) of sequentially Cohen-Macaulay simplicial complexes. This result characterizes the array of numbers of faces of various dimensions and codimensions in such a complex, generalizing Theorem 17.2.3 to the nonpure case.

Cohen-Macaulay complexes are closely related to certain commutative rings [Sta96], and via this connection such complexes have also been of use in the theory of splines; see [Sta96, Section III.5] and also Chapter 56 of this Handbook.

Theorem 17.2.6 was conjectured by Eckhoff and proved by Kalai [Kal84, Kal86].

17.3 SIMPLICIAL POLYTOPES, SPHERES, AND MANIFOLDS

GLOSSARY

A ***triangulated d-ball*** is a simplicial complex Δ whose realization $\|\Delta\|$ is homeomorphic to the ball $\{x \in \mathbb{R}^d \mid x_1^2 + \cdots + x_d^2 \leq 1\}$. A ***triangulated (d−1)-sphere*** is a simplicial complex whose realization is homeomorphic to the sphere $\{x \in \mathbb{R}^d \mid x_1^2 + \cdots + x_d^2 = 1\}$. Equivalently, it is the boundary of a triangulated d-ball. Examples of triangulated $(d-1)$-spheres are given by the boundary complexes of simplicial d-polytopes.

A ***pseudomanifold*** is a pure simplicial complex Δ such that

(i) each face of codimension 1 is contained in precisely two maximal faces; and

(ii) the dual graph (whose vertices are the maximal faces of Δ and whose edges are the faces of codimension 1) is connected.

An ***Eulerian pseudomanifold*** is a pseudomanifold Δ such that Δ itself and the link of each face have the Euler characteristic of a sphere of the corresponding dimension.

A pure $(d-1)$-dimensional simplicial complex Δ is a ***homology manifold*** if it is connected and the link of each nonempty face has the Betti numbers of a sphere of the same dimension. It is a ***homology sphere*** if, in addition, Δ itself has the Betti numbers of a $(d-1)$-sphere. Examples of homology manifolds are given by triangulations of compact connected topological manifolds, i.e., spaces that are locally Euclidean.

The ***cyclic d-polytope with n vertices*** $C_d(n)$ is the convex hull of any n points on the moment curve in $\mathbb{R}^d$. (See Section 15.1.4.)

The following implications hold among these various classes, all of them strict:

polytope boundary ⇒ *sphere* ⇒ *homology sphere* ⇒
Eulerian pseudomanifold ⇒ *pseudomanifold*

homology sphere ⇒ *homology manifold* ⇒ *pseudomanifold*

homology sphere ⇒ *Cohen-Macaulay complex*

PSEUDOMANIFOLDS

The following results give the basic lower and upper bounds on f-vectors of pseudomanifolds.

THEOREM 17.3.1 *Lower Bound Theorem*

For a $(d-1)$-dimensional pseudomanifold Δ with n vertices,

$$f_k(\Delta) \geq \begin{cases} \binom{d}{k}n - \binom{d+1}{k+1}k & \text{for } 1 \leq k \leq d-2 \\ (d-1)n - (d-2)(d+1) & \text{for } k = d-1. \end{cases}$$

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THEOREM 17.3.2 *Upper Bound Theorem*

Let Δ be a $(d-1)$-dimensional homology manifold with n vertices, such that either

(i) *d is even, or*

(ii) *$d = 2k+1$ is odd, and either $\chi(\Delta) = 2$ or $\beta_k \le 2\beta_{k-1} + 2\sum_{i=0}^{k-3} \beta_i$.*

Then $f_k(\Delta) \le f_k(C_d(n))$ for $1 \le k \le d-1$.

This upper bound theorem applies when the homology manifold is Eulerian (irrespective of dimension); in particular, it applies to all simplicial polytopes and spheres. By the geometric operation of "pulling vertices" (Section 16.2), one can extend this to all convex polytopes.

THEOREM 17.3.3

If P is any *convex d-polytope with n vertices, then $f(P) \le f(C_d(n))$.*

The given lower and upper bounds are best possible within the class of simplicial polytope boundaries. The lower bound is attained by the class of stacked polytopes (Sections 16.4.2 and 19.2). To make the upper bound numerically explicit, we give the formula for the f-vector of a cyclic polytope.

THEOREM 17.3.4

For $d \ge 2$ and $0 \le k \le d-1$, the number of k-faces of the cyclic polytope $C_d(n)$ with n vertices is

$$f_k(C_d(n)) \quad = \quad \frac{n-\delta(n-k-2)}{n-k-1} \sum_{j=0}^{\lfloor d/2 \rfloor} \binom{n-1-j}{k+1-j} \binom{n-k-1}{2j-k-1+\delta},$$

where $\delta = d - 2\lfloor d/2 \rfloor$.

In particular,

$$f_{d-1}(C_d(n)) \quad = \quad \binom{n-\lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} \quad + \quad \binom{n-\lceil \frac{d+1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor},$$

which shows that for fixed d the number of facets is $O(n^{\lfloor d/2 \rfloor})$.

POLYTOPES AND SPHERES

For boundaries of simplicial d-polytopes and, more generally, for Eulerian pseudomanifolds, we have the following basic relations.

THEOREM 17.3.5 *Dehn-Sommerville Equations*

For d-dimensional Eulerian pseudomanifolds,

$$h_i = h_{d-i} \qquad \textit{for all } 0 \le i \le d.$$

These equations give a complete description of the linear span of all f-vectors of d-polytopes (equivalently, $(d-1)$-spheres). (The affine span is defined by including the relation $h_0 = 1$.)

One consequence of the Dehn-Sommerville equations is the following relation between the h-vector of a triangulated ball K and the g-vector of its boundary ∂K.

