# 16 SUBDIVISIONS AND TRIANGULATIONS OF POLYTOPES

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## INTRODUCTION

We are interested in the set of all subdivisions or triangulations of a given polytope $P$ and with a fixed finite set $V$ of points that can be used as vertices. $V$ must contain the vertices of $P$, and it may or may not contain additional points; these additional points are vertices of some, but not all, the subdivisions that we can form. This setting has interest in several contexts:

- In computational geometry there is often a set of *sites* $V$ and one wants to find the triangulation of $V$ that is optimal with respect to certain criteria.
- In algebraic geometry and in integer programming one is interested in triangulations of a lattice polytope $P$ using only lattice points as vertices.
- Subdivisions of some particular polytopes using only vertices of the polytope turn out to be interesting mathematical objects. For example, for a convex $n$-gon and for the prism over a $d$-simplex they are isomorphic to the face posets of two remarkable polytopes, the associahedron and the permutahedron.

Our treatment is very combinatorial. In particular, instead of regarding a subdivision as a set of polytopes we regard it as a set of subsets of $V$, whose convex hulls subdivide $P$. This may appear to be an unnecessary complication at first, but it has advantages in the long run. It also relates this chapter to Chapter 6 (oriented matroids). For more application-oriented treatments of triangulations see Chapters 27 and 29. A general reference for the topics in this chapter is [DRS10].

## 16.1 BASIC CONCEPTS

### GLOSSARY

***Affine span:*** The affine span of a set $V \subset \mathbb{R}^d$ is the smallest affine space, or flat, containing $V$. It is denoted by $\mathrm{aff}\,(V)$.

***Convex hull:*** The convex hull of a set $V \subset \mathbb{R}^d$ is the smallest convex set containing $V$. It is denoted by $\mathrm{conv}\,(V)$.

***Polytope:*** A polytope $P$ is the convex hull of a finite set $V$ of points. Its ***dimension*** is the dimension of its affine span $\mathrm{aff}\,(P) = \mathrm{aff}\,(V)$. A ***face*** of $P$ is the set $P^f := \{x \in P : f(x) \geq f(y)\ \forall y \in P\}$ that maximizes a linear functional $f$. The empty set and $P$ are considered faces and every face is a polytope, of dimension



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ranging from $-1$ (empty set), 0 (vertices), 1 (edges), 2, ..., to $d-1$ (facets), and $d$ ($P$ itself). The set of vertices will be denoted by vert $(P)$. The ***boundary*** of a $d$-dimensional polytope is the union of all its proper faces. See Chapter 15.

***Polytopal complex:*** A polytopal complex is a finite, nonempty collection $S = \{P_1, \ldots, P_k\}$ of polytopes in $\mathbb{R}^d$ such that every face of each $P_i \in S$ is in $S$, and such that $P_i \cap P_j$ is always a common face of both (possibly empty). The dimension of $S$, $\dim(S)$, is the largest dimension of $P_i$. $S$ is ***pure*** if all maximal polytopes in $S$ have the same dimension [Zie95]. The ***k-skeleton*** of $P_i$ is the $k$-dimensional complex consisting of faces of dimension at most $k$.

***Faces of a set:*** Let $S$ be a subset of a finite set $V$ of points in $\mathbb{R}^d$. We say $S$ is a face of $V$ if there is a face $F$ of the polytope $P = \operatorname{conv}(V)$ for which $S = V \cap F$. Note that $S$ may include points that are not vertices of $F$. The dimension of $S$ is the dimension of $\operatorname{conv}(S)$, and faces of dimension 0, 1, and $\dim(V) - 1$ are referred to as vertices, edges, and facets, respectively, of the set $V$.

***Subdivision:*** Suppose $V$ is a finite set of points in $\mathbb{R}^d$ such that $P = \operatorname{conv}(V)$ is $d$-dimensional. A subdivision of $V$ is a finite collection $S = \{S_1, \ldots, S_m\}$ of subsets of $V$, called ***cells***, such that:

(DP) for each $i \in \{1, \ldots, m\}$, $P_i := \operatorname{conv}(S_i)$ is $d$-dimensional (a $d$-polytope);

(UP) $P$ is the union of $P_1, \ldots, P_m$; and

(IP) if $i \neq j$ then $F := S_i \cap S_j$ is a common (possibly empty) proper face of $S_i$ and $S_j$ and $P_i \cap P_j = \operatorname{conv}(F)$.

We will also say that $S$ is a subdivision of the polytope $P$. The collection of polytopes $P_1, \ldots, P_m$, together with their faces, is a pure polytopal complex.

***Trivial subdivision:*** The trivial subdivision of $V$ is the subdivision $\{V\}$.

***Simplex:*** A $d$-dimensional simplex is a $d$-polytope with exactly $d+1$ vertices. Equivalently, it is the convex hull of a set of affinely independent points in $\mathbb{R}^d$. We will also refer to the set of vertices of a $d$-simplex as a $d$-simplex.

***Triangulation:*** A subdivision of $V$ is a triangulation if every cell is a simplex.

***Faces:*** The faces of a subdivision $\{S_1, \ldots, S_m\}$ are $S_1, \ldots, S_m$ and all their faces.

## EXAMPLES

In Figure 16.1.1, (a) shows a set of six points in $\mathbb{R}^2$. The collection of three polygons in (b) is not a subdivision of that set since not every pair of polygons meets along a common edge or vertex; (c) shows a subdivision that is not a triangulation; and (d) gives a triangulation.

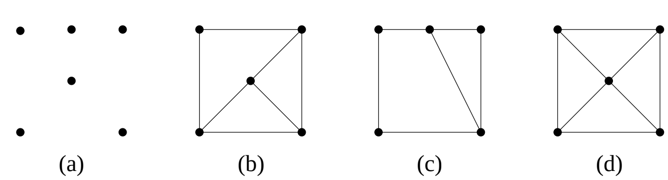


FIGURE 16.1.1
(a) *A set of points.*
(b) *A nonsubdivision.*
(c) *A subdivision.*
(d) *A triangulation.*

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# 16.2 BASIC CONSTRUCTIONS AND PROPERTIES

## GLOSSARY

The ***size of a subdivision*** is its number of cells (full dimensional faces). That is, the size of $S = \{S_1, \dots, S_m\}$ is $m$.

***Diameter of a subdivision:*** Let $S = \{S_1, \dots, S_m\}$ be a subdivision and let $P_i = \operatorname{conv}(S_i)$, $1 \le i \le m$. Polytopes $P_i \ne P_j$ are ***adjacent*** if they share a common facet. A sequence $P_{i_0}, \dots, P_{i_k}$ is a ***path*** if $P_{i_j}$ and $P_{i_{j-1}}$ are adjacent for each $j$, $1 \le j \le k$. The ***length*** of such a path is $k$. The ***distance*** between $P_i$ and $P_j$ is the length of the shortest path connecting them. The diameter of $S$ is the maximum distance occurring between pairs of polytopes $P_i, P_j$.

***Refinement of a subdivision:*** Suppose $S = \{S_1, \dots, S_l\}$ and $T = \{T_1, \dots, T_m\}$ are two subdivisions of $V$. Then $T$ is a refinement of $S$ if for each $j$, $1 \le j \le m$, there exists $i$, $1 \le i \le l$, such that $T_j \subseteq S_i$. In this case we will write $T \le S$.

***Visible facet:*** Let $P = \operatorname{conv}(V)$ be a $d$-polytope in $\mathbb{R}^d$, $F$ a facet of $P$, and $v$ a point in $\mathbb{R}^d$. We say that $F$ (or that $F \cap V$, as a facet of $V$) is visible from $v$ if the unique hyperplane containing $F$ has $v$ and the interior of $P$ in opposite sides. If $P$ is a $k$-polytope in $\mathbb{R}^d$ with $k < d$ and $v \in \operatorname{aff}(P)$, then the above definition is modified in the obvious way, taking $\operatorname{aff}(P)$ as the ambient space.

***Placing a vertex:*** Suppose $S = \{S_1, \dots, S_m\}$ is a subdivision of $V$ and $v \notin V$. The subdivision $T$ of $V \cup \{v\}$ that results from placing $v$ is obtained as follows:

- If $v \notin \operatorname{aff}(V)$, then $T = \{S_i \cup \{v\} : S_i \in S\}$ (cone over $S$ with apex $v$).
- If $v \in \operatorname{aff}(V)$, then $T$ equals $S$ together with the faces $F \cup \{v\}$ for each $(d-1)$-face $F$ in $S$ that is contained in a facet of $\operatorname{conv}(V)$ visible from $v$.

Note that if $v \in \operatorname{conv}(V)$, then $S = T$ (that is, $T$ does not use $v$).

***Pulling a vertex:*** Suppose $S = \{S_1, \dots, S_m\}$ is a subdivision of $V$ and $v \in S_1 \cup \dots \cup S_m$. The result of pulling $v$ is the refinement $T$ of $V$ obtained by modifying each $S_i \in S$ as follows. It was described in [Hud69, Lemma 1.4].

- If $v \notin S_i$, then $S_i \in T$.
- If $v \in S_i$, then for every facet $F$ of $S_i$ not containing $v$, $F \cup \{v\} \in T$.

***Pushing a vertex:*** Suppose $S = \{S_1, \dots, S_m\}$ is a subdivision of $V$ (where $\dim(\operatorname{conv}(V)) = d$) and $v \in S_1 \cup \dots \cup S_m$. The result of pushing $v$ is the refinement $T$ of $V$ obtained by modifying each $S_i \in S$ as follows:

- If $v \notin S_i$, then $S_i \in T$.
- If $v \in S_i$ and $S_i \setminus \{v\}$ is $(d-1)$-dimensional (i.e., $\operatorname{conv}(S_i)$ is a pyramid with apex $v$), then $S_i \in T$.
- If $v \in S_i$ and $S_i \setminus \{v\}$ is $d$-dimensional, then $S_i \setminus \{v\} \in T$ and for every facet $F$ of $S_i \setminus \{v\}$ that is visible from $v$, $F \cup \{v\} \in T$.

***Lexicographic subdivisions:*** A subdivision $T$ of $V$ is lexicographic if it can be obtained from the trivial subdivision by pushing and/or pulling some of the points in $V$ in some order. If only pushings (resp. pullings) are used, we call it a pushing (resp. pulling) subdivision.

## 16.2.1 LEXICOGRAPHIC SUBDIVISIONS

Refinement of subdivisions of $V$ is a partial order with a unique maximal element, the trivial subdivision, and whose minimal elements are the triangulations of $V$: Every subdivision that is not a triangulation can be lexicographically refined.

