15 BASIC PROPERTIES OF CONVEX POLYTOPES

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INTRODUCTION

Convex polytopes are fundamental geometric objects that have been investigated since antiquity. The beauty of their theory is nowadays complemented by their importance for many other mathematical subjects, ranging from integration theory, algebraic topology, and algebraic geometry to linear and combinatorial optimization.

In this chapter we try to give a short introduction, provide a sketch of "what polytopes look like" and "how they behave," with many explicit examples, and briefly state some main results (where further details are given in subsequent chapters of this Handbook). We concentrate on two main topics:

- Combinatorial properties: faces (vertices, edges, ..., facets) of polytopes and their relations, with special treatments of the classes of low-dimensional polytopes and of polytopes "with few vertices;"
- Geometric properties: volume and surface area, mixed volumes, and quermassintegrals, including explicit formulas for the cases of the regular simplices, cubes, and cross-polytopes.

We refer to Grünbaum [Grü67] for a comprehensive view of polytope theory, and to Ziegler [Zie95] respectively to Gruber [Gru07] and Schneider [Sch14] for detailed treatments of the combinatorial and of the convex geometric aspects of polytope theory.

15.1 COMBINATORIAL STRUCTURE

GLOSSARY

$\mathcal{V}$-polytope: The convex hull of a finite set $X = \{x^1, \ldots, x^n\}$ of points in $\mathbb{R}^d$,

$$P = \mathrm{conv}(X) \ := \ \Big\{ \sum_{i=1}^{n} \lambda_i x^i \ \Big|\ \lambda_1, \ldots, \lambda_n \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \Big\}.$$

$\mathcal{H}$-polytope: The solution set of a finite system of linear inequalities,

$$P = P(A, b) \ := \ \big\{ x \in \mathbb{R}^d \mid a_i^T x \leq b_i \text{ for } 1 \leq i \leq m \big\},$$

with the extra condition that the set of solutions is bounded, that is, such that there is a constant N such that $||x|| \leq N$ holds for all $x \in P$. Here $A \in \mathbb{R}^{m \times d}$ is a real matrix with rows a_i^T, and $b \in \mathbb{R}^m$ is a real vector with entries b_i.

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Polytope: A subset P of some $\mathbb{R}^d$ that can be presented as a $\mathcal{V}$-polytope or (equivalently, by the main theorem below) as an $\mathcal{H}$-polytope.

Affine hull $\mathrm{aff}(S)$ ***of a set*** S***:*** The inclusion-minimal affine subspace of $\mathbb{R}^d$ that contains S, which is given by $\{ \sum_{j=1}^p \lambda_j x^j \mid p > 0,\, x^1, \dots, x^p \in S, \lambda_1, \dots, \lambda_p \in \mathbb{R}, \sum_{j=1}^p \lambda_j = 1 \}$.

Dimension: The dimension of an arbitrary subset S of $\mathbb{R}^d$ is defined as the dimension of its affine hull: $\dim(S) := \dim(\mathrm{aff}(S))$.

d-polytope: A d-dimensional polytope. The prefix "d-" denotes "d-dimensional." A subscript in the name of a polytope usually denotes its dimension. Thus "d-cube C_d" will refer to a d-dimensional incarnation of the cube.

Interior and ***relative interior:*** The interior $\mathrm{int}(P)$ is the set of all points $x \in P$ such that for some $\varepsilon > 0$, the ε-ball $B_\varepsilon(x)$ around x is contained in P.

Similarly, the relative interior $\mathrm{relint}(P)$ is the set of all points $x \in P$ such that for some $\varepsilon > 0$, the intersection $B_\varepsilon(x) \cap \mathrm{aff}(P)$ is contained in P.

Affine equivalence: For polytopes $P \subseteq \mathbb{R}^d$ and $Q \subseteq \mathbb{R}^e$, the existence of an affine map $\pi : \mathbb{R}^d \longrightarrow \mathbb{R}^e$, $x \longmapsto Ax + b$ that maps P bijectively to Q. The affine map π does not need to be injective or surjective. However, it has to restrict to a bijective map $\mathrm{aff}(P) \longrightarrow \mathrm{aff}(Q)$. In particular, if P and Q are affinely equivalent, then they have the same dimension.