THEOREM 17.3.6

For a triangulated d-ball K and its boundary $(d-1)$-sphere ∂K,

$$g_i(\partial K) = h_i(K) - h_{d+1-i}(K) \qquad \text{for } i \geq 1.$$

A complete characterization of the f-vectors of simplicial (and, by duality, simple) convex polytopes is given in terms of the h-vector and g-vector.

THEOREM 17.3.7 *g-Theorem*

A nonnegative integer vector $h = (h_0, \ldots, h_d)$ is the h-vector of a simplicial convex d-polytope if and only if

(i) $h_i = h_{d-i}$, *and*

(ii) $(g_0, \ldots, g_{\lfloor d/2 \rfloor})$ *is an M-sequence.*

One consequence of (ii) is that $g_i \geq 0$, which was known as the *generalized lower bound conjecture*. The case of equality in this conjecture has only recently been settled.

THEOREM 17.3.8

A simplicial d-polytope P satisfies $g_k(P) = h_k(P) - h_{k-1}(P) = 0$ for some k, $1 \leq k \leq \lfloor \frac{d}{2} \rfloor$, if and only if P is k-stacked, i.e., *P can be triangulated so that every face of dimension $d-k$ or less is on the boundary of P.*

For centrally symmetric polytopes, we get a better lower bound.

THEOREM 17.3.9

For centrally symmetric simplicial d-polytopes,

$$g_i = h_i - h_{i-1} \geq \binom{d}{i} - \binom{d}{i-1} \qquad \text{for } i \leq \lfloor d/2 \rfloor.$$

The following arithmetic property of the numbers of k-faces of all simplicial d-polytopes is a consequence of the g-theorem.

THEOREM 17.3.10

Given $0 \leq k < d$ there exist positive integers $G(k,d)$ and $N(k,d)$ such that

(i) *$G(k,d)$ divides $f_k(P)$ for every simplicial d-polytope P, and*

(ii) *if $G(k,d)$ divides n and $n > N(k,d)$, then $n = f_k(P)$ for some simplicial d-polytope P.*

MANIFOLDS

For face numbers of triangulations of a $(d-1)$-dimensional manifold X we have the following generalization of the Dehn-Sommerville equations.

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THEOREM 17.3.11

If $X = |\Delta|$ is a $(d-1)$-dimensional manifold, then

$$h_{d-i} - h_i = (-1)^i \binom{d}{i} (\chi(\Delta) - \chi(S^{d-1})).$$

If Δ is a $(d-1)$-dimensional simplicial complex, then define

$$h_i' = h_i + \binom{d}{i} (\beta_{i-2} - \beta_{i-3} + \cdots \pm \beta_0)\,,$$

where the $\beta_j = \beta_j(\Delta)$ are the reduced homology Betti numbers of Δ.

THEOREM 17.3.12

If $X = |\Delta|$ is a $(d-1)$-dimensional manifold, then

$$h_i' \geq \binom{d}{i} \beta_{i-1}.$$

COMMENTS

The Lower Bound Theorem 17.3.1 is due to Kalai and Gromov in the generality given here; see [Kal87] including the note added in proof. The $k = d-1$ case had earlier been done by Klee and the case of polytope boundaries by Barnette. See [Kal87] for a discussion of the history of this result.

The Upper Bound Theorem 17.3.2 is due to Novik [Nov98]. See also [NS12]. The case of polytopes (Theorem 17.3.3) was first proved by McMullen (see [MS71]), and extended to spheres by Stanley (see [Sta96]). The computation of the f-vector of the cyclic polytope can be found in [Grü67, Sections 4.7.3 and 9.6.1] or [MS71]. More results on comparing face numbers of a simplicial polytope to those of the cyclic polytope can be found in [Bjö07].

The Dehn-Sommerville equations for polytopes are classical; proofs can be found in [Grü67, Sta86, Zie95]. The extension to Eulerian pseudomanifolds is due to Klee [Kle64]; an equivariant version appears in [Bar92]. The D-S equations imply an upper bound on the average number of j-faces contained in a k-face of a simple polytope (roughly, the number of j-faces of a k-dimensional cube) due to Nikulin. This has been useful in the theory of hyperbolic reflection groups. See [Nik87, Theorem C] for references and ramifications; see also Theorem 17.5.17, which is a similar result for arrangements and zonotopes.

The g-theorem was conjectured by McMullen and proved by Billera, Lee, and Stanley [BL81a, Sta80]. Another proof of the necessity of these conditions was given by McMullen [McM93]. More recently, a self-contained, elementary proof of necessity was given by Fleming and Karu [FK08]. It is not known whether the second condition of Theorem 17.3.7 holds for general triangulated spheres. The g-theorem has a convenient reformulation as a one-to-one correspondence (via matrix multiplication) between f-vectors of simplicial polytopes and M-sequences, see [Bjö87, Zie95]. Theorem 17.3.8, the equality case of the generalized lower bound theorem, was conjectured by McMullen and Walkup in 1971 and recently proved by Murai and Nevo [MN13]. Theorem 17.3.9 was proved by Stanley [Sta87a]; for another proof see [Nov99]. Theorem 17.3.10 is from Björner and Linusson [BL99], where also an explicit expression for the modulus $G(k, d)$ is given.

Theorems 17.3.11 and 17.3.12 are due to Klee [Kle64] and Novik and Swartz [NS09], respectively. For simplicity, they are here not stated in their maximal generality. That $h'_i \geq 0$ for manifolds was originally shown by Schenzel in 1981.

The question of characterizing f-vectors for compact manifolds more general than spheres is at the present far beyond our reach. However, much interesting work has been done on the more restrictive question of minimizing the number of vertices of triangulations for given manifolds, see e.g., [Küh90, Küh95, BL00, Lut05]. This is of interest for efficient presentations of manifolds to computers. For information about face numbers of manifolds, see [Swa09].