Lexicographic triangulations were introduced by Sturmfels [Stu91] and studied in detail by Lee [Lee91]. The following results show that they are a quite versatile way of constructing subdivisions. For more details see [DRS10, Sect. 4.3]:

1. Placing and pushing are closely related: the triangulation obtained by placing the points of $V$ in a certain order is the same as obtained starting with the trivial subdivision of $V$ and pushing the points of $V$ in the opposite order.
2. Placing all elements of $V$ in any given order produces a triangulation of $V$. If the order is chosen so that no point is in the convex hull of the previously placed ones (e.g., ordering the points with respect to a generic linear functional) then the triangulation obtained uses all points of $V$ as vertices. This shows that every finite set $V$ is the vertex set of some triangulation of $V$.
3. Pulling or pushing a vertex $v$ in a lexicographic subdivision in which $v$ had already been pushed or pulled produces no effect. Hence, every lexicographic subdivision can be determined as an ordered subset of $V$ indicating, for each point in it, whether it is to be pulled or pushed.
4. After all but an affinely independent subset of $V$ have been pulled or pushed the lexicographic subdivision is a triangulation. For pullings the converse does not hold, but for pushings it does: For every ordering $\{v_1, \ldots, v_n\}$ of the points in $V$ such that the last $d+1$ are affinely independent, each of the first $n-d-1$ pushings produces a proper refinement. In particular, the poset of subdivisions of $V$ has chains of length at least $n-d-1$, for every $V$.
5. In a pulling subdivision, all cells contain the first point that is pulled.
6. In a pushing subdivision $S$, if all points except those of a subset $F \subset V$ are pushed, then $F$ is a face of $S$.
7. Both operations may produce subdivisions that do not use all the points of $V$: if a $v \in V$ is pushed before it is a vertex, then it disappears from all cells containing it. The same happens if $v \in V$ is in the relative interior of a face $F$ of a subdivision and another point of $F$ is pulled. In particular, in a pulling triangulation at most one point in the interior of $\operatorname{conv}(V)$ is used.
8. If $\operatorname{card}(V) \leq d+3$, then every triangulation of $V$ is lexicographic [Lee91]:
   - If $\operatorname{card}(V) = d+1$, then $V$ has a unique triangulation, the trivial one.
   - If $\operatorname{card}(V) = d+2$, let $(\lambda_1, \ldots, \lambda_{d+2})$ be the unique (up to rescaling) affine dependence among $V = \{v_1, \ldots, v_{d+2}\}$. (That is, the solution to $\sum_i \lambda_i = 0$ and $\sum_i \lambda_i v_i = 0$.) Then, $V$ has exactly two triangulations
$$T^+ = \{V \setminus \{v_i\} \mid \lambda_i > 0\} \qquad \text{and} \qquad T^- = \{V \setminus \{v_i\} \mid \lambda_i < 0\}.$$

$T^+$ (resp. $T^-$) is obtained by pushing any $v_i$ with $\lambda_i > 0$ (resp. $\lambda_i < 0$) or by pulling any $v_i$ with $\lambda_i < 0$ (resp. $\lambda_i > 0$). See Figure 16.3.1.

- If card $(V) = d+3$, then $V$ has at most $d+3$ triangulations, with equality if (but not only if) no $d+1$ points lie in a hyperplane. They can all be obtained by pushing the points in specific orders, but not always by pulling them. See an example in Figure 16.4.1.

The triangulations in Figure 16.3.2, with $d+4$ points, are not lexicographic.

## EXAMPLES

FIGURE 16.2.1
(a) *Pulling point 1 already gives a triangulation.*
(b) *Pushing triangulation for the order* 1234567. *Equivalently, placing triangulation for the order* 7654321.
(c) *Pushing at* 1 *then pulling at* 2.

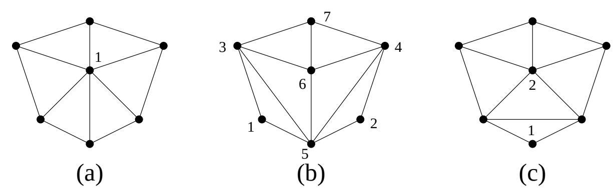



Figure 16.2.1 gives three lexicographic triangulations. (a) is obtained by pulling point 1, but cannot be obtained by pushing alone. (b) is obtained by pushing the points in the indicated order, but cannot be obtained by pulling points alone. (c) is obtained by pushing point 1 and then pulling point 2.

## 16.2.2 NUMBER, SIZE, AND DIAMETER OF TRIANGULATIONS

Size and diameter of subdivisions are monotone with respect to refinement, so the maximum is always achieved at a triangulation.

Every $d$-dimensional triangulation with $n$ vertices has size bounded above by $O(n^{\lceil d/2 \rceil})$, achieved for example for (some) triangulations of cyclic polytopes. See Section 16.7. Note that asymptotic bounds in this section consider $d$ fixed.

Every set $V$ has triangulations of diameter at most $2(n-d-1)$, since pushing a point in a subdivision increases its diameter by at most two units [Lee91]. No good upper bound is known for the diameters of all triangulations. In particular, no upper bound for the diameter that is polynomial in both $d$ and $n$ is known (obtaining them is essentially as difficult as solving the polynomial Hirsch conjecture for polytopes), and no construction of triangulations with diameter greater than a small constant times $n-d$ is known. See [San13] and the references therein.

A triangulation of $V$ is completely determined if we know its faces up to dimension $d/2$ [Dey93]. Hence the number of different triangulations $V$ can have is bounded above by $2^{\binom{n}{d/2+1}}$. This bound is not far from the number of triangulations of a cyclic $d$-polytope with $n$ vertices, which is in $2^{\Omega(n^{\lfloor d/2 \rfloor})}$ [DRS10, Sect. 6.1.6].

## 16.2.3 TRIANGULATIONS AND ORIENTED MATROIDS

Checking whether a given collection $S$ of subsets of $V$ is a subdivision (or a triangulation) can be done knowing the oriented matroid $M$ of $V$ alone. (We refer to

Chapter 6 or to [BLS$^+$99] for details on oriented matroids). Indeed, properties (1) and (2) of the following theorem are respectively equivalent to (DP) and (IP) in the definition of subdivision. Once (IP) and (DP) hold, (3) is equivalent to (UP).

**THEOREM 16.2.1** [DRS10, Theorems 4.1.31 and 4.1.32]
*A collection $S = \{S_1, \ldots, S_m\}$ of subsets of $V \subset \mathbb{R}^d$ is a subdivision if and only if:*

1. *Every $S_i$ is spanning in (that is, contains a basis of) $M$.*
2. *For every oriented circuit $C = (C^+, C^-)$ in $M$ with $C^+ \subset S_i$ for some $S_i \in S$, either $C^- \subset S_i$ or $C^-$ is not contained in any $S_j \in S$.*
3. *$S$ is not empty and for every $S_i$ in $S$ and facet $F$ of $S_i$, either $F$ is contained in a facet of $M$ or there is another $S_j \in M$ having $F$ as a facet and lying in the other side of $F$. (Facets of $V$ can easily be detected via cocircuits of $M$.)*

This led Billera and Munson to introduce triangulations of (perhaps not realizable) oriented matroids [BM84], including notions of placing, pulling and pushing for them. See also [BLS$^+$99, Ch. 9] and [San02]. The oriented matroid approach to triangulations is implemented in the software package TOPCOM [PR03, Ram02], which is currently part of the distribution of `polymake` (see Chapter 67).

## 16.3 REGULAR TRIANGULATIONS AND SUBDIVISIONS

One way to construct a subdivision of a point set $V \subset \mathbb{R}^d$ is to lift it to $\mathbb{R}^{d+1}$ and then look at the projection of the lower facets (facets visible from below) of the lifted point set. This allows any convex hull algorithm in $\mathbb{R}^{d+1}$ (see Chapter 26 of this Handbook) to be used to compute subdivisions in $\mathbb{R}^d$. The subdivisions obtained in this way are called regular, and they have some special properties.

### GLOSSARY

***Regular subdivision:*** Let $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ and let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ be any vector. The ***regular subdivision*** of $V$ obtained by the ***lifting vector*** $\alpha$ is defined as follows [GKZ94, Lee91, Zie95, DRS10]:

(i) Let $\tilde{v}_i = (v_i, \alpha_i)$ for each $i$ and compute the facets of $\tilde{V} = \{\tilde{v}_1, \ldots, \tilde{v}_n\}$.
(ii) Project the lower facets of $\tilde{V}$ onto $\mathbb{R}^d$.

Here, a ***lower facet*** of $\tilde{V}$ is a facet that is visible from below. That is, a facet whose outer normal vector has its last coordinate negative. Observe that the "projection" step is combinatorially trivial. For each lower facet $\{\tilde{v}_{i_1}, \ldots, \tilde{v}_{i_k}\}$ of $\tilde{V}$ we simply make $\{v_{i_1}, \ldots, v_{i_k}\}$ a cell in the subdivision.

***Combinatorially isomorphic subdivisions:*** Let $V$ and $V'$ be point sets. A subdivision $S$ of $V$ and a subdivision $S'$ of $V'$ are ***combinatorially isomorphic*** if there is a bijection between $V$ and $V'$ such that for every face $F$ of $S$ the corresponding subset $F' \subseteq V'$ is a face of $S'$, and vice-versa. See Figure 16.3.2.

***Shellable:*** A pure polytopal complex $S$ is ***shellable*** if it is 0-dimensional (i.e., a nonempty finite set of points) or else $\dim(S) = k > 0$ and $S$ has a ***shelling***, i.e., an ordering $P_1, \dots, P_m$ of its maximal faces such that for $2 \le j \le m$ the intersection of $P_j$ with $P_1 \cup \dots \cup P_{j-1}$ is nonempty and is the beginning segment of a shelling of the $(k-1)$-dimensional boundary complex of $P_j$ [Zie95].

***Nonconvex polytope:*** The region of $\mathbb{R}^d$ enclosed by a $(d-1)$-dimensional pure polytopal complex homeomorphic to a sphere.

### 16.3.1 PROPERTIES AND EXAMPLES OF REGULAR SUBDIVISIONS

All regular subdivisions are shellable. To see this, consider a ray in the direction $(0, \dots, 0, -1)$ emitted from a point in the interior of $\mathrm{conv}(\tilde{V})$ and in sufficiently general position. The order in which the ray crosses the supporting hyperplanes of the lower facets of $\tilde{V}$ is a shelling order. (This is an example of a *line shelling* of $\mathrm{conv}(\tilde{V})$; see [BM71, Zie95]). In contrast, there exist nonshellable subdivisions, starting in dimension 3. The first example was Rudin's nonshellable triangulation of a tetrahedron [Rud58]. For some additional discussion, including a nonshellable triangulation of the 3-cube, see Ziegler [Zie95].

Regular subdivisions include all lexicographic ones. The subdivision obtained by pushing/pulling $v_{i_1}, \dots, v_{i_k}$ in that order coincides with the regular subdivision constructed by choosing $|\alpha_{i_1}| \gg \dots \gg |\alpha_{i_k}| \gg 0$, where $\alpha_i > 0$ if $v_i$ is pushed and $\alpha_i < 0$ if $v_i$ is pulled, and choosing $\alpha_i = 0$ if $v_i$ is neither pulled nor pushed. Pulling or pushing points in a regular subdivision produces a regular subdivision.

Non-regular subdivisions exist for all dimensions $d \ge 2$ and number of points $n \ge d + 4$ [Lee91] (see Figure 16.3.2(b) for a smallest example). In dimension 2 all subdivisions are combinatorially isomorphic to regular ones as a consequence of Steinitz's Theorem (see [Grü67, Zie95] and Chapter 15 of this Handbook). For $d \ge 3$ and $n \ge 7$ the same is not true [Lee91] (see Figure 16.3.3(b)).

There are, in general, many fewer regular than nonregular triangulations:

- For fixed $n - d$, the number of regular subdivisions of $V$ is bounded above by a polynomial of degree $(n - d - 1)^2$ in $d$ [BFS90]. In contrast, the number of non-regular triangulations can grow exponentially, even fixing $n - d = 4$ [DHSS96].
- For fixed $d$, the number of regular triangulations is bounded above by $2^{O(n \log n)}$ while the number of nonregular ones can grow as $2^{\Omega(n^{\lfloor d/2 \rfloor})}$ [DRS10, Sec. 6.1].

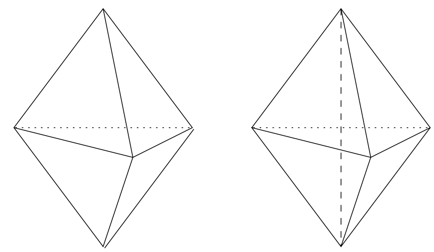
A prime example of a regular subdivision is the Delaunay subdivision, obtained with $\alpha_i = \|v_i\|^2$. In fact, regular subdivisions are sometimes called *weighted Delaunay subdivisions*. The regular subdivision obtained with $\alpha_i = -\|v_i\|^2$ is the "farthest site" Delaunay subdivision. See Chapter 27 of this Handbook.

Regularity of a subdivision of $V$ cannot be decided based only on the oriented matroid of $V$: the two point sets in Figure 16.3.2 have the same oriented matroid, yet the triangulation in (a) is regular and the triangulation in (b) is not.