THEOREM 15.1.1 *Main Theorem of Polytope Theory* (cf. [Zie95, Sect. 1.1])
The definitions of $\mathcal{V}$-polytopes and of $\mathcal{H}$-polytopes are equivalent. That is, every $\mathcal{V}$-polytope has a description by a finite system of inequalities, and every $\mathcal{H}$-polytope can be obtained as the convex hull of a finite set of points (its vertices*).*

Any $\mathcal{V}$-polytope can be viewed as the image of an $(n-1)$-dimensional simplex under an affine map $\pi : x \mapsto Ax + b$, while any $\mathcal{H}$-polytope is affinely equivalent to an intersection $\mathbb{R}^m_{\geq 0} \cap L$ of the positive orthant in m-space with an affine subspace [Zie95, Lecture 1]. To see the Main Theorem at work, consider the following two statements. The first one is easy to see for $\mathcal{V}$-polytopes, but not for $\mathcal{H}$-polytopes, and for the second statement we have the opposite effect:

1. *Projections:* Every image of a polytope P under an affine map is a polytope.
2. *Intersections:* Every intersection of a polytope with an affine subspace is a polytope.

However, the computational step from one of the main theorem's descriptions of polytopes to the other—a "convex hull computation"—is often far from trivial. Essentially, there are three types of algorithms available: inductive algorithms (inserting vertices, using a so-called beneath-beyond technique), projection algorithms (known as Fourier–Motzkin elimination or double description algorithms), and reverse search methods (as introduced by Avis and Fukuda [AF92]). For explicit computations one can use public domain codes as the software package `polymake` [GJ00] that we use here, or `sage` [SJ05]; see also Chapters 26 and 67.

In each of the following definitions of d-simplices, d-cubes, and d-cross-polytopes we give both a $\mathcal{V}$- and an $\mathcal{H}$-presentation. From this one can see that the $\mathcal{H}$-presentation can have exponential size (number of inequalities) in terms of the size (number of vertices) of the $\mathcal{V}$-presentation (e.g., for the d-cross-polytopes), and vice versa (e.g., for the d-cubes).

Definition: A (regular) d-dimensional ***simplex*** (or ***d-simplex***) in $\mathbb{R}^d$ is given by

$$\begin{aligned} T_d &:= \operatorname{conv}\big\{e^1, e^2, \dots, e^d, \frac{1-\sqrt{d+1}}{d}(e^1+\dots+e^d)\big\} \\ &= \big\{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i \le 1,\ -(1+\sqrt{d+1}+d)x_k + \sum_{i=1}^d x_i \le 1 \text{ for } 1 \le k \le d\big\}, \end{aligned}$$

where $e^1, \dots, e^d$ denote the coordinate unit vectors in $\mathbb{R}^d$.

The simplices T_d are ***regular polytopes*** (with a symmetry group that is flag-transitive—see Chapter 18): The parameters have been chosen so that all edges of T_d have length $\sqrt{2}$. Furthermore, the origin $0 \in \mathbb{R}^d$ is in the interior of T_d: This is clear from the $\mathcal{H}$-presentation.

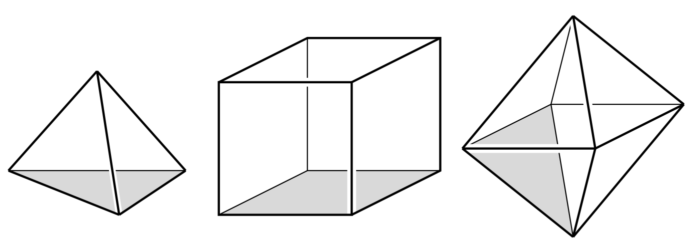
For the combinatorial theory one considers polytopes that differ only by an affine change of coordinates or—more generally—a projective transformation to be equivalent. Combinatorial equivalence is, however, still stronger than projective equivalence. In particular, we refer to any d-polytope that can be presented as the convex hull of $d{+}1$ affinely independent points as a ***d-simplex***, since any two such polytopes are equivalent with respect to an affine map. Other standard choices include

$$\begin{aligned} \Delta_d &:= \operatorname{conv}\{0, e^1, e^2, \dots, e^d\} \\ &= \big\{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i \le 1, \quad x_k \ge 0 \text{ for } 1 \le k \le d\big\} \end{aligned}$$

and the $(d{-}1)$-dimensional simplex in $\mathbb{R}^d$ given by

$$\begin{aligned} \Delta'_{d-1} &:= \operatorname{conv}\{e^1, e^2, \dots, e^d\} \\ &= \big\{x \in \mathbb{R}^d \mid \sum_{i=1}^d x_i = 1, \quad x_k \ge 0 \text{ for } 1 \le k \le d\big\}. \end{aligned}$$

FIGURE 15.1.1
A 3-simplex, a 3-cube, and a 3-cross-polytope (octahedron).