The study of f-vectors of unbounded polyhedra can be approached by studying the f-vectors of ***polytope pairs*** (P, F), where P is a polytope and F is a maximal face of P. See [BL81b, BaL93] for a summary of such results.

17.4 CELL COMPLEXES

GLOSSARY

Convex polytopes and *faces* of such are defined in Chapter 15.

A ***polyhedral complex*** Γ is a finite collection of convex polytopes in $\mathbb{R}^n$ such that (i) if $\pi \in \Gamma$ and σ is a face of π, then $\sigma \in \Gamma$; and (ii) if $\pi, \sigma \in \Gamma$ and $\pi \cap \sigma \neq \emptyset$, then $\pi \cap \sigma$ is a face of both. The ***space*** of Γ is $\|\Gamma\| = \bigcup_{\pi \in \Gamma} \pi$, a subspace of $\mathbb{R}^n$. Examples of polyhedral complexes are given by ***boundary complexes*** ∂P of convex polytopes P (i.e., the collection of all proper faces). A geometric simplicial complex (defined in Section 17.1) is a polyhedral complex all of whose cells are simplices. A ***cubical complex*** is a polyhedral complex all of whose cells are (combinatorially isomorphic to) cubes.

A ***regular cell complex*** Γ is a family of closed balls (homeomorphs of $\{x \in \mathbb{R}^j \mid |x| \leq 1\}$) in a Hausdorff space $\|\Gamma\|$ such that (i) the interiors of the balls partition $\|\Gamma\|$ and (ii) the boundary of each ball in Γ is a union of other balls in Γ. The members of Γ are called (closed) ***cells*** or ***faces***. The ***dimension*** of a cell is its topological dimension and $\dim \Gamma = \max_{\sigma \in \Gamma} \dim \sigma$.

A ***Gorenstein* complex*** is a regular cell complex whose poset of faces has an order complex that is a homology sphere. These include all triangulations of spheres.

A regular cell complex has the ***intersection property*** if, whenever the intersection of two cells is nonempty, then this intersection is also a cell in the complex. Polyhedral complexes are examples of regular cell complexes with the intersection property. Regular cell complexes with the intersection property can be reconstructed up to homeomorphism from the corresponding "abstract" complex consisting of the family of vertex sets of its cells.

For a regular cell complex Γ, let f_i be the number of i-dimensional cells, and let $\beta_i = \dim_{\mathbb{Q}} \widetilde{H}_i(\|\Gamma\|, \mathbb{Q})$. The latter denotes i-dimensional reduced singular homology with rational coefficients of the space $\|\Gamma\|$; see [Mun84, Spa66] for explanations of this concept. Then we have the ***f-vector*** $f = (f_0, f_1, \ldots)$ and the ***Betti sequence*** $\beta = (\beta_0, \beta_1, \ldots)$ of Γ. These definitions generalize those previously given in the simplicial case.

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BASIC f-VECTOR RELATIONS

Among the classes of complexes

- simplicial complexes
- polyhedral complexes
- regular cell complexes with the intersection property
- regular cell complexes

each is a proper subclass of its successor. Thus one may wonder how many of the relations for f-vectors of simplicial complexes given in Sections 17.1–17.3 can be extended to these broader classes of complexes. Also, what new phenomena (not visible in the simplicial case) arise? Some answers are given in this section and the following one, but current knowledge is quite fragmentary. We begin here with the most general relations.

THEOREM 17.4.1

$(f_0, \ldots, f_d)$ is the f-vector of a d-dimensional regular cell complex if and only if $f_d \geq 1$ and $f_i \geq 2$ for all $0 \leq i < d$.

THEOREM 17.4.2

f is the f-vector of a regular cell complex with the intersection property if and only if f is a K-sequence.

Let $\beta = (\beta_0, \beta_1, \ldots) \in \mathbb{N}^{(\infty)}$ be fixed, and for every sequence $f = (f_0, f_1, \ldots)$ let

$$\chi_{k-1} = \sum_{j \geq k} (-1)^{j-k}(f_j - \beta_j) \qquad \text{for } k \geq 0.$$

THEOREM 17.4.3

$(f_0, \ldots, f_d)$ is the f-vector of a d-dimensional regular cell complex with Betti sequence β if and only if $\chi_{-1} = 1$ and $\chi_k \geq 1$ for $0 \leq k < d$.

THEOREM 17.4.4

For $f \in \mathbb{N}^{(\infty)}$ the following are equivalent:

(i) *f is the f-vector of a regular cell complex with the intersection property and with Betti sequence β;*

(ii) *$\chi_{-1} = 1$ and $\partial_{k+1}(\chi_k + \beta_k) \leq \chi_{k-1}$ for all $k \geq 1$.*

These results show that the f-vectors of regular cell complexes (with or without Betti number constraints) are considerably more general than the f-vectors of simplicial complexes, but that the two classes of f-vectors agree in the presence of the intersection property.

COMMENTS

Regular cell complexes are known as ***regular CW complexes*** in the topological literature [LW69]. The nonregular CW complexes offer an even more general class of cell complexes [LW69, Mun84, Spa66], but there is very little one can say about

f-vectors in that generality. See [BLS$^+$93, Section 4.7] for a detailed discussion of regular cell complexes from a combinatorial point of view.

For the results of this section see [BK88, BK91, BK89]. A characterization of f-vectors of (cubical) subcomplexes of a cube can be found in [Lin71], and of regular cell decompositions of spheres in [Bay88].