Checking regularity is equivalent to feasibility of a linear program on $n$ variables (the $\alpha_i$'s) with one constraint for each pair of adjacent cells (local convexity of the lift) [DRS10, Sec. 8.2]. On the other hand, checking whether a triangulation or subdivision is combinatorially isomorphic to a regular one is very hard, as difficult as the existential theory of the reals (determining feasibility of systems of real polynomial inequalities). See comments on the Universality Theorem in Chapters 6 and 15 of this Handbook, and in [Zie95]).

## EXAMPLES

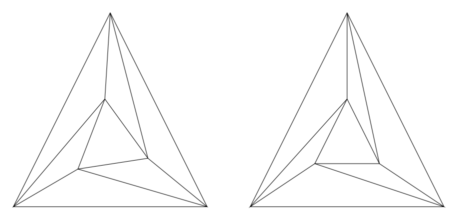
Figure 16.3.1 shows the two triangulations (both regular) of the vertices of a 3-dimensional bipyramid over a triangle. In (a) there are two tetrahedra, sharing a common internal triangle; in (b) there are three, sharing a common internal edge.



(a) (b)

FIGURE 16.3.1
*The two triangulations of a set of* 5 *points in* $\mathbb{R}^3$.

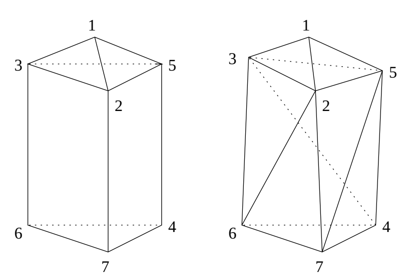
Figure 16.3.2 shows the same triangulation for two different sets of 6 points in $\mathbb{R}^2$ having the same oriented matroid. Only the first triangulation is regular.



(a) (b)

FIGURE 16.3.2
*A regular and a nonregular (but combinatorially isomorphic) triangulation.*

Figure 16.3.3 shows two 3-polytopes, both with 7 vertices. The "capped triangular prism" in (a) admits two nonregular triangulations: $\{1257, 1457, 1236, 1267, 1345, 1346, 1467\}$ and $\{1245, 1247, 1237, 1367, 1356, 1456, 1467\}$. Both triangulations are combinatorially isomorphic to regular ones. The polytope in (b) is obtained from the capped triangular prism by slightly rotating the top triangle. It has one nonregular triangulation, not combinatorially isomorphic to a regular one: $\{1245, 1247, 1237, 1367, 1356, 1456, 1467, 2457, 2367, 2345\}$. See [Lee91].




(a) (b)

FIGURE 16.3.3
*Two polytopes with nonregular triangulations.*

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### 16.3.2 TRIANGULATING REGIONS BETWEEN POLYTOPES

Not every nonconvex polytope can be triangulated without additional vertices. One classical example is Schönhardt's 3-polytope [Sch28] [DRS10, Example 3.6.1]: a nonconvex octahedron obtained by slightly rotating with respect to one another the two triangular facets of a triangular prism. However, regular triangulations can sometimes be used to triangulate nonconvex regions:

- Suppose $P$ and $Q$ are two $d$-polytopes in $\mathbb{R}^d$ with $Q$ contained in $P$. If we start with the trivial subdivision and push all vertices of $P$, then we get a subdivision in which the region of $P$ outside $Q$ is triangulated [GP88].
- Now suppose $P$ and $Q$ are two disjoint $d$-polytopes in $\mathbb{R}^d$ with vertex sets $V$ and $W$, respectively. One can triangulate the region in $\operatorname{conv}(P \cup Q)$ that is exterior to both $P$ and $Q$ by the following procedure [GP88]:
  1. Let $H$ be a hyperplane for which $P$ and $Q$ are contained in opposite open halfspaces. Construct a regular subdivision of $V \cup W$ by setting $\alpha_i$ equal to the distance of $v_i$ to $H$ for each $v_i \in V \cup W$. For example, if $H = \{x \mid a \cdot x = \beta\}$, then $\alpha_i$ can be taken to equal $|a \cdot v_i - \beta|$.
  2. Refine this arbitrarily and ignore the simplices within $P$ or $Q$.

However, if we are given *three* mutually disjoint polytopes $P$, $Q$ and $R$ it may be impossible to triangulate, without additional vertices, the region in $\operatorname{conv}(P \cup Q \cup R)$ that is exterior to the three. As an example, removing the three tetrahedra $2457, 2367, 2345$ from the convex hull of $234567$ in Figure 16.3.3(b) gives Schönhardt's nontriangulable 3-polytope [Sch28].

## 16.4 SECONDARY AND FIBER POLYTOPES

This section deals with the structure of the collection of all regular subdivisions of a given finite set of points $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$. The main result is that the poset of regular triangulations of $V$ is isomorphic to the face poset of a certain polytope, the secondary polytope of $V$. This polytope plays an important role in the study of generalized discriminants and determinants [GKZ94] and Gröbner bases [Stu96]. Secondary polytopes are studied in detail in [DRS10, Ch. 5].

### 16.4.1 SECONDARY POLYTOPES

#### GLOSSARY

***Volume vector:*** Suppose $T$ is a triangulation of $V = \{v_1, \ldots, v_n\}$. Define the volume vector $z(T) = (z_1, \ldots, z_n) \in \mathbb{R}^n$ by $z_i = \sum_{v_i \in F \in T} \operatorname{vol}(F)$, where the sum is taken over all $d$-simplices $F$ in $T$ having $v_i$ as a vertex. $z(T)$ is sometimes called the ***GKZ-vector*** of $T$, to honor Gelfand, Kapranov, and Zelevinsky.

***Secondary polytope:*** The secondary polytope $\Sigma(V)$ is the convex hull of the volume vectors of all triangulations of $V$.

***Link:*** The link of a face $F$ of a triangulation $T$ is $\{G \mid F \cup G \in T, F \cap G = \emptyset\}$.

**THEOREM 16.4.1** Gelfand, Kapranov, and Zelevinsky [GKZ94]

1. *$\Sigma(V)$ has dimension $n-d-1$. Its affine span is defined by the $d+1$ equations*

$$\sum_{i=1}^{n} z_i = (d+1)\mathrm{vol}\,(P), \quad \text{and} \quad \sum_{i=1}^{n} z_i v_i = (d+1)\mathrm{vol}\,(P)c, \tag{16.4.1}$$

*where $c$ is the centroid of $P = \mathrm{conv}\,(V)$.*

2. *The poset of (nonempty) faces of $\Sigma(V)$ is isomorphic to the poset of all regular subdivisions of $V$, partially ordered by refinement:*
   - *For a given regular subdivision $S$ of $V$, the volume vectors of all regular triangulations that refine $S$ are the vertices of a face $F_S$ of $\Sigma(V)$. This face contains also the volume vectors of nonregular triangulations refining $S$, but these are never vertices of it.*
   - *A lifting vector $(\alpha_1, \ldots, \alpha_n)$ produces $S$ as a regular subdivision if and only if it lies in the relatively open normal cone of $F_S$ in $\Sigma(V)$.*

The secondary polytope $\Sigma(V)$ can also be expressed as a discrete or continuous Minkowski sum of polytopes coming from a representation of $V$ as a projection of the vertices of an $(n-1)$-dimensional simplex. See Section 16.4.3.

The following are consequences of Theorem 16.4.1:

1. The vertices of $\Sigma(V)$ are the volume vectors of the regular triangulations.

2. Two nonregular triangulations can have the same volume vector, but two regular ones, or a regular and a nonregular one, cannot. This implies that the triangulation of Figure 16.3.2(b) is nonregular: Flipping three diagonals in it produces another triangulation with the same volume vector.

Lifting vectors, as used in the definition of regular subdivision, correspond to linear functionals in the ambient space of $\Sigma(V)$: Suppose $S = \{S_1, \ldots, S_m\}$ is a regular subdivision of $V = \{v_1, \ldots, v_n\} \subset \mathbb{R}^d$ determined by lifting numbers $\alpha_1, \ldots, \alpha_n$. Let $f : \mathrm{conv}\,(V) \to \mathbb{R}$ be the piecewise-linear convex function whose graph is given by the lower facets of $Q = \mathrm{conv}\,(\{(v_1, \alpha_1), \ldots, (v_n, \alpha_n)\})$. Define $c_j$ to be the centroid of the polytope $P_j = \mathrm{conv}\,(S_j)$, $1 \le j \le m$. Then the inequality

$$\sum_{i=1}^{n} \alpha_i z_i \ge (d+1) \sum_{j=1}^{m} \mathrm{vol}\,(P_j) f(c_j)$$

is valid on the secondary polytope and holds with equality at the volume vector of a triangulation $T$ if and only if $T$ refines $S$. This allows for a local monotone algorithm to construct the regular triangulation corresponding to a certain lifting vector $\alpha$: start with any regular triangulation $T$ of $V$ (for example, a lexicographic one) and do flips in it (see Section 16.4.2) always decreasing $\sum_{i=1}^{n} \alpha_i z_i$ and keeping the regularity property. When such flips no longer exist we have the desired regular triangulation. For Delaunay triangulations this procedure was first described in [ES96].

There are two ubiquitous polytopes that can be constructed as secondary polytopes (see Chapter 15 of this Handbook):

- When $V$ is the set of vertices of a convex $n$-gon, $\Sigma(V)$ is the ***associahedron*** of dimension $n-3$ [Lee89]. Explicit coordinates and inequalities for $\Sigma(V)$ can be found in [Zie95]. See also Section 16.7.3.

- When $V$ is the vertex set of the Cartesian product of a $d$-simplex and a segment, $\Sigma(V)$ is a $d$-dimensional ***permutahedron***, affinely isomorphic to the convex hull of the $(d+1)!$ vectors obtained by permuting the coordinates in $(1, 2, 3, \ldots, d)$. See Section 16.7.1.

## 16.4.2 THE GRAPH OF TRIANGULATIONS OF $V$

## GLOSSARY

***(Oriented) circuits, Radon partitions:*** A circuit is a set $C = \{c_1, \ldots, c_k\}$ of affinely dependent points such that every proper subset is affinely independent. This implies, in particular, that $k = \dim(C) + 2$ and that there is a unique (up to rescaling) affine dependence $\lambda = (\lambda_1, \ldots, \lambda_k)$ of $C$. (That is, a solution to $\sum_i \lambda_i = 0$ and $\sum_i \lambda_i c_i = 0$.) Since $\lambda$ has no zero entries, this produces a natural (and unique) way of partitioning $C$ as the disjoint union of $C^+ = \{c_i \mid \lambda_i > 0\}$ and $C^- = \{c_i \mid \lambda_i < 0\}$ with the property that $\operatorname{conv}(C^+) \cap \operatorname{conv}(C^-) \neq \emptyset$. The pair $(C^+, C^-)$ is the ***oriented circuit*** or ***Radon partition*** of $C$.

***Triangulations of a circuit***: A circuit $C$ with Radon partition $(C^+, C^-)$ has exactly two triangulations

$$T^+ = \{C \setminus \{c_i\} \mid c_i \in C^+\} \qquad \text{and} \qquad T^- = \{C \setminus \{c_i\} \mid c_i \in C^-\}.$$

See Figure 16.3.1 for an example.

***Adjacent triangulations, bistellar flips, graph of triangulations:*** Let $T$ be a triangulation of $V$. Suppose there is a circuit $C$ in $V$ such that $T$ contains one of the two triangulations, say $T^+$, of $C$, and suppose further that the links in $T$ of all the cells of $T^+$ are identical. Then it is possible to construct a new triangulation $T'$ of $V$ by removing $T^+$ (together with its link) and inserting $T^-$ (with the same link). This operation is called a ***(geometric bistellar) flip***, and $T'$ is said to be adjacent to $T$. The set of all triangulations of $V$, under adjacency by flips, forms the graph of triangulations, or flip-graph, of $V$.

Flips correspond to "next-to-minimal" elements in the refinement poset of subdivisons of $V$: If $T_1$ and $T_2$ are two adjacent triangulations of $V$, then there is a subdivision $S$ whose only two proper refinements are $T_1$ and $T_2$. Conversely, if all proper refinements of a subdivision $S$ are triangulations then $S$ has exactly two such refinements, which are adjacent triangulations [DRS10, Sec. 2.4].