Definition: A ***d-cube*** (a.k.a. the d-dimensional ***hypercube***) is

$$\begin{aligned} C_d &:= \operatorname{conv}\big\{\alpha_1 e^1 + \alpha_2 e^2 + \dots + \alpha_d e^d \mid \alpha_1, \dots, \alpha_d \in \{+1, -1\}\big\} \\ &= \big\{x \in \mathbb{R}^d \mid -1 \le x_k \le 1 \text{ for } 1 \le k \le d\big\}. \end{aligned}$$

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Again, there are other natural choices, among them the d-dimensional ***unit cube***

$$[0,1]^d \quad = \quad \mathrm{conv}\Big\{ \sum_{i \in S} e^i \;\Big|\; S \subseteq \{1,2,\dots,d\} \Big\}$$
$$= \quad \big\{ x \in \mathbb{R}^d \;\big|\; 0 \le x_k \le 1 \text{ for } 1 \le k \le d \big\}.$$

A d-***cross-polytope*** in $\mathbb{R}^d$ (for $d = 3$ known as the ***octahedron***) is given by

$$C_d^{\Delta} \quad := \quad \mathrm{conv}\{\pm e^1, \pm e^2, \dots, \pm e^d\}$$
$$= \quad \Big\{ x \in \mathbb{R}^d \;\Big|\; \sum_{i=1}^{d} \alpha_i x_i \le 1 \text{ for all } \alpha_1,\dots,\alpha_d \in \{-1,+1\} \Big\}.$$

To illustrate concepts and results we will repeatedly use the unnamed polytope with six vertices shown in Figure 15.1.2.

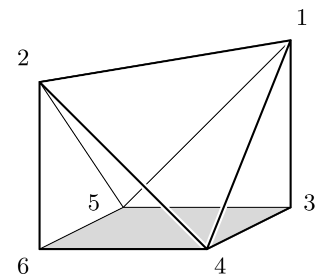

FIGURE 15.1.2
Our unnamed "typical" 3-polytope. It has 6 vertices, 11 edges and 7 facets.

This polytope without a name can be presented as a $\mathcal{V}$-polytope by listing its six vertices. The following coordinates make it into a ***subpolytope*** of the 3-cube C_3: The vertex set consists of all but two vertices of C_3. Our list below (on the left) shows the vertices of our unnamed polytope in a format used as input for the `polymake` program, i.e., the vertices are given in homogeneous coordinates with an additional 1 as first entry. From these data the `polymake` program produces a description (on the right) of the polytope as an $\mathcal{H}$-polytope, i.e., it computes the facet-defining hyperplanes with respect to homogeneous coordinates. For instance, the entries in the last row of the section FACETS describe the halfspace $1\,x_0 - 1\,x_1 + 1\,x_2 - 1\,x_3 \ge 0$, which with $x_0 \equiv 1$ corresponds to the facet-defining inequality $x_1 - x_2 + x_3 \le 1$ of our 3-dimensional unnamed polytope.

```
POINTS                                  FACETS
1  1  1  1                              1  0 -1  0
1 -1 -1  1                              1 -1  0  0
1  1  1 -1                              1  1  0  0
1  1 -1 -1                              1  0  1  0
1 -1  1 -1                              1  0  0  1
1 -1 -1 -1                              1  1 -1 -1
                                        1 -1  1 -1
```

Any unbounded pointed polyhedron (that is, the set of solutions of a system of inequalities, which is not bounded but does have a vertex) is, via a projective transformation, equivalent to a polytope with a distinguished facet; see [Zie95, Sect. 2.9 and p. 75]. In this respect, we do not lose anything on the combinatorial level if we restrict the following discussion to the setting of *full-dimensional* convex polytopes, that is, d-polytopes embedded in $\mathbb{R}^d$.