17.5 GENERAL POLYTOPES AND SPHERES

GLOSSARY

A ***flag*** of faces in a (polyhedral) $(d-1)$-complex Δ is a chain $F_1 \subsetneq F_2 \subsetneq \cdots \subsetneq F_k$ of faces F_i in Δ. It is an ***S-flag*** if

$$S = \{\dim F_1, \dots, \dim F_k\} \subseteq \{0, 1, \dots, d-1\}.$$

Let $f_S = f_S(\Delta)$ denote the number of S-flags in Δ. The function $S \mapsto f_S$, $S \subseteq \{0, 1, \dots, d-1\}$, is called the ***flag f-vector*** of Δ. If

$$h_S = \sum_{T \subseteq S} (-1)^{|S|-|T|} f_T,$$

then the function $S \mapsto h_S$, $S \subseteq \{0, 1, \dots, d-1\}$, is called the ***flag h-vector***.

For $S \subseteq \{0, \dots, d-1\}$ and noncommuting symbols $\boldsymbol{a}$ and $\boldsymbol{b}$, let $u_S = u_0 u_1 \cdots u_{d-1}$ be the $\boldsymbol{ab}$-word defined by $u_i = \boldsymbol{a}$ if $i \notin S$ and $u_i = \boldsymbol{b}$ otherwise. When Δ is spherical (or, more generally, Eulerian), then the $\boldsymbol{ab}$-polynomial $\sum h_S u_S$ is also a polynomial in $\boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$ and $\boldsymbol{d} = \boldsymbol{ab} + \boldsymbol{ba}$. (Note that the degree of $\boldsymbol{c}$ is 1 and the degree of $\boldsymbol{d}$ is 2.) The resulting $\boldsymbol{cd}$-polynomial

$$\sum h_S u_S = \sum \phi_w w,$$

where the right-hand sum is over all $\boldsymbol{cd}$-words w of degree d, is called the ***cd-index*** $\Phi(\Delta)$ of Δ. For 2-, 3-, and 4-polytopes, the $\boldsymbol{cd}$-index is $\boldsymbol{c}^2 + (f_0 - 2)\boldsymbol{d}$, $\boldsymbol{c}^3 + (f_0 - 2)\boldsymbol{dc} + (f_2 - 2)\boldsymbol{cd}$, and $\boldsymbol{c}^4 + (f_0 - 2)\boldsymbol{dc}^2 + (f_1 - f_0)\boldsymbol{cdc} + (f_3 - 2)\boldsymbol{c}^2\boldsymbol{d} + (f_{02} - 2f_2 - 2f_0 + 4)\boldsymbol{d}^2$, respectively.

For *any* convex d-polytope P, we define the ***toric h-vector*** and ***toric g-vector*** recursively by $h(P, x) = \sum_{i=0}^{d} h_i x^{d-i}$ and $g(P, x) = \sum_{i=0}^{\lfloor d/2 \rfloor} g_i x^i$, where $g_i = h_i - h_{i-1}$ and the following relations hold:

(i) $g(\emptyset, x) = h(\emptyset, x) = 1$; and

(ii) $h(P, x) = \sum_{G \text{ face of } P,\, G \neq P} g(G, x)(x-1)^{d-1-\dim G}$.

(Compare to Section 16.4.1, where this toric h-vector is defined for any polyhedral complex. In the notation given there, we have defined h and g for the complex ∂P.) When P is simplicial, this definition coincides with that of the usual h-vector, as defined in Section 17.2. For 2-, 3-, and 4-polytopes, the g-polynomial is $1 + (f_0 - 3)x$, $1 + (f_0 - 4)x$, and $1 + (f_0 - 5)x + (10 - 3f_0 - 3f_3 + f_{03})x^2$, respectively.

A ***rational polytope*** is one whose vertices all have rational coordinates. Equivalently, all maximal faces are determined by linear forms with rational coefficients.

A ***cubical polytope*** is one that has a cubical boundary complex. For any cubical $(d-1)$-complex with f-vector $(f_0, \dots, f_{d-1})$, define the ***cubical h-vector*** $h^c = (h_0^c, \dots, h_d^c)$ by

$$h_i^c = (-1)^i 2^{d-1} + \sum_{j=1}^{i} (-1)^{i-j} 2^{j-1} f_{j-1} \sum_{k=0}^{i-j} \binom{d-j}{k} \qquad \text{for } i = 0, \ldots, d.$$

The ***cubical g-vector*** $g^c = (g_0^c, \ldots, g_{\lfloor d/2 \rfloor}^c)$ is defined by $g_0^c = h_0^c = 2^{d-1}$ and $g_i^c = h_i^c - h_{i-1}^c$ for $i \geq 1$.

An ***Eulerian polyhedral complex*** is one whose first barycentric subdivision is an Eulerian pseudomanifold. Examples are boundary complexes of polytopes and ***spherical*** polyhedral complexes, i.e., those whose underlying space is homeomorphic to a sphere.

A (central) ***hyperplane arrangement*** is a collection $\mathcal{H}$ of n linear hyperplanes in $\mathbb{R}^d$, given by normal vectors $x_1, \ldots, x_n$ (see Section 6.1.3). The arrangement is ***essential*** if the normals x_i span $\mathbb{R}^d$. The associated ***zonotope*** is the Minkowski sum of the n line segments $[-x_i, x_i]$, i.e., $Z = \{\sum \lambda_i x_i \mid -1 \leq \lambda_i \leq 1\}$ (see Section 15.1.4).

LINEAR RELATIONS

We give the linear equalities on the invariants defined above that are known to hold for all boundary complexes of polytopes and, more generally, for all Eulerian polyhedral complexes.