In particular, all edges of the secondary polytope $\Sigma(V)$ correspond to adjacency between regular triangulations. That is, the 1-skeleton of $\Sigma(V)$ is a subgraph of the graph of triangulations of $V$. But it may not be an induced subgraph: sometimes two regular triangulations $T_1$ and $T_2$ are adjacent but the intermediate subdivision $S$ is not regular, hence the flip between $T_1$ and $T_2$ does not correspond to an edge of $\Sigma(V)$ [DRS10, Examples 5.3.4 and 5.4.16].

Since the 1-skeleton of every $(n-d-1)$-polytope is $(n-d-1)$-connected and has all vertices of degree at least $n-d-1$, all regular triangulations of $V$ have at least $n-d-1$ flips and the adjacency graph of regular triangulations of $V$ is $(n-d-1)$-connected. For general triangulations the following is known:

- When $|V| \leq \dim(V)+3$ all triangulations are regular [Lee91]. When $|V| = \dim(V)+4$ every triangulation has at least three flips and the graph of all triangulations of $V$ is 3-connected [AS00].
- When $\dim(V) \leq 2$ the graph of triangulations is known to be connected [Law72], and triangulations are known to have at least $n-3$ flips. Whether the flip-graph is $(n-3)$-connected for every $V$ is an open question.
- Flip-deficient triangulations (that is, triangulations with fewer than $n-d-1$ flips) exist starting in dimension three and with $|V| = 8$ [DRS10, Ex. 7.1.1]. Triangulations exist in dimension three with $O(\sqrt{n})$ flips, in dimension four with $O(1)$ flips, and in dimension six without flips [San00].
- Point sets exist with disconnected graphs of triangulations in dimension five and higher [San05b]. In dimension six they can be constructed in general position [San06]. Whether they exist in dimensions three and four is open.

The graph of triangulations of $V$ is known to be connected for the vertex sets of cyclic polytopes [Ram97], of Cartesian products of two simplices if one of them has dimension at most three [San05a, Liu16a] and of regular cubes up to dimension four [Pou13]. It is known to be disconnected for the vertex set of the Cartesian product of a 4-simplex and a $k$-simplex, for sufficiently large $k$ [Liu16b].

Figure 16.4.1 shows the five regular triangulations of a set of 5 points in $\mathbb{R}^2$, marking which pairs of triangulations are adjacent.

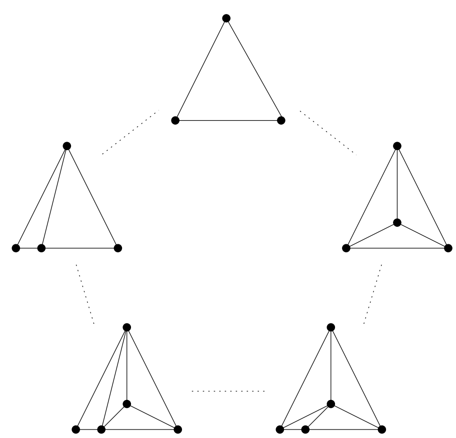

FIGURE 16.4.1
*A polygon of regular triangulations.*

See [DHSS96] for properties of the polytope that is the convex hull of the $(0,1)$ incidence vectors of all triangulations of $V$, and for the relationship of it to $\Sigma(V)$. For the vertex-set of a convex $n$-gon this polytope was first described in [DHH85].

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### 16.4.3 FIBER POLYTOPES

## GLOSSARY

***Fiber polytope:*** Let $\pi : P \to Q$ be an affine surjective map (a projection) from a polytope $P$ to a polytope $Q$. A ***section*** of $\pi$ is a continuous map $\gamma : Q \to P$ with $\pi(\gamma(x)) = x$ for all $x \in Q$. The fiber polytope of $\pi$ is defined to be the set of all average values of the sections of $\pi$:

$$\Sigma(P, Q) = \left\{ \frac{1}{\mathrm{vol}\,(Q)} \int_Q \gamma(x)dx \mid \gamma \text{ is a section of } \pi \right\}.$$

Equivalently, $\Sigma(P, Q)$ equals the Minkowski average of all fibers of the map $\pi$.

***$\pi$-induced and $\pi$-coherent subdivisions:*** A subdivision $S$ of $Q$ is $\pi$-induced if each cell in $S$ equals the image of the vertex set of a face of $P$. It is $\pi$-coherent if $\pi$ factors as $Q \to P' \to P$ for a polytope $P'$ of dimension $\dim(P) + 1$ and $S$ equals the lower part of the convex hull of $P'$.

***Baues poset:*** The poset of all nontrivial $\pi$-induced subdivisions, under refinement, is called the Baues poset of $\pi$.

The fiber polytope $\Sigma(P, Q)$ has dimension $\dim(P) - \dim(Q)$, and its face poset equals the refinement poset of $\pi$-coherent subdivisions. Billera, Kapranov and Sturmfels [BKS94] conjectured the Baues poset to be homotopy equivalent to a $(\dim P - \dim Q - 1)$-sphere, and they proved the conjecture for $\dim Q = 1$. Although the conjecture in its full generality was soon disproved [RZ96], the following so-called ***Generalized Baues Problem*** received attention: *When is the Baues poset homotopy equivalent to a sphere of dimension* $\dim P - \dim Q - 1$*?* See [Rei99] for a survey of this problem and [BS92, BKS94, DRS10, HRS00, RGZ94, Zie95] for general information on fiber polytopes. The following three cases are of special interest:

1. When $P$ is a simplex all the subdivisions of $V = \pi(\mathrm{vert}(P))$ are $\pi$-induced subdivisions, and $\pi$-coherent ones are the regular ones [BFS90, BS92, Zie95]. $\Sigma(P, Q)$ is the secondary polytope of $V$. Examples in which the Baues poset is disconnected exist [San06].

2. When $\dim(Q) = 1$ the finest $\pi$-induced subdivisions are ***monotone paths*** in the 1-skeleton of $P$ with respect to the linear functional that is constant on each fiber of $\pi$. $\Sigma(P, Q)$ is the ***monotone path polytope*** of $P$ and the Baues complex is homotopy equivalent to a $(\dim(P) - 2)$-sphere [BKS94].

3. When $P$ is a regular $k$-cube, $Q$ is a zonotope (a Minkowski sum of segments). $\pi$-induced subdivisions are the ***zonotopal tilings*** of $Q$, the finest ones being ***cubical tilings***. The Bohne-Dress Theorem [RGZ94, HRS00] states that the Baues poset coincides with the *extension space* of the oriented matroid dual to that of $Q$. Examples of disconnected extension spaces (with $\dim(Q) = 3$) have been recently announced [Liu16c].

The $d$-dimensional permutahedron stands out as a fiber polytope belonging to the three cases above: it is the monotone path polytope of the $(d+1)$-cube projected to a line via the sum of coordinates, and it is the secondary polytope of the vertex set of $\Delta_d \times I$ for a $d$-simplex $\Delta_d$ and a segment $I$. This coincidence is a special instance of the ***combinatorial Cayley Trick***: If $\pi_i : P_i \to Q_i$, $i = 1, \ldots, k$ are polytope projections with $Q_i \in \mathbb{R}^d$ for all $i$ then the fiber polytopes of the following two projections, defined from the $\pi_i$'s in the natural way, coincide [HS95, HRS00]:

$$\begin{aligned} \Pi_C : P_1 * \cdots * P_k &\to \operatorname{conv}(Q_1 \times \{e_1\} \cup \cdots \cup Q_k \times \{e_k\}) \subset \mathbb{R}^{d+k}, \\ \Pi_M : P_1 \times \cdots \times P_k &\to Q_1 + \cdots + Q_k \subset \mathbb{R}^d. \end{aligned}$$

Here $Q_1 + \cdots + Q_k$ is a ***Minkowski sum***, $P_1 \times \cdots \times P_k$ is a ***Cartesian product***, $P_1 * \cdots * P_k := \operatorname{conv}((P_1 \times \cdots \times \{0\} \times \{e_1\}) \cup \cdots \cup (\{0\} \times \cdots \times P_k \times \{e_k\})) \subset \mathbb{R}^{d_1+\cdots+d_k+k}$ is the ***join*** of the polytopes $P_i \subset \mathbb{R}^{d_i}$ and $\operatorname{conv}(Q_1 \times \{e_1\} \cup \cdots \cup Q_k \times \{e_k\})$ is called the ***Cayley polytope*** or ***Cayley sum*** of $Q_1, \ldots, Q_k$.

When all the $P_i$'s are simplices we have that $P_1 * \cdots * P_k$ is a simplex, so that all subdivisions of the Cayley polytope are $\Pi_C$-induced, and the $\Pi_M$-induced subdivisions of $Q_1 + \cdots + Q_k$ are the ***mixed subdivisions***, of interest in algebraic geometry [HS95]. Hence, the Cayley Trick gives a bijection between all subdivisions of a Cayley polytope and mixed subdivisions of the corresponding Minkowski sum.

If we further assume that all $P_i$'s are segments then $\Pi_M$-induced subdivisions are zonotopal tilings $Q_1 + \cdots + Q_k$, and the Cayley polytope of a set of segments is called a ***Lawrence polytope***. (Equivalently, a Lawrence polytope is a polytope with a centrally symmetric Gale diagram. See Chapters 6 and 15 for the definition of Gale diagrams). In particular, the Cayley Trick and the Bohne-Dress Theorem imply the following three posets to be isomorphic, for a set $Q_1, \ldots, Q_k$ of segments:

- The Baues poset of the Lawrence polytope $\operatorname{conv}(Q_1 \times \{e_1\} \cup \cdots \cup Q_k \times \{e_k\})$.
- The poset of zonotopal tilings of the zonotope $Q_1 + \cdots + Q_k$.
- The poset of extensions of the oriented matroid dual to $\{Q_1, \ldots, Q_k\}$.

## 16.5 FACE VECTORS OF SUBDIVISIONS

In this section we examine some properties of the numbers of faces of different dimensions of a triangulation or subdivision. More information on $f$-vectors, $g$-vectors, and $h$-vectors can be found in Chapter 17 of this Handbook. But note that the symbol $d$ is here shifted by one unit with respect to the conventions there, since there the $h$- and $g$-vector are usually meant for the boundary of a $d$-polytope.

### GLOSSARY

***Boundary and interior:*** Every face of dimension $d-1$ of a pure $d$-dimensional complex $S \subset \mathbb{R}^d$ is contained in exactly one or two cells. Those contained in one cell, together with their faces, form the ***boundary*** $\partial S$ of $S$, which is a pure polytopal $(d-1)$-complex. The faces of $S$ that are not in the boundary form the ***interior*** of $S$, which is not a polytopal complex. The boundary complex of a subdivision $S$ of $V$ equals $\{F \mid$ is a face of $S$ and $F \subseteq G$ for some facet $G$ of $V\}$.

***f-vector:*** Let $f_j(S)$ denote the number of $j$-dimensional faces of $S$, $-1 \le j \le d$. Note that $f_{-1}(S) = 1$ since the empty set is the unique face of $S$ of dimension $-1$. The $f$-vector of $S$ is $f(S) = (f_0(S), \ldots, f_d(S))$. In an analogous way we define $f(\partial S)$ and $f(\operatorname{int} S)$. Note that $f_{-1}(\partial S) = 1$ and $f_{-1}(\operatorname{int} S) = 0$.

***Simplicial polytope:*** A polytope all of whose faces are simplices.

***(Geometric) simplicial complex:*** A polytopal complex all of whose faces are simplices.

***h-vector and g-vector:*** For a $d$-dimensional simplicial complex $S$ we define the $h$-vector $h(S) = (h_0(S), \dots, h_{d+1}(S))$ with generating function $h(S,x) = \sum_{i=0}^{d+1} h_i x^{d+1-i}$ as

$$\sum_{i=0}^{d+1} h_i x^{d+1-i} = \sum_{i=0}^{d+1} f_{i-1}(S)(x-1)^{d+1-i}.$$

We define the $g$-vector $g(S) = (g_0(S), \dots, g_{\lfloor (d+1)/2 \rfloor}(S))$ as

$$g_i(S) = h_i(S) - h_{i-1}(S), \quad 1 \le i \le \lfloor (d+1)/2 \rfloor.$$

Take $h_i(S) = 0$ if $i < 0$ or $i > d+1$, and $g_i(S) = 0$ if $i < 0$ or $i > \lfloor (d+1)/2 \rfloor$.