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15.1.1 FACES

GLOSSARY

Support function: Given a polytope $P \subseteq \mathbb{R}^d$, the function

$$h(P,\cdot)\colon \mathbb{R}^d \to \mathbb{R}, \qquad h(P,x) := \sup\{\langle x,y\rangle \mid y \in P\},$$

where $\langle x,y\rangle$ denotes a fixed inner product on $\mathbb{R}^d$. Since P is compact one may replace sup by max.

Supporting hyperplane of P: A hyperplane

$$H(P,v) := \{x \in \mathbb{R}^d \mid \langle x,v\rangle = h(P,v)\},$$

for $v \in \mathbb{R}^d \setminus \{0\}$. Note that $H(P,\mu v) = H(P,v)$ for $\mu \in \mathbb{R}$, $\mu > 0$. For a vector u of the $(d-1)$-dimensional ***unit sphere*** S^{d-1}, $h(P,u)$ is the signed distance of the supporting plane $H(P,u)$ from the origin. For $v = 0$ we set $H(P,0) := \mathbb{R}^d$, which is not a hyperplane.

Face: The intersection of P with a supporting hyperplane $H(P,v)$. If P is full-dimensional, then this is a nontrivial face of P. We call it a ***k-face*** if the dimension of $\operatorname{aff}(P \cap H(P,v))$ is k. Each face is itself a polytope.

The set of all k-faces is denoted by $\mathcal{F}_k(P)$ and its cardinality by $f_k(P)$.

The empty set $\emptyset$ and the polytope P itself are also defined to be faces of P, called the ***trivial faces*** of P, of dimensions -1 and $\dim(P)$, respectively. Thus the ***nontrivial faces*** F of a d-polytope have dimensions $0 \le \dim(F) \le d-1$. All faces other than P are referred to as ***proper faces***.

The faces of dimension 0 and 1 are called ***vertices*** and ***edges***, respectively. The $(d-1)$-faces and $(d-2)$-faces of a d-polytope P are called ***facets*** and ***ridges***, respectively.

f-vector: The vector of face numbers $\boldsymbol{f}(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P))$ associated with a d-polytope.

Facet-vertex incidence matrix: The matrix $M \in \{0,1\}^{f_{d-1}(P)\times f_0(P)}$ that has an entry $M(F,v) = 1$ if the facet F contains the vertex v, and $M(F,v) = 0$ otherwise.

Graded poset: A partially ordered set $(P,\le)$ with a unique minimal element $\hat{0}$, a unique maximal element $\hat{1}$, and a ***rank function*** $r : P \longrightarrow \mathbb{N}_0$ that satisfies

(1) $r(\hat{0}) = 0$, and $p < p'$ implies $r(p) < r(p')$, and
(2) $p < p'$ and $r(p') - r(p) > 1$ implies that there is a $p'' \in P$ with $p < p'' < p'$.

Lattice L: A partially ordered set $(P,\le)$ in which every pair of elements $p,p' \in P$ has a unique maximal lower bound, called the ***meet*** $p \wedge p'$, and a unique minimal upper bound, called the ***join*** $p \vee p'$.

Atom, coatom: If L is a graded lattice, the minimal elements of $L \setminus \{\hat{0}\}$ (i.e., the elements of rank 1) are the atoms of L. Similarly, the maximal elements of $L \setminus \{\hat{1}\}$ (i.e., the elements of rank $r(\hat{1})-1$) are the coatoms of L. A graded lattice is ***atomic*** if every element is a join of a set of atoms, and it is ***coatomic*** if every element is a meet of a set of coatoms.

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Face lattice $L(P)$: The set of all faces of P, partially ordered by inclusion.

Combinatorially equivalent: Polytopes whose face lattices are isomorphic as abstract (i.e., unlabeled) partially ordered sets/lattices.

Equivalently, P and P' are combinatorially equivalent if their facet-vertex incidence matrices differ only by column and row permutations.

Combinatorial type: An equivalence class of polytopes under combinatorial equivalence.

THEOREM 15.1.2 *Face Lattices of Polytopes* (cf. [Zie95, Sect. 2.2])

The face lattices of convex polytopes are finite, graded, atomic, and coatomic lattices. The meet operation $G \wedge H$ is given by intersection, while the join $G \vee H$ is the intersection of all facets that contain both G and H. The rank function on $L(P)$ is given by $r(G) = \dim(G) + 1$.