THEOREM 17.5.1

For $(d-1)$-dimensional Eulerian polyhedral complexes, the following relations always hold for the flag h, the toric h, and the flag f:

(i) $h_S = h_{\{0,\ldots,d-1\} \setminus S}$ *for all* $S \subseteq \{0, \ldots, d-1\}$;

(ii) $h_i = h_{d-i}$ *for* $0 \leq i \leq d$; *and*

(iii) $\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup \{j\}} = (1 - (-1)^{k-i-1}) f_S$ *whenever* $i, k \in S \cup \{-1, d\}$ *with* $i \leq k-2$ *and* $S \cap \{i+1, \ldots, k-1\} = \emptyset$.

It is known that the relations in Theorem 17.5.1(iii), the ***generalized Dehn-Sommerville equations***, completely describe the linear span of all flag f-vectors of Eulerian complexes, and so they imply those in (i). Since the toric h is known to be a linear function of the flag f, they imply those in (ii) as well. The linear span of flag f-vectors has dimension e_d, where e_d is the dth Fibonacci number (defined by the recurrence $e_d = e_{d-1} + e_{d-2}$, $e_0 = e_1 = 1$). There are e_d ***cd***-words of degree d. Furthermore, the coefficients ϕ_w of the ***cd***-index, considered as linear expressions in the f_S, form a linear basis for the span of flag f-vectors of d-polytopes. The affine span of all flag f-vectors is defined by including the relation $f_\emptyset = 1$.

For cubical polytopes and spheres, the cubical h-vector satisfies the analogue of the Dehn-Sommerville equations.

THEOREM 17.5.2

For cubical d-polytopes and cubical $(d-1)$-spheres,

$$h_i^c = h_{d-i}^c \qquad \textit{for all } 0 \leq i \leq d.$$

These give all linear relations satisfied by f-vectors of cubical polytopes and spheres. The cubical h-vector satisfies, as well, the equations of Theorem 17.3.6, linking the h of a cubical ball to the g of its boundary sphere.

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LINEAR INEQUALITIES

Some linear inequalities that hold for flag f-vectors of all polytope boundaries are given in this section. The list is not thought to be complete, although there are no conjectures for what the complete set might be.

For a Cohen-Macaulay polyhedral complex, i.e., one whose first barycentric subdivision is a Cohen-Macaulay simplicial complex, the flag h is always nonnegative.

THEOREM 17.5.3

For a Cohen-Macaulay polyhedral $(d-1)$-complex Γ, we have $h_S(\Gamma) \geq 0$ for all $S \subseteq \{0, \dots, d-1\}$.

For general convex polytopes, we also have nonnegativity of the ***cd***-index. In fact, the ***cd***-index of any d-polytope is minimized termwise by the ***cd***-index of the d-simplex $\Delta^{(d)}$.

THEOREM 17.5.4

(i) *If P is a convex d-polytope (or, more generally a Gorenstein* complex), then*

$$\phi_w(P) \geq 0$$

*for all **cd**-words w of degree d.*

(ii) *If P is a convex d-polytope (or, more generally a Gorenstein* complex whose face poset is a lattice), then*

$$\phi_w(P) \geq \phi_w(\Delta^{(d)})$$

*for all **cd**-words w of degree d.*

Note that Theorem 17.5.4(i) gives the most general possible linear inequalities for flag f-vectors of spherical regular cell complexes (i.e., regular cell complexes homeomorphic to the sphere).

There are also relations between the ***cd***-coefficients ϕ_w for any polytope.

THEOREM 17.5.5

For any d-polytope P

$$\phi_{\boldsymbol{u}\boldsymbol{d}\boldsymbol{v}}(P) \geq \phi_{\boldsymbol{u}\boldsymbol{c}^2\boldsymbol{v}}(P),$$

*for any **cd**-words u and v with* $\deg u + \deg v = d-2$.

For all convex polytopes, it is known, further, that the toric h is unimodal.

THEOREM 17.5.6

For a convex d-polytope, $g_i \geq 0$ for $i \leq \lfloor d/2 \rfloor$.

Related to this is the following *nonlinear* inequality holding between the g-vectors of a polytope P and any of its faces F. We denote by P/F the ***link*** of F in P, i.e., the polytope whose lattice of faces is (isomorphic to) the interval $[F, P]$ in the face lattice of P.

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THEOREM 17.5.7

For a polytope P and any face F, we have the polynomial inequality

$$g(P,t) - g(F,t)g(P/F,t) \geq 0,$$

i.e., all coefficients of this polynomial are nonnegative.

We have a similar relation between the ***cd***-index of a polytope and that of any face.

THEOREM 17.5.8

For a polytope P and any face F, we have the polynomial inequalities

$$\Phi(P) \geq \begin{cases} \boldsymbol{c} \cdot \Phi(F) \cdot \Phi(P/F) \\ \Phi(F) \cdot \boldsymbol{c} \cdot \Phi(P/F) \\ \Phi(F) \cdot \Phi(P/F) \cdot \boldsymbol{c} \end{cases}$$

*where $\Phi(P)$, $\Phi(F)$, and $\Phi(P/F)$ are the **cd**-indices of P, F, and P/F, respectively.*

As with f-vectors of polytopes, their flag f-vectors, flag h-vectors and ***cd***-indices satisfy the upper bound theorem.

THEOREM 17.5.9

If P is a d-dimensional polytope with n vertices, then for any S,

$$\begin{array}{rcl} f_S(P) & \leq & f_S(C_d(n)), \\ h_S(P) & \leq & h_S(C_d(n)), \end{array}$$

and termwise as polynomials

$$\Phi(P) \quad \leq \quad \Phi(C_d(n)),$$

where $C_d(n)$ is the cyclic d-polytope with n vertices.

There are the following relations between invariants of a polytope P and its dual polytope P^*. If $w = w_1, \ldots, w_n$ is a ***cd***-word, then $w^* := w_n, \ldots, w_1$, the reverse word.