## 16.5.1 $h$-VECTORS and $g$-VECTORS

The $f$-vector and the $h$-vector of a simplicial complex carry the same information on $S$, since the definition of the $h$-vector can be inverted to give

$$\sum_{i=0}^{d+1} f_{i-1} x^{d+1-i} = \sum_{i=0}^{d+1} h_i(S)(x+1)^{d+1-i}.$$

But the $h$-vector more directly captures topological properties of $S$. For example:

$$(-1)^d h_{d+1} = -f_{-1} + f_0 - f_1 + \cdots + (-1)^{d-1} f_{d-1} + (-1)^d f_d = \chi(S) - 1,$$

where $\chi(S)$ is the ***Euler characteristic*** of $S$. In particular, $h_{d+1} = 0$ if $S$ is a ball and $h_{d+1} = 1$ if $S$ is a sphere, of whatever dimension.

For every triangulation $T$ of a point configuration the following hold:

1. The sum $\sum_{i=0}^{d+1} h_i(T)$ of the components of the $h$-vector equals $f_d(T)$.

2. $h_0(T) = f_{-1}(T) = 1$ and $h_i(T) \ge 0$ for all $i$ [Sta96].

3. The $h$-vector of $\partial T$ is symmetric; i.e., $h_i(\partial T) = h_{d-i}(\partial T)$, $0 \le i \le d$. These are the Dehn-Sommerville equations; see [MS71, Sta96, Zie95] and Chapter 17 of this Handbook. The case $h_d(\partial T) = h_0(\partial T) = 1$ is Euler's formula.

4. The $h$-vectors of $T$, $\partial T$, and $\operatorname{int} T$ are related in the following ways [MW71]:

$$h_i(T) - h_{d+1-i}(T) = h_i(\partial T) - h_{i-1}(\partial T) = g_i(\partial T),\ 0 \le i \le d+1.$$

$$h_i(T) = h_{d+1-i}(\operatorname{int} T),\ 0 \le i \le d+1.$$

   In particular, the $h$-vectors and the $f$-vectors of $\partial T$ and $\operatorname{int} T$ are completely determined by the $h$-vector (and hence the $f$-vector) of $T$.

5. Assume further that $T$ is shellable and that $P_1, \dots, P_m$ is a shelling order of the $d$-dimensional simplices in $T$. In particular, each $P_j$ meets $\bigcup_{i=1}^{j-1} P_i$ in some positive number $s_j$ of facets of $P_j$, $2 \le j \le m$. Define also $s_1 = 0$. Then $h_i(T)$ equals $\operatorname{card}\{j \mid s_j = i\}$, $0 \le i \le d+1$ [McM70, MS71, Sta96].

6. Assume further that $T$ is regular. Then, for every integer $0 \leq k \leq d+2$, the vector $(h_0(T)-h_{d+k+1}(T), h_1(T)-h_{d+k}(T), h_2(T)-h_{d+k-1}(T), \ldots, h_{\lfloor(d+k+1)/2\rfloor}(T)-h_{\lfloor(d+k+2)/2\rfloor}(T))$ is an $M$-sequence [BL81]. (See Chapter 17 of this Handbook for the definition of $M$-sequence.)

Properties (1) to (5) above hold for any *simplicial ball* (simplicial complex that is topologically a $d$-ball). Property (6) follows from the *g-theorem*, and it would hold for all simplicial balls if the *g-conjecture* holds for simplicial sphere (see Chapter 17 for details). In the other direction, Billera and Lee [BL81] conjectured the conditions in part (6) to be also *sufficient* for a vector to be the $h$-vector of a regular triangulation. In dimensions up to four the conditions indeed characterize $h$-vectors of balls [LS11, Kol11], but in dimensions five and higher Kolins [Kol11] has shown that some vectors satisfying property (6) are not the $h$-vectors of any ball, let alone regular triangulation.

TABLE 16.5.1 $h$- and $g$-vectors of polytopal complexes.

| $S$ | $h$-vector | $g$-vector |
|---|---|---|
| $\{\emptyset\}$ | $(1)$ | $(1)$ |
| Set of $n$ points | $(1, n-1)$ | $(1)$ |
| Line segment | $(1, 0, 0)$ | $(1, -1)$ |
| Boundary of convex $n$-gon | $(1, n-2, 1)$ | $(1, n-3)$ |
| Trivial subdivision of convex $n$-gon | $(1, n-3, 0, 0)$ | $(1, n-4)$ |
| Boundary of tetrahedron | $(1, 1, 1, 1)$ | $(1, 0)$ |
| Trivial subdivision of tetrahedron | $(1, 0, 0, 0, 0)$ | $(1, -1, 0)$ |
| Boundary of cube | $(1, 5, 5, 1)$ | $(1, 4)$ |
| Trivial subdivision of cube | $(1, 4, 0, 0, 0)$ | $(1, 3, -4)$ |
| Triangulation of cube into 6 tetrahedra (See Figure 16.7.3(a)) | $(1, 4, 1, 0, 0)$ | $(1, 3, -3)$ |
| Boundary of triangular prism | $(1, 3, 3, 1)$ | $(1, 2)$ |
| Trivial subdivision of triangular prism | $(1, 2, 0, 0, 0)$ | $(1, 1, -2)$ |
| Triangulation of triangular prism into 3 tetrahedra (See Figure 16.7.1) | $(1, 2, 0, 0, 0)$ | $(1, 1, -2)$ |

The definitions of $h$- and $g$-vectors can be extended to arbitrary polytopal complexes in the following recursive way:

1. $g_0(S) = h_0(S)$.
2. $g_i(S) = h_i(S) - h_{i-1}(S)$, $1 \leq i \leq \lfloor (d+1)/2 \rfloor$.
3. $g(\emptyset, x) = h(\emptyset, x) = 1$. (Here $\emptyset$ denotes the empty polytopal complex, not to be confused with $\{\emptyset\}$, the polytopal complex consisting of a the empty set.)
4. $h(S, x) = \sum_{G \text{ face of } S} g(\partial G, x)(x-1)^{d-\dim(G)}$.

This restricts to the previous definition since $g(\partial\Delta, x) = 1$ for every simplex $\Delta$.

For example, the $h$-vector of the trivial subdivision of a point set $V$ equals:

$$h_i(\{V\}) = \begin{cases} g_i(\partial(\{V\})), & 1 \leq i \leq \lfloor d/2 \rfloor, \\ 0, & \lfloor d/2 \rfloor < i \leq d. \end{cases}$$

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where $\partial(\{V\})$ denotes the complex of proper faces of $V$ [Bay93]. For any subdivision $S$ of $V$ one has: $h_i(S) \geq h_i(P)$ and $h_i(\partial S) \geq h_i(\partial P)$ for all $i$. In particular, $f_d(S) \geq h_{\lfloor d/2 \rfloor}(\partial S) \geq h_{\lfloor d/2 \rfloor}(\partial P)$ [Bay93, Sta92, Kar04].

Table 16.5.1 lists the $h$-vectors and $g$-vectors of some polytopal complexes.

## 16.5.2 STACKED AND EQUIDECOMPOSABLE POLYTOPES

### GLOSSARY

In the following definitions $V$ is the vertex set of a convex $d$-polytope $P$.

***Shallow triangulation:*** A triangulation $T$ of $V$ is called ***shallow*** if every face $F$ of $T$ is contained in a face of $P$ of dimension at most $2\dim(F)$.

***Weakly neighborly:*** A polytope $P$ is ***weakly neighborly*** if every set of $k+1$ vertices is contained in a face of dimension at most $2k$ for all $k \leq d/2$.

***Equidecomposable:*** If all triangulations of $V$ have the same $f$-vector, then $P$ is ***equidecomposable***.

***Stacked, k-stacked:*** If $P$ is simplicial and it has a triangulation in which there are no interior faces of dimension smaller than $d-k$, then $P$ is $k$***-stacked***. A 1-stacked polytope is simply called ***stacked***.

Shallow triangulations were introduced in order to understand the case of equality in the last result mentioned in Section 16.5.1: a shallow triangulation $T$ has $h(T) = h(P)$ and $h(\partial T) = h(\partial P)$. See [Bay93, BL93]. Stackedness and neighborliness are somehow opposite properties for a polytope: if $P$ is stacked then its $f$-vector is as small as can be (for a given dimension and number of vertices) and if it is neighborly (and simplicial, see Chapter 15) then it is as big as can be. Both properties have implications for triangulations of $P$, in particular for shallow ones.

A polytope $P$ is weakly neighborly if and only if all its triangulations are shallow [Bay93]. In this case $P$ is equidecomposable. Equidecomposability admits the following characterization via circuits.

**THEOREM 16.5.1** [DRS10, Section 8.5.3]

*The following are equivalent for a point configuration $V$:*

1. *All triangulations of $V$ have the same $f$-vector ($V$ is equidecomposable).*
2. *All triangulations of $V$ have the same number of $d$-simplices.*
3. *Every circuit $(C^+, C^-)$ of $V$ is balanced:* $\operatorname{card}(C^+) = \operatorname{card}(C^-)$.

The following are examples of weakly neighborly, hence equidecomposable, polytopes [Bay93]: Cartesian products of two simplices of any dimensions, Lawrence polytopes (see Section 16.4 for the definition), pyramids over weakly neighborly polytopes, and subpolytopes of weakly neighborly polytopes. The only simplicial weakly neighborly polytopes are simplices and even-dimensional neighborly polytopes (those for which every $d/2$ vertices form a face of the polytope; see Chapter 15). The only weakly neighborly 3-polytopes are pyramids and the triangular prism. Regular octahedra are equidecomposable, but not weakly neighborly.

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Assume now that $P$ is simplicial. In this case having a shallow triangulation is equivalent to being $\lfloor d/2 \rfloor$-stacked [Bay93]. McMullen [McM04] calls a triangulation of $P$ *small-face-free*, abbreviated *sff*, if it has no interior faces of dimension less than $d/2$. That is, $P$ has an sff triangulation if and only if it is $(\lceil d/2 \rceil - 1)$-stacked. Observe that shallow and sff are the same if $d$ is odd, but they differ by one unit in the dimension of the allowed interior faces if $d$ is even. The sff-triangulation of $P$ is unique, in case it exists [McM04]. Its existence and the minimum size of its interior feces are related to the $g$-vector of $\partial P$:

**THEOREM 16.5.2** Generalized lower bound theorem [MW71, Sta80, MN13]
*For any simplicial $d$-polytope $P$ and any $k \in \{2, \ldots, \lfloor d/2 \rfloor\}$ one has $g_k(\partial P) \geq 0$, with equality if and only if $P$ is $(k-1)$-stacked.*

The inequality was proved by Stanley [Sta80] and the 'if' part of the equality was already established in [MW71]. The 'only if' part was recently proved by Murai and Nevo. The case $k = 2$ is the lower bound theorem, proved by Barnette [Bar73].

## 16.6 TRIANGULATIONS OF LATTICE POLYTOPES

A ***lattice polytope*** or an *integral polytope* is a polytope with vertices in $\mathbb{Z}^d$ (or, more generally, in a lattice $\Lambda \subset \mathbb{R}^d$). Lattice polytopes and their triangulations have interest in algebraic geometry and in integer optimization. See [BR07, CLO11, Stu96, BG09, HPPS14].

### GLOSSARY

***Normalized volume:*** The ***normalized volume*** of a lattice polytope $P \subset \mathbb{R}^d$ is its Euclidean volume multiplied by $d!$. It is always an integer. All references to volume in this section are meant normalized.

***Empty simplex:*** A lattice simplex with no lattice points apart from its vertices.

***Unimodular simplex:*** A lattice simplex $\Delta \subset \mathbb{R}^d$ whose vertices are an affine lattice basis of $\mathbb{Z}^d \cap \operatorname{aff}(\Delta)$. If $\dim(\Delta) = d$ this is equivalent to $\operatorname{vol}(\Delta) = 1$.