THEOREM 17.5.10

For a d-polytope P,

(i) $\phi_w(P^*) = \phi_{w^*}(P)$,

(ii) $g_k(P) = 0$ *if and only if* $g_k(P^*) = 0$ *and*

(iii) $\sum_{\emptyset \subseteq F \subseteq P} (-1)^{\dim F} g(F^*, t)\; g(P/F, t) = 0.$

For a $(2k)$-polytope P,

(iv) $g_k(P) = g_k(P^*)$.

Finally, we have the following lower bounds for the number of vertices of polytopes with no triangular faces (this includes the class of cubical polytopes), and for the combined numbers of vertices and facets of centrally symmetric polytopes.

THEOREM 17.5.11

A d-polytope with no triangular 2-face has at least 2^d vertices.

THEOREM 17.5.12

There exists a constant $c > 0$ such that

$$\log f_0 \cdot \log f_{d-1} > cd,$$

for any centrally symmetric d-polytope.

HYPERPLANE ARRANGEMENTS AND ZONOTOPES

An essential hyperplane arrangement $\mathcal{H}$ defines a decomposition of $\mathbb{R}^d$ into polyhedral cones (as in Section 6.1.3). This decomposition $\Gamma_{\mathcal{H}}$, a regular cell complex if intersected with the unit sphere, has a flag f-vector dual to that of its associated zonotope Z, in the sense that $f_S(\Gamma_{\mathcal{H}}) = f_{d-S}(Z)$, where $S = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$ and $d - S = \{d - i_k, \ldots, d - i_1\}$.

THEOREM 17.5.13

The flag f-vector of an arrangement (or zonotope) depends only on the matroid (linear dependency structure) of the underlying point configuration $\{x_1, \ldots, x_n\}$.

Although a fairly special subclass of polytopes, the zonotopes nonetheless are varied enough to carry all the linear information carried by flag numbers of general polytopes.

THEOREM 17.5.14

The flag f-vectors of zonotopes (and thus of hyperplane arrangements) satisfy the generalized Dehn-Sommerville equations, and there are no other linear relations not implied by these.

When it comes to linear *inequalities*, however, a difference between zonotopes and general polytopes emerges. As with general convex polytopes, we have nonnegativity of the $\boldsymbol{cd}$-index for zonotopes. However, the $\boldsymbol{cd}$-index of any d-zonotope is minimized termwise by the $\boldsymbol{cd}$-index of the d-cube $C^{(d)}$.

THEOREM 17.5.15

For a convex d-zonotope Z, $\phi_w(Z) \geq \phi_w(C^{(d)}) \geq 0$ for all $\boldsymbol{cd}$-words w of degree d. Further, if the word w has k $\boldsymbol{d}$'s, then 2^k divides $\phi_w(Z)$.

There is also a strengthening of Theorem 17.5.5 for zonotopes.

THEOREM 17.5.16

For any d-zonotope Z

$$\phi_{\boldsymbol{u}\boldsymbol{d}\boldsymbol{v}}(Z) - \phi_{\boldsymbol{u}\boldsymbol{c}^2\boldsymbol{v}}(Z) \geq \phi_{\boldsymbol{u}\boldsymbol{d}\boldsymbol{v}}(C^{(d)}) - \phi_{\boldsymbol{u}\boldsymbol{c}^2\boldsymbol{v}}(C^{(d)})$$

for any $\boldsymbol{cd}$-words u and v with $\deg u + \deg v = d - 2$.

The following result has the most direct interpretation when it is stated for arrangements, where it bounds the average number of $\{i_1, \ldots, i_k\}$-flags in an i_k-face by the number of $\{i_1{-}1, \ldots, i_k{-}1\}$-flags in an $(i_k{-}1)$-cube.

THEOREM 17.5.17

For a hyperplane arrangement $\mathcal{H}$ in $\mathbb{R}^d$ and $S = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, d\}$ with $k \geq 2$,

$$\frac{f_S(\Gamma_{\mathcal{H}})}{f_{i_k}(\Gamma_{\mathcal{H}})} < \binom{i_k - 1}{i_1 - 1, i_2 - i_1, \ldots, i_k - i_{k-1}} 2^{i_k - i_1}.$$

There is a straightforward reformulation of Theorem 17.5.17 for zonotopes that is easily seen not to be valid for all polytopes.

GENERAL 3- AND 4-POLYTOPES

We describe here the situation for flag f-vectors of 3- and 4-polytopes. The equations in Theorem 17.5.1(iii) reduce consideration to (f_0, f_2) when $d = 3$ and to (f_0, f_1, f_2, f_{02}) when $d = 4$.

THEOREM 17.5.18

For 3-polytopes, the following is known about the vector (f_0, f_2).

(i) *An integer vector (f_0, f_2) is the f-vector of a 3-polytope if and only if $f_0 \leq 2f_2 - 4$ and $f_2 \leq 2f_0 - 4$.*

(ii) *An integer vector (f_0, f_2) is the f-vector of a cubical 3-polytope if and only if $f_2 = f_0 - 2$, $f_0 \geq 8$, and $f_0 \neq 9$.*

(iii) *If $(f_0, f_2) = (f_0(Z), f_2(Z))$ for a 3-zonotope Z, then f_0 and f_1 are both even integers, $f_0 \leq 2f_2 - 4$, and $f_2 \leq f_0 - 2$.*

For 4-polytopes, much less is known.

THEOREM 17.5.19

Flag f-vectors (f_0, f_1, f_2, f_{02}) of 4-polytopes satisfy the following inequalities.

(i) $f_{02} \geq 3f_2$

(ii) $f_{02} \geq 3f_1$

(iii) $f_{02} + f_1 + 10 \geq 3f_2 + 4f_0$

(iv) $6f_1 \geq 6f_0 + f_{02}$

(v) $f_0 \geq 5$

(vi) $f_0 + f_2 \geq f_1 + 5$

(vii) $2(f_{02} - 3f_2) \leq \binom{f_0}{2}$

(viii) $2(f_{02} - 3f_1) \leq \binom{f_2 - f_1 + f_0}{2}$

(ix) $f_{02} - 4f_2 + 3f_1 - 2f_0 \leq \binom{f_0}{2}$

(x) $f_{02} + f_2 - 2f_1 - 2f_0 \leq \binom{f_2 - f_1 + f_0}{2}$.

It is not known, for example, whether (i)–(vi) give all linear inequalities holding for flag f-vectors of 4-polytopes.

COMMENTS

It is thought that the best route to an eventual characterization of f-vectors of general polytopes lies in an understanding of their flag f-vectors. The latter inherit many of the algebraic properties of f-vectors of simplicial polytopes that led to their characterization, while having a rich theory of their own.

The relations in Theorem 17.5.1 hold more generally for the case of enumeration of chains in Eulerian posets; see the article by Stanley in [BMSW94]. The relations in Theorem 17.5.1(iii) are proved in [BaB85]. An expression for the toric h in terms of the flag f can be found in the article by Bayer in [BMSW94]. The article by Kalai in the same volume contains an extensive discussion of g-vectors for both simplicial and general polytopes. The existence of the ***cd***-index was established in [BaK91]. Expressions for the (toric) g and h-vectors in terms of the flag h-vector or the ***cd***-index can be found in [BaE00]. The form of the cubical Dehn-Sommerville equations given in Theorem 17.5.2 appeared in [Adi96].

Theorem 17.5.3 can be found in [Sta96, Theorem III.4.4] (where h_S is denoted $\beta(S)$). The nonnegativity of the ***cd***-index for polytopes in Theorem 17.5.4(i) was proved as well for certain shellable spheres by Stanley (see [Sta96, Section III.4]); it was extended to all Gorenstein* complexes (and so all spherical complexes) by Karu [Kar06]. Theorem 17.5.4(ii), that the ***cd***-index is minimized over polytopes by simplices, is shown in [BE00]; the proof for Gorenstein* lattices is in [EK07]. Theorem 17.5.5 is proved in [Ehr05a], where one can find a list of the currently best known inequalities for ***cd***-coefficients of polytopes of low dimensions (through $d = 8$). Theorem 17.5.6 was proved by Stanley [Sta87b] for rational polytopes, and extended to all polytopes by Karu [Kar04]. Relationships between these classes of inequalities and those that can be derived from them are discussed in [Ste04]. Nonnegativity of *certain* ***cd***-coefficients for odd-dimensional simplicial manifolds is shown in [Nov00].

The problem of determining all linear inequalities for flag f-vectors has been considered for classes of partially ordered sets more general than the face posets of polytopes and spheres. In [BH00a], the (Catalan many) extreme rays are determined for the closed convex cone determined by flag f-vectors of all graded posets (posets with a rank function and having minimum and maximum elements). A nice description of the finite minimum set of inequalities is lacking, however. In [BaH01], a partial family of extreme rays is determined for the subcone determined by all Eulerian posets. See [BH00b] for more such results.

There is a notion of convolution product of flag f numbers, originally due to Kalai [Kal88], that can be used to produce new linear inequalities from given ones; see, for example, [BaL93, Section 3.10]. The algebraic properties of this product have been developed in [BiL00]; this has led to a deeper understanding of the combinatorial and algebraic properties of the ***cd***-index via duality of Hopf algebras (see [BHW02]). For a discussion of these developments, including an extension to enumeration in Bruhat intervals in Coxeter groups, see [Bil10] and [BiB11].

Theorem 17.5.7, due to Braden and MacPherson [BM99] (for rational polytopes and [Kar04] in general), gives a connection between the g-vector of a polytope P and that of one of its faces. The analogous Theorem 17.5.8 for ***cd***-indices can be found in [BE00] (as can the upper bound theorem, Theorem 17.5.9). These are examples of "monotonicity theorems" related to face numbers. For similar theorems relating h-vectors of subcomplexes and subdivisions of a simplicial complex Δ, see Sections III.9–10 of [Sta96] and the references given there.

Theorem 17.5.10(i) and (iv) can be found in [BaK91]. Theorem 17.5.10(ii) is proved in [Bra06, Theorem 4.5], where it is attributed to Kalai (unpublished). Theorem 17.5.10(iii) is due to Stanley [Sta92, Proposition 8.1]; this form of it is [Bra06, (20)].

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Theorems 17.5.11 and 17.5.12 are due to Blind and Blind [BB90] and Figiel, Lindenstrauss, and Milman [FLM77], respectively.

For the fact that the flag f-vector of a zonotope or arrangement (or, more generally, of an oriented matroid) depends only on the underlying matroid, see [BLS$^+$93, Cor. 4.6.3]. For expressions giving the $\boldsymbol{cd}$-index of a zonotope in terms of the flag h-vector of its underlying geometric lattice, see [BER97, Corollary 3.2] and [BHW02, Proposition 3.5]. That the only linear relations satisfied by zonotopes are the generalized Dehn-Sommerville equations of Theorem 17.5.1(iii), as well as the divisibility property in Theorem 17.5.15, is proved in [BER98]. The bounds on the $\boldsymbol{cd}$-indices of zonotopes in Theorem 17.5.15 are proved in [BER97]; the bounds in Theorem 17.5.16 can be found in [Ehr05b]. Theorem 17.5.17 is due to Varchenko for the case $k = 2$ (see [BLS$^+$93, Proposition 4.6.9]) and to Liu. The stronger version given here is due to Stenson; in fact, [Ste05, Theorem 9] gives a stronger inequality (see also [Ste01]).