***Unimodular triangulation:*** A triangulation into unimodular simplices.

***Flag triangulation:*** A triangulation (or a more general simplicial complex) in which all *minimal non-faces* (i.e., minimal subsets of vertices that are not faces) have at most size two. A flag triangulation is the clique complex of its 1-skeleton.

***Width:*** The width of a lattice polytope $P$ with respect to an integer linear functional $f : \mathbb{Z}^d \to \mathbb{Z}$ equals $\max_{p \in P} f(p) - \min_{p \in P} f(p)$. The width of $P$ itself is the minimum width taken over all non-zero integer linear functionals.

### 16.6.1 EMPTY SIMPLICES AND UNIMODULAR TRIANGULATIONS

Every lattice polytope $P$ can be triangulated into empty simplices, via any triangulation of $A := P \cap \mathbb{Z}^d$ that uses all points (e.g., placing them with respect to a suitable ordering). One central question on lattice polytopes is whether they have unimodular triangulations, and how to construct them.

In dimension two, every lattice polygon has unimodular triangulations since every empty triangle is unimodular (by Pick's Theorem, see [BR07]). The set of unimodular triangulations of a lattice polygon is known to be connected under bistellar flips, and the number of them is at most $2^{3i+b-3}$, where $i$ and $b$ are the numbers of lattice points in the interior and the boundary of $P$, respectively [Anc03].

In dimension three and higher there are empty non-unimodular simplices, which implies that not every polytope has unimodular triangulations. Empty 3-simplices are well understood, but higher dimensional ones are not:

1. Every 3-dimensional empty simplex is equivalent (modulo an affine integral automorphism $\mathbb{Z}^d \to \mathbb{Z}^d$) to the following $\Delta_{p,q}$, for some $0 \le p < q$ with $\gcd(p,q) = 1$ [Whi64]:

$$\Delta_{p,q} := \mathrm{conv}\,\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\}.$$

   In particular, they all have width one. Moreover, $\Delta_{p,q} \cong \Delta_{p',q'}$ if and only if $q' = q$ and $p' = \pm p^{\pm 1} \pmod{q}$. Observe that $\mathrm{vol}\,(\Delta_{p,q}) = q$.

2. All but finitely many empty 4-simplices have width one or two [BHHS16]. The exceptions have been computed up to volume 1000. The maximum volume among them is 179 and the maximum width is 4, achieved at a unique empty 4-simplex [HZ00]. It is conjectured that no larger exceptions exist.

3. The quotient group of $\mathbb{Z}^d$ by the sublattice generated by vertices of an empty $d$-simplex is cyclic if $d \le 4$ but not always so if $d \ge 5$ [BBBK11].

4. The maximum width of empty $d$-simplices lies between $2\lfloor d/2\rfloor - 1$ and $O(d \log d)$ [Seb99, BLPS99] (see also Chapter 7).

The most general result about existence of unimodular triangulations is:

**THEOREM 16.6.1** *(Knudson-Mumford-Waterman [Knu73])*
*For each lattice polytope $P$ there is an integer $k \in \mathbb{N}$ such that the dilation $kP$ has a unimodular triangulation.*

The original proof of this theorem does not lead to a bound on $k$. That the following $k$ is valid for lattice $d$-polytopes of volume $v$ was proved in [HPPS14]:

$$k = (d+1)!^{v!(d+1)^{(d+1)^2 v}}.$$

It is easy to show that the set $\{k \in \mathbb{N} \mid kP \text{ has some unimodular triangulation}\}$ is closed under taking multiples, for every $P$ [DRS10, Thm. 9.3.17]. But it is unknown whether it contains all sufficiently large choices of $k$. It is also unknown whether there is a global value $k_d$ that works for all polytopes of fixed dimension $d \ge 4$. These two open questions have a positive answer in dimension three, where every $k \in \mathbb{N}\setminus\{1,2,3,5\}$ works for every lattice 3-polytope [KS03, SZ13], or if the requirement is relaxed to $kP$ having unimodular ***covers*** (sets of unimodular simplices contained in $P$ and covering it), which is weaker than unimodular triangulations: there is a $k_d \in O(d^6)$ such that $kP$ has unimodular covers for every $k \ge k_d$ and every $d$-polytope $P$ [BG99, Theorem 3.23].

There is some literature on the existence of unimodular triangulations for particular lattice polytopes. See [HPPS14] for a recent survey. A notable open question is the following (see definition of smooth polytope in Section 16.6.2):

### QUESTION 16.6.2

*Does every smooth polytope have a (regular) unimodular triangulation?*

## 16.6.2 RELATION TO TORIC VARIETIES AND GRÖBNER BASES

Let $\mathbb{K}[x_1, \dots, x_n]$ be the polynomial ring over an algebraically closed field. For a nonnegative integer vector $u \in \mathbb{N}^n$ we denote as $x^u$ the monomial $\prod_i x_i^{u_i}$ and to each $u \in \mathbb{Z}^n$ we associate the binomial $x^{u_+} - x^{u_-}$, where $u = u_+ - u_-$ is the minimal decomposition of $u$ with $u_+, u_- \in \mathbb{N}^n$. (This minimal decomposition is the unique one in which $u_+$ and $u_-$ have disjoint supports.) General references for the topics in this section are [CLO11, Stu96]. See also Chapter 7 in this Handbook.

### GLOSSARY

Let $A \in \mathbb{Z}^{k \times n}$ be an integer matrix.

***Toric ideal:*** The toric ideal of $A$, denoted $I_A$, is the ideal generated by the binomials $\{x^{u_+} - x^{u_-} : Au = 0\}$. The variety cut out by $I_A$ is the (perhaps not normal) ***affine toric variety*** of $A$, denoted $X_A$.

***Smooth polytope:*** A lattice polytope $P$ is smooth if it is simple and the primitive normals to the facets at each vertex form a lattice basis. Equivalently, if each vertex $v$ of $P$ together with the first lattice point along each edge incident to $v$ forms a unimodular simplex.

***Normal polytope:*** $P$ is ***normal*** or ***integrally closed*** if every integer point in $k \cdot \mathrm{conv}\,(V)$ can be written as the sum of $k$ (perhaps repeated) points of $V$.

To emphasize the relations to triangulations of point sets, we assume that the columns of $A$ are $\{(a_1, 1), \dots, (a_n, 1)\}$ for a point set $V = \{a_1, \dots, a_n\} \in \mathbb{Z}^d$. Then all binomials defining $I_A$ are homogeneous, so besides the affine toric variety $X_A$ we have a projective variety $Y_A$. We also assume that $V$ generates $\mathbb{Z}^d$ as an affine lattice (it is not contained in a proper sublattice). In these conditions:

1. The normalizations $\widetilde{X}_A$ and $\widetilde{Y}_A$ of $X_A$ and $Y_A$ are the toric varieties associated in the standard way to $\mathbb{R}_{\geq 0}(A)^\vee$ and to the normal fan of $\mathrm{conv}\,(V)$ [Stu96, Cor. 13.6]. Here $\mathbb{R}_{\geq 0}(A)$ is the cone generated by the columns of $A$ and $\mathbb{R}_{\geq 0}(A)^\vee$ is its polar cone. $\widetilde{Y}_A$ is smooth if and only if $\mathrm{conv}\,(V)$ is smooth.

2. $X_A$ is normal if and only if the semigroup $\mathbb{Z}_{\geq 0}(A)$ is normal (that is, $\mathbb{R}_{\geq 0}A \cap \mathbb{Z}A = \mathbb{Z}_{\geq 0}A$). Equivalently, if $V = \mathrm{conv}\,(V) \cap \mathbb{Z}^d$ and $\mathrm{conv}\,(V)$ is normal. In this case $Y_A$ is called ***projectively normal***.

3. $Y_A$ is normal if the same happens for sufficiently large $k$ [Stu96, Thm. 13.11].

If $V$ has a unimodular triangulation then the condition in (2) holds. Hence, a positive answer to Question 16.6.2 is weaker than a positive answer to the following:

### QUESTION 16.6.3 *(Oda's question)*

*Is every smooth projective toric variety projectively normal? That is, is every smooth lattice polytope normal?*

Reduced Gröbner bases of $I_A$ are related to regular triangulations of $V$ as follows: Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ be a generic weight vector. We can use $\alpha$ to define a regular triangulation $T_\alpha$ of $V$, and also to define a monomial order in $K[x_1, \dots, x_n]$, which in turn defines a monomial initial ideal $\mathrm{in}_\alpha(I_A)$ (and a Gröbner basis) of $I_A$. Then:

**THEOREM 16.6.4** Sturmfels [Stu91], see also [Stu96, Chapter 8]

*For every subset $S \subset V$ we have that $S$ is* not *a face in $T_\alpha$ if and only if $S$ is the support of a monomial in $\mathrm{in}_\alpha(I_A)$. Said in a more algebraic language: the radical of $\mathrm{in}_\alpha(I_A)$ equals the Stanley-Reisner ideal of $T_\alpha$.*

*Moreover, if $T_\alpha$ is unimodular, then $\mathrm{in}_\alpha(I_A)$ is square-free (it equals its own radical). In particular, the maximum degree of a generator in $\mathrm{in}_\alpha(I_A)$ (and in the associated reduced Gröbner basis) equals the maximum size of a set $S$ that is not a face in $T_\alpha$ but such that every proper subset of $S$ is a face.*

For example, if $V$ has a regular, unimodular and flag triangulation, then $I_A$ has a Gröbner basis consisting of binomials of degree two. For this reason regular flag unimodular triangulations of lattice polytopes are called ***quadratic***.

Observe that this theorem induces a surjective map from the monomial initial ideals of $I_A$ to the regular triangulations of $V$. In case $\alpha$ is not generic then $T_\alpha$ may be a subdivision instead of a triangulation, and $\mathrm{in}_\alpha(I_A)$ may not be a monomial ideal, but the above map extends to this case. That is to say, *the Gröbner fan* of $I_A$ refines the secondary fan of $V$ [Stu91].

In [Stu96, Ch. 10], Sturmfels further extends the correspondence in Theorem 16.6.4 to a map from *all* the subdivisions of $V$ to the set of $A$-graded ideals, which generalize initial ideals of $I_A$ (see the definition in [Stu96]). This map is no longer surjective, but its image contains all regular subdivisions and all unimodular triangulations. The set of all $A$-graded ideals has a natural algebraic structure called the ***toric Hilbert scheme*** of $A$ and, using a notion of "flip" between radical monomial $A$-graded ideals, Maclagan and Thomas showed:

**THEOREM 16.6.5** Maclagan and Thomas [MT02]

*If two unimodular triangulations lie in different connected components of the graph of triangulations of $V$ then the corresponding monomial radical $A$-graded ideals lie in different connected components of the toric Hilbert scheme of $A$.*

Point configurations with unimodular triangulations not connected by a sequence of flips are known to exist [San05b]. Very recently Gaku Liu [Liu16b] has announced that this happens also for the Cartesian product $\Delta_4 \times \Delta_N$ of a 4-simplex and an $N$-simplex, which is a smooth polytope whose associated toric variety is the Cartesian product $\mathbb{P}^4 \times \mathbb{P}^N$ of two projective spaces. That is to say, the toric Hilbert scheme of $\mathbb{P}^4 \times \mathbb{P}^N$ is disconnected, for sufficiently large $N$.

Another relation between triangulations and toric varieties comes from looking at the affine toric variety $U_A$ associated to the cone $\mathbb{R}_{\geq 0}(A)$ (recall $X_A$ was associated to the dual cone $\mathbb{R}_{\geq 0}(A)^\vee$). This can be defined for every integer matrix $A$ for which $\mathbb{R}_{\geq 0}(A)$ is a *pointed cone*, but the case where the columns of $A$ are of the form $(a_i, 1)$ for a point configuration $V = \{a_1, \dots, a_n\}$ gives us the extra property that $U_A$ is *Gorenstein*. Since this construction depends only on $\mathrm{conv}\,(V)$ and not $V$ itself, we now assume $V$ to be the set of all lattice points in $\mathrm{conv}\,(V)$.