Theorem 17.5.18(i) can be found in [Grü67, Section 10.3]; 17.5.18(ii) appears in dual form (for 4-valent 3-polytopes) in [Bar83]; 17.5.18(iii) can be derived using the methods of [Grü67, Section 18.2] (see also [BER98]). Theorem 17.5.19 can be found in [Bay87]; see also [HZ00]. An interesting general discussion of f-vectors of 4-polytopes (ordinary and flag) and an up-to-date survey of this topic is given by Ziegler [Zie02]. In particular, a good case is made there that the situation for f-vectors of 4-polytopes is much more complicated than that for polytopes in dimension 3. One reason for this is that *neighborly cubical* d-polytopes begin to exist for $d = 4$: for any $n \geq d \geq 2r + 2$, there is a cubical convex d-polytope whose r-skeleton is combinatorially equivalent to that of the n-dimensional cube [JZ00] (see also [BBC97], where spheres having this property are constructed). In particular, for any $n \geq 4$, there is a cubical 4-polytope with the graph of the n-cube. These polytopes show that the ratio f_3/f_0 is not bounded over cubical 4-polytopes.

17.6 OPEN PROBLEMS

PROBLEM 17.6.1

Characterize the f-vectors of triangulations of the $(d-1)$-sphere. (It has been conjectured that the conditions of the g-theorem provide the answer. See [Swa14] for recent results and an overview of the g-conjecture.)

PROBLEM 17.6.2

Characterize the f-vectors of triangulations of the d-ball. (See [Kol11a, Kol11b, Kol11c, Mur13a, Mur13b] for recent results on this and related questions.)

PROBLEM 17.6.3

Characterize the f-vectors of triangulations of the d-torus. (It is known that $f(\text{2-torus}) = \{(n, 3n, 2n) \mid n \geq 7\}$, but the question is open for $d \geq 3$.)

PROBLEM 17.6.4

Characterize the f-vectors of d-polytopes. (The answer is known for $d \leq 3$, cf. Theorem 17.5.18(i), but for $d \geq 4$ there is not even a conjectured answer.)

PROBLEM 17.6.5 *I. Bárány*

Does there exist a constant $c_d > 0$ such that $f_i \geq c_d \cdot \min\{f_0, f_{d-1}\}$ for all d-polytopes and all i? Will $c_d = 1$ do?

PROBLEM 17.6.6

Characterize the f-vectors of centrally symmetric d-polytopes. (The question is open in the simplicial as well as in the general case. For results and conjectures on upper bounds, see [BN08, BLN13].)

PROBLEM 17.6.7 *Conjecture of G. Kalai*

The total number of faces (counting P but not $\emptyset$) of a centrally symmetric convex d-polytope P is at least 3^d. (Verified in the simplicial case as a consequence of Theorem 17.3.8.)

PROBLEM 17.6.8

Characterize the f-vectors of clique complexes.

PROBLEM 17.6.9 *Conjecture of Charney and Davis* [Sta96, p. 100]

Let $(g_0, \ldots, g_k)$ be the g-vector of a clique complex homeomorphic to the sphere S^{2k-1}. Then $g_k - g_{k-1} + \ldots + (-1)^k g_0 \geq 0$. (The case $k = 2$ was proved by Davis and Okun [DO01].)

PROBLEM 17.6.10 *Conjecture of Ehrenborg*

*For d-polytopes P (and more generally for simplicial $(d-1)$-spheres) the **cd**-index satisfies*

$$\phi_{\boldsymbol{udv}}(P) - \phi_{\boldsymbol{uc}^2\boldsymbol{v}}(P) \geq \phi_{\boldsymbol{udv}}(\Delta^{(d)}) - \phi_{\boldsymbol{uc}^2\boldsymbol{v}}(\Delta^{(d)}),$$

where $\deg u + \deg v = d - 2$, and $\Delta^{(d)}$ is the d-simplex. (This is a special case of [Ehr05a, Conj. 6.1].)

PROBLEM 17.6.11 *Adin* [Adi96]

The "generalized lower bound conjecture" for cubical d-polytopes and $(d-1)$-spheres: $g_i^c \geq 0$ for $i \leq \lfloor d/2 \rfloor$. (This has been shown to be the best possible set of linear inequalities for cubical $(d-1)$-spheres [BBC97]. The case $i = 1$ is implied by Theorem 17.5.11.) *More generally, characterize the f-vectors of cubical polytopes.*

PROBLEM 17.6.12

Characterize the flag f-vectors of polytopes and of zonotopes. In particular, determine a complete set of linear inequalities holding for flag f-vectors of polytopes and of zonotopes.

PROBLEM 17.6.13

Characterize (toric) h-vectors of general polytopes.

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PROBLEM 17.6.14

Characterize flag f-vectors of colored complexes (here f_S is the number of simplices with color set S) (see [Cho15, Fro12a, Fro12b, Wal07])*; of* pure *colored complexes; of graded posets* (all linear inequalities are known here [BH00a])*; of Eulerian posets* (see [BaH01])*; of Eulerian lattices.*

17.7 SOURCES AND RELATED MATERIAL

FURTHER READING

Surveys of f-vector theory are given in [Bil16, BaL93, Bjö87, BK89, KK95, Sta85]. Books treating f-vectors (among other things) include [And87, BMSW94, Grü67, MS71, Sta96, Zie95].

RELATED CHAPTERS

Chapter 6: Oriented matroids
Chapter 15: Basic properties of convex polytopes
Chapter 16: Subdivisions and triangulations of polytopes
Chapter 56: Splines and geometric modeling

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