Then, every polyhedral subdivision $T$ of $V$, considered as a polyhedral fan covering $\mathbb{R}_{\geq 0}(A)$, induces an affine toric variety $U_T$ and a toric morphism $U_T \to U_A$. If $T$ is a triangulation then $U_T$ only has quotient singularities, which are *terminal* if $T$ uses all lattice points in $V$. If $T$ is a unimodular triangulation then $U_T$ is smooth; that is, it is a resolution of the singularity of $U_A$ at the origin. These resolutions are called *crepant* and they do not always exist (since not every lattice polytope has unimodular triangulations). See [DHZ98, DHZ01] for more details on this.

### 16.6.3 RELATION TO COUNTING LATTICE POINTS

For an integral polytope $P$ and a nonnegative integer $n$, let $i(P,n)$ be the number of integer points in $nP$. Equivalently, it is the number of points $x \in P$ for which $nx$ has integer coordinates. Ehrhart (1962) showed that $i(P,n)$ is a polynomial in $n$ of degree $d$. This implies that the generating function $J(P,t) = \sum_{n=0}^{\infty} i(P,n)t^n$ can be rewritten as a rational function with denominator $(1-t)^{d+1}$ and numerator of degree at most $d$. That is:

$$J(P,t) = \frac{\sum_{j=0}^{d} h_j^* t^j}{(1-t)^{d+1}},$$

for a certain rational vector $h^*(P) = (h_0^*, \ldots, h_d^*)$. $i(P,n)$ and $J(P,t)$ are called, respectively, the ***Ehrhart polynomial*** and ***Ehrhart series*** of $P$.

Stanley [Sta96] proved that $h^*(P)$ is a nonnegative integer vector and that:

**THEOREM 16.6.6** Stanley [Sta96]

*For every triangulation $T$ of $P$ one has $h(T) \leq h^*(P)$ coordinate-wise, with equality if and only if $T$ is unimodular. In particular, for a unimodular triangulation $T$:*

$$i(P,n) = \sum_{i=0}^{d} \binom{n-1}{i} f_i(T).$$

See [Sta96, BR07] and Chapter 7 of this Handbook for more details on Ehrhart polynomials. As an example, if $P$ is the standard unit 3-cube, then its unimodular triangulations have $h(T) = (1,4,1,0,0)$. Thus $J(P,t) = (1+4t+t^2)/(1-t)^4 = (1+4t+t^2)(1+4t+10t^2+20t^3+35t^4+\cdots) = 1+8t+27t^2+64t^3+125t^4+\cdots$, which agrees with $i(P,n) = (n+1)^3$. On the other hand, $i(P^o,n) = (n-1)^3 = -i(P,-n)$.

## 16.7 TRIANGULATIONS OF PARTICULAR POLYTOPES

### 16.7.1 PRODUCT OF TWO SIMPLICES

Consider the $(k+l)$-polytope $\Delta_k \times \Delta_l$, the product of a $k$-dimensional simplex $\Delta_k$ and an $l$-dimensional simplex $\Delta_l$. We look at triangulations of its vertex set $V_{k,l}$.

Triangulations of the product of two simplices have interest from several perspectives: They can be used as building blocks to triangulate more complicated polytopes [Hai91, OS03, San00]. In toric geometry they correspond to Gröbner bases of the toric ideal of the product of two projective spaces [Stu96]. Via the

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Cayley Trick, they correspond to mixed subdivisions of a dilated simplex [HRS00, San05a]. This connection also relates them to tropical geometry, where they correspond to tropical hyperplane arrangements, tropical convexity, and tropical oriented matroids [AD09, Hor16]. In optimization, their regular triangulations are closely related to dual transportation polytopes. See also [BCS88, DeL96, GKZ94] or [DRS10, Sect. 6.2].

If $\Delta_k$ and $\Delta_l$ are taken unimodular then $\Delta_k \times \Delta_l$ is ***totally unimodular***; that is, all maximal simplices with vertices in $V_{k,l}$ are unimodular. This implies that $\Delta_k \times \Delta_l$ is equidecomposable. In fact, for every triangulation $T$ of $\Delta_k \times \Delta_l$, we have $f_{k+l}(T) = (k+l)!/(k!l!)$, and $h_i(T) = \binom{k}{i}\binom{l}{i}$ for $0 \le i \le k+l$ (with $h_i(T)$ taken to be zero if $i > \min\{k,l\}$) [BCS88]. For small values of $k$ and/or $l$ the following is known. Most of it is proved via the Cayley Trick mentioned in Section 16.4:

1. All triangulations of $\Delta_k \times \Delta_1$ are affinely equivalent. Hence, they are all lexicographic. There are $k!$ of them and the secondary polytope is (affinely equivalent to) the $k$-dimensional ***permutahedron***, the convex hull of the points obtained permuting the coordinates of $(1, 2, \ldots, k+1)$. See Chapter 15.

2. All triangulations of $\Delta_3 \times \Delta_2$ and $\Delta_4 \times \Delta_2$ are regular. But all $\Delta_k \times \Delta_l$ with $\min\{k,l\} \ge 3$ or $k-3 \ge l = 2$, have nonregular triangulations [DeL96].

3. The number of triangulations of $\Delta_k \times \Delta_2$ grows as $2^{\Theta(k^2)}$ [San05a].

4. The flip-graphs of $\Delta_k \times \Delta_2$ and of $\Delta_k \times \Delta_3$ are connected [San05a, Liu16a], but that of $\Delta_k \times \Delta_4$ is not, for large $k$ [Liu16b].

The ***staircase triangulation*** of $\Delta_k \times \Delta_l$ is easy to describe explicitly [BCS88, GKZ94, San05a]: By ordering the vertices of $\Delta_k$ and (independently) of $\Delta_l$ we have a natural bijection between $V_{k,l} = \{(v_i, w_j) : i = 0, \ldots, k, j = 0, \ldots, l\}$ and the integer points in $[0,k] \times [0,l]$. Then, the vertices in each of the $\binom{k+l}{k}$ monotone paths from $(0,0)$ to $(k,l)$ form a full-dimensional simplex in $\Delta_k \times \Delta_l$, and these simplices form a triangulation. The same triangulation is obtained starting with the trivial subdivision of $P$ and pulling the vertices in any order compatible with the product order. That is, if $i \le i'$ and $j \le j'$, then $(v_i, w_j)$ is pulled before $(v_{i'}, w_{j'})$. Figure 16.7.1 shows the staircase triangulation for $\Delta_2 \times \Delta_1$, a triangular prism. Its three tetrahedra are $\{00, 10, 20, 21\}$, $\{00, 10, 11, 21\}$ and $\{00, 01, 11, 21\}$, where $ij$ is an abbreviation for $(v_i, w_j)$.

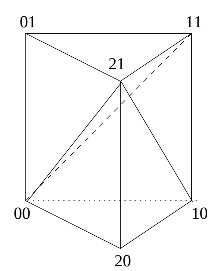


FIGURE 16.7.1
*A triangulation of* $\Delta_2 \times \Delta_1$.

The staircase triangulation generalizes to a product $\Delta_{l_1} \times \cdots \times \Delta_{l_n}$ of $n$ simplices: consider the natural bijection between vertices of $\Delta_{l_1} \times \cdots \times \Delta_{l_n}$ and integer points in $[0, l_1] \times \cdots \times [0, l_n]$ and take as simplices the monotone paths from $(0, \ldots, 0)$

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to $(l_1, \ldots, l_n)$. Staircase triangulations are the natural way to refine a Cartesian product of simplicial complexes to become a triangulation, by triangulating each individual product of simplices [ES52].

## 16.7.2 $d$-CUBES

The unit $d$-dimensional ***cube*** $I^d$ is the $d$-fold product of the unit interval $I = [0, 1]$ with itself. Here we consider triangulations of it using only its set $V$ of vertices. Up to $d = 4$ they have been completely enumerated: The 3-cube has precisely 74 triangulations, all regular, falling into 6 classes modulo affine symmetries of the cube. Figure 16.7.2 shows the unique (modulo symmetry) triangulation of size 5. The 4-cube has 92 487 256 triangulations in total (247 451 symmetry classes) [PR03, Pou13] of which 87 959 448 are regular (235 277 symmetry classes) [HSYY08]. Nonregular triangulations of it were first described in [DeL96]. Still, its graph of triangulations is connected [Pou13].

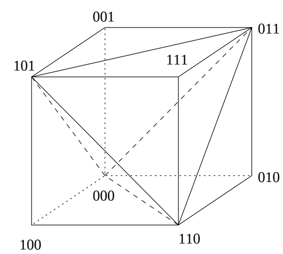


FIGURE 16.7.2
*A minimum size triangulation of the 3-cube.*

The maximum size of a triangulation of $I^d$ is $d!$, achieved by unimodular triangulations. These include all pulling triangulations. Every unimodular triangulation $S$ of $I^d$ has $h_d(T) = h_{d+1}(T) = 0$ and $h_i(T) = A(d, i)$, $0 \leq i \leq d-1$, where $A(d, i)$ is the Eulerian number (the number of permutations of $\{1, \ldots, d\}$ having exactly $i$ descents). Finding small triangulations of the $d$-cube is interesting in finite element methods [Tod76]. The minimum possible size is called the ***simplexity*** of the $d$-cube, which we denote $\varphi(d)$. The following summarizes what we know about it:

- Exact values are known up to dimension 7 [Hug93, HA96]. See Table 16.7.1.

TABLE 16.7.1 Minimal triangulations of $d$-cubes.

| $d$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\varphi(d)$ | 1 | 2 | 5 | 16 | 67 | 308 | 1493 |

- Up to $d = 5$ a minimum triangulation can be obtained by Sallee's ***corner slicing*** idea [Hai91, Sal82]: if the $2^{d-1}$ vertices of $I^d$ with an odd number of nonzero coordinates are sliced and the rest of $I^d$ is triangulated arbitrarily, a triangulation $T$ of $I^d$ arises in which all cells except the first $2^{d-1}$ have volume

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at least $2/d!$. Hence, $T$ has size at most $(d! + 2^{d-1})/2$. This equals $\varphi(d)$ up to $d = 4$ and is off by one for $d = 5$ (where a corner-slicing triangulation of size $\varphi(5) = 67$ still exists). For $d = 6, 7$ the minimum corner-slicing triangulations have sizes 324 and 1820 [Hug93, HA96], much greater than $\varphi(d)$.

- The Hadamard bound for matrices implies that no simplex contained in the cube has volume greater than $(d+1)^{(d+1)/2}/(2^d d!)$ [Hai91]. Hence,

$$\varphi(d) \geq 2^d d!(d+1)^{-(d+1)/2}.$$

A better bound of

$$\varphi(d) \geq \frac{1}{2}\sqrt{6}^d d!(d+1)^{-(d+1)/2}$$

is obtained with the same argument with respect to hyperbolic volume [Smi00].

- For a triangulation $S$ of size $|S|$, let

$$\rho(S) := (|S|/d!)^{1/d}.$$

This parameter is called the ***efficiency*** of $S$ [Tod76]. It is at most one for every $S$, with equality if and only if $S$ is unimodular. If $I^d$ has a triangulation $S$ of a certain efficiency then any triangulation of $I^{kd}$ obtained by pulling refinement of the $k$-fold Cartesian product of $S$ with itself has exactly the same efficiency [Hai91]. This shows that $\lim_{d\to\infty}(\varphi(d)/d!)^{1/d}$ exists, and that it is less or equal than the efficiency of any triangulation of any $I^d$. The best known upper bound for this limit is [OS03]

$$\lim_{d\to\infty}(\varphi(d)/d!)^{1/d} \leq 0.816,$$

but no strictly positive lower bound is known. Observe, for example, that the Hadamard bound only says $(\varphi(d)/d!)^{1/d} \gtrsim \frac{2}{\sqrt{d+1}}$. The improvement in [Smi00] merely changes the constant 2 in the numerator to a $\sqrt{6}$.

## SOME SPECIFIC TRIANGULATIONS OF $I^d$

***Standard*** or ***staircase*** triangulation: Consider the $d!$ monotone paths from $(0,\ldots,0)$ to $(1,\ldots,1)$ obtained by changing one coordinate from 0 to 1 at a time. The vertices in each such path form a unimodular simplex, that we call the monotone-path simplex corresponding to that permutation. The $d!$ simplices obtained in this way form a triangulation of $I^d$, which is nothing but the staircase triangulation of $I^d$ regarded as the product of $d$ segments. It is also known as Kuhn's triangulation [Tod76] and it admits the following alternative descriptions:

- It is the subdivision obtained slicing $I^d$ by all hyperplanes of the form $x_i = x_j$.
- It is the pulling triangulation for any ordering of vertices with the following property: for every face $F$ of $I^d$, the first vertex of $F$ to be pulled is either the vertex with minimum or maximum support.
- It is the regular triangulation for the height function $f(v) = -(\sum v_i)^2$.
- It is the flag triangulation containing the edge $uv$ for two vertices $u$ and $v$ if and only if $u - v$ is nonnegative (or nonpositive).

- It is a special case of the triangulation of an order polytope by linear extensions: the order polytope $P(\mathcal{O})$ of a poset $\mathcal{O}$ on $d$ elements $\{a_1, \ldots, a_d\}$ is the subpolytope of $I^d$ cut by the inequalities $x_i \leq x_j$ for every $a_i < a_j$. (This is the whole of $I^d$ when $\mathcal{O}$ is an antichain.) Linear extensions of $\mathcal{O}$ are in bijection to permutations whose associated monotone-path simplex are contained in $P(\mathcal{O})$, and these simplices triangulate $P(\mathcal{O})$.

***Alcoved*** triangulation: If $I^d$ is sliced by all hyperplanes of the form $x_i+\cdots+x_j = m$ for $1 \leq i < j \leq d$ and $m \in \mathbb{N}$ another regular, unimodular triangulation of $I^d$ is obtained. It was first described by Stanley, who showed a piecewise linear map from $I^d$ to itself sending it to the standard triangulation. It was then studied in detail in [LP07]. It is the flag triangulation whose edges are the pairs $uv$ such that the nonzero coordinates in $u - v$ alternate between $+1$ and $-1$.

***Sallee's middle cut triangulation:*** Assume $d \geq 2$. Slice the cube into two polytopes by the hyperplane $x_1 + \cdots + x_d = \lfloor d/2 \rfloor$. Refine this subdivision to a triangulation by pulling the vertices in the following order: pull $(v_1, \ldots, v_d)$ in step $1 + \sum_{i=0}^{d-1} v_{i+1} 2^i$. This triangulation has size $O(d!/d^2)$ [Sal84].

## EXAMPLES

Figure 16.7.3 shows two triangulations of the 3-cube: (a) the one resulting from pulling the vertices in order of increasing distance to the origin, which equals the standard triangulation. And (b), one resulting from pushing the following vertices in order: 000, 100, 101, 001.

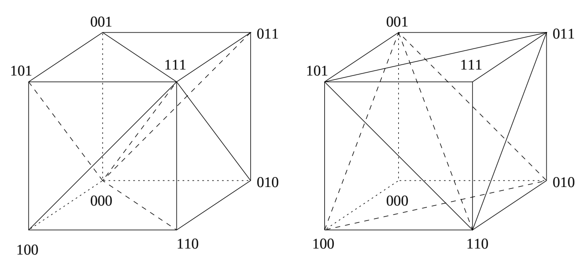


FIGURE 16.7.3
(a) *A pulling triangulation of the* 3*-cube.*
(b) *A pushing triangulation of the* 3*-cube.*

## 16.7.3 CONVEX $n$-GONS

Let $V_n$ be the set of vertices of a convex $n$-gon. A subdivision of $V_n$ is determined by a collection of mutually noncrossing internal diagonals, and vice versa. That is, the Baues poset of $V_n$ equals the poset of non-crossing sets of diagonals of the $n$-gon (with respect to *reverse inclusion*). All subdivisions are regular (in fact, they are all pushing), so this is also the face poset of the secondary polytope of $V_n$, the ***associahedron*** of dimension $n - 3$. See [Lee89, GKZ94, Zie95, DRS10].

The number of triangulations of the $n$-gon is the Catalan number

$$C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2} \in \Theta\left(\frac{4^n}{n^{3/2}}\right).$$

This counts many other combinatorial structures, including the ways to parenthesize a string of $n-1$ symbols, Dyck paths from $(0,0)$ to $(n-2, n-2)$, and rooted binary trees with $n-2$ nodes. More generally, there are $\frac{1}{n-1}\binom{n-3}{j}\binom{n+j-1}{j+1}$ subdivisions of a convex $n$-gon having exactly $j$ diagonals, $0 \le j \le n-3$. This equals the number of $(n-3-j)$-faces of the associahedron.

Two triangulations are adjacent if they share all but one diagonal. In particular, the graph of triangulations is $(n-3)$-regular. Its diameter equals $2n-10$ for all $n > 12$ [STT88, Pou14]. The flip-distance between two triangulations equals the *rotation distance* between the corresponding binary trees [STT88].

The associahedron is an ubiquitous polytope. It was first described by Tamari (1951) and arose in works of Stasheff and Milnor in the 1960's. The first explicit constructions of it as a polytope (and not only as a cell complex) are by Haiman (1984, unpublished) and Lee [Lee89]. Besides being a secondary polytope, it is a *generalized permutahedron*, and dual to a *cluster complex* of type $A$. The latter means that it can be realized with facet normals equal to the nonnegative roots of type $A$ in a way that captures the combinatorics of the corresponding cluster algebra. See [CSZ15] and the references therein for more details.

## 16.7.4 CYCLIC POLYTOPES

Cyclic polytopes are neighborly polytopes with very nice combinatorial properties. Being neighborly means their $f$-vectors and $h$-vectors are as big as can be, which reflects in their triangulations having number, size and flip-graph diameter also (asymptotically) as big as can be.

## GLOSSARY

***Cyclic polytope:*** The standard ***cyclic polytope*** $C_d(n)$ of dimension $d$ with $n$ vertices is the convex hull of the points $c(1), \dots, c(n)$ on the $d$-dimensional ***moment curve***, the parametrized curve $c : \mathbb{R} \to \mathbb{R}^d$ defined as

$$c(t) = (t, t^2, \dots, t^d).$$

The convex hull of any $n$ points in the curve has the same combinatorics (both as a polytope and as an oriented matroid) as $C_d(n)$. Depending on the context, one calls cyclic polytope the convex hull of any $n$ points in this curve, any polytope with the same oriented matroid, or any polytope with the same face lattice.

Cyclic polytopes are simplicial and are the prime examples of ***neighborly polytopes***: $d$-polytopes in which every $\lfloor d/2 \rfloor$ vertices form a face. For even $d$ this implies they are weakly neighborly in the sense of Section 16.5 (hence equidecomposable).

Consider the natural projection $\pi : C_{d+1}(n) \to C_d(n)$ that forgets the last coordinate. Every triangulation $T$ of $C_d(n)$ is a section of this projection, and every flip to another triangulation $T'$ gives a section that is pointwise above or below $T$. This allows us to speak of upward and downward flips, and to give a structure of poset to the set of triangulations of $C_d(n)$: $T$ is below $T'$ in the poset if there is a sequence of upward flips from $T$ to $T'$. This poset is called the ***first Stasheff-Tamari order***, denoted $T <_1 T'$. The ***second Stasheff-Tamari order*** is given by $T <_2 T'$ if the section produced by $T$ is pointwise below that of $T'$. Clearly, $T <_1 T'$ implies $T <_2 T'$, but the converse is not known in general.

The first Stasheff-Tamari order has a unique minimum and a unique maximum triangulation, which we denote $T_{\hat{0}}$ and $T_{\hat{1}}$: The lower and upper envelope of $C_{d+1}(n)$, which coincide with the pushing and pulling triangulations of $C_d(n)$ with respect to the natural ordering on the vertices. Other properties of this poset are as follows (see [Ram97] or [DRS10, Ch. 6]):

- Every triangulation of $C_d(n)$ lies in some monotone path from the $T_{\hat{0}}$ to $T_{\hat{1}}$.
- Every maximal chain of flips from $T_{\hat{0}}$ to $T_{\hat{1}}$ corresponds to a triangulation of $C_{d+1}(n)$, so that the length of the chain equals the size of the triangulation.
- In odd dimension upward flips decrease size by exactly one, so that $T_{\hat{0}}$ and $T_{\hat{1}}$ are maximum and minimum in size, respectively.

These properties imply the following [DRS10, Corollary 6.1.20]:

1. The minimum and maximum sizes of triangulations of $C_d(n)$ equal the numbers of upper and lower facets of $C_{d+1}(n)$, which are:
$$\binom{n - \lfloor (d+1)/2 \rfloor - 1}{\lceil (d+1)/2 \rceil - 1} \quad \text{and} \quad \binom{n - \lceil (d+1)/2 \rceil}{\lfloor (d+1)/2 \rfloor}.$$
2. If $d$ is odd, the diameter of the flip graph of $C_d(n)$ equals $\binom{n-(d+1)/2-1}{(d+1)/2}$ (the size of every triangulation of $C_{d+1}(n)$). Indeed, this is the length of every monotone chain from $T_{\hat{0}}$ to $T_{\hat{1}}$, and for every two triangulations $T$ and $T'$ there is a cycle in the flip-graph going through them and of twice that length.
3. If $d$ is even, every triangulation is at flip-distance at most $\binom{n-d/2-2}{d/2}$ from $T_{\hat{1}}$, with equality for $T_{\hat{0}}$. Hence, the graph of flips of $C_d(n)$ has diameter between $\binom{n-d/2-2}{d/2}$ and $2\binom{n-d/2-2}{d/2}$.

Exact formulas exist for the number of triangulations of $C_{n-4}(n)$ (the first non-trivial case of few more vertices than the dimension). For even $n$, $C_{n-4}(n)$ has exactly $(n+4)2^{(n-4)/2}-n$ triangulations. Of these, at most $6\binom{n/2}{4}+3\binom{n/2}{3}+4\binom{n/2}{2}-n/2+2$ are regular, and this number is exact for sufficiently generic coordinatizations of the oriented matroid of $C_d(n)$ [AS02]. Similar but different formulas exist when $n$ is odd. For arbitrary number of vertices the following is proved in [DRS10]:

**THEOREM 16.7.1** [DRS10, Thm. 6.1.22]

*The cyclic polytope $C_d(n)$ has at least $\Omega\left(2^{n^{\lfloor d/2 \rfloor}}\right)$ triangulations, for $d$ fixed.*

# 16.8 SOURCES AND RELATED MATERIAL

## FURTHER READING

Chapter 29 discusses triangulations of more general (e.g., nonconvex) objects. Chapter 27 provides details on Delaunay triangulations and Voronoi diagrams. We refer also to Chapter 15, on basic properties of convex polytopes. For the topics of Sections 16.5 and 16.6; see also Chapters 17 and 7, respectively.

A recent book covering the topics in this chapter is [DRS10]. A section on triangulations and subdivisions of convex polytopes can be found in the survey article [BL93]. The book [Zie95] and the article [Lee91] contain information on regular subdivisions and triangulations; for their important role in generalized discriminants and determinants see the book [GKZ94], and for their significance in computational algebra see the book [Stu96]. Additional references can be found in the above-mentioned sources, as well as the citations given in this chapter.

## RELATED CHAPTERS

Chapter 3: Tilings
Chapter 6: Oriented matroids
Chapter 7: Lattice points and lattice polytopes
Chapter 15: Basic properties of convex polytopes
Chapter 17: Face numbers of polytopes and complexes
Chapter 20: Polyhedral maps
Chapter 26: Convex hull computations
Chapter 27: Voronoi diagrams and Delaunay triangulations
Chapter 29: Triangulations and mesh generation
Chapter 36: Computational convexity

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