The minimal nonempty faces of a polytope are its vertices: They correspond to atoms of the lattice $L(P)$. Every face is the join of its vertices, hence $L(P)$ is atomic. Similarly, the maximal proper faces of a polytope are its facets: They correspond to the coatoms of $L(P)$. Every face is the intersection of the facets it is contained in, hence face lattices of polytopes are coatomic.

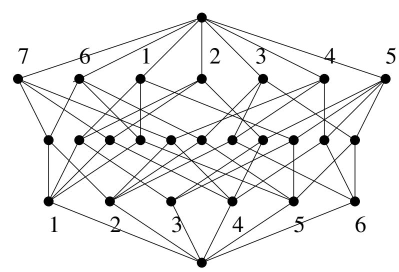

FIGURE 15.1.3
The face lattice of our unnamed 3-polytope. The seven coatoms (facets) and the six atoms (vertices) have been labeled in the order of their appearance in the lists on page 386. Thus, the downwards-path from the coatom "4" to the atom "6" represents the fact that the fourth facet contains the sixth vertex.

The face lattice is a complete encoding of the combinatorial structure of a polytope. However, in general the encoding by a facet-vertex incidence matrix is more efficient. The following matrix—also provided by `polymake`—represents our unnamed 3-polytope:

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ \mathbf{1} & \mathbf{1} & 0 & 0 & 1 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

How do we decide whether a set of vertices $\{v^1, \dots, v^k\}$ is (the vertex set of) a face of P? This is the case if and only if no other vertex v^0 is contained in all the facets that contain $\{v^1, \dots, v^k\}$. This criterion makes it possible, for example, to derive the edges of a polytope P from a facet-vertex matrix. The four 1's printed in boldface in the above matrix M thus certify that the vertices $1, 2$ lie on a face in the unnamed polytope, which is contained in the intersection of the facets $6, 7$. By dimension reasons, the face in question is an edge that connects the vertices $1, 2$.

For low-dimensional polytopes, the criterion can be simplified: If $d \leq 4$, then two vertices are connected by an edge if and only if there are at least $d-1$ different facets that contain them both. However, the same is no longer true for 5-dimensional polytopes, where vertices may be nonadjacent despite being contained in many common facets. The best way to see this is by using polarity, which is discussed next.

15.1.2 BASIC CONSTRUCTIONS

GLOSSARY

Polarity: If $P \subseteq \mathbb{R}^d$ is a d-polytope with the origin in its interior, then the ***polar*** of P is the d-polytope

$$P^\Delta \ := \ \{y \in \mathbb{R}^d \mid \langle y, x\rangle \leq 1 \text{ for all } x \in P\}.$$

Stacking onto a facet F: A polytope $\operatorname{conv}(P \cup x^F)$, where x^F is a point of the form $y^F - \varepsilon(y^P - y^F)$, where y^P is in the interior of P, y^F is in the relative interior of the facet F, and $\varepsilon > 0$ is small enough.

Note that this definition specifies the combinatorial type of the resulting polytope completely, but not the geometric realization; similarly some of the following constructions are not specified completely.

Vertex figure P/v: If v is a vertex of P, then $P/v := P \cap H$ is a polytope obtained by intersecting P with a hyperplane H that has v on one side and all the other vertices of P on the other side.

Cutting off a vertex: A polytope $P \cap H^-$ obtained by intersecting P with a closed halfspace H^- that does not contain the vertex v, but contains all other vertices of P in its interior.

Quotient of P: A k-polytope obtained from a d-polytope by repeatedly ($d-k$ times) taking a vertex figure.

Simplicial polytope: A polytope all of whose facets (equivalently, proper faces) are simplices. Examples: the d-cross-polytopes.

Simple polytope: A polytope all of whose vertex figures (equivalently, proper quotients) are simplices. Examples: the d-cubes.

Polarity is a fundamental construction in the theory of polytopes. One always has $P^{\Delta\Delta} = P$, under the assumption that P has the origin in its interior. This condition can always be obtained by a change of coordinates. In particular, we speak of (combinatorial) polarity between d-polytopes Q and R that are combinatorially equivalent to P and P^Δ, respectively.

Any $\mathcal{V}$-presentation of P yields an $\mathcal{H}$-presentation of P^Δ, and vice versa, via

$$P = \operatorname{conv}\{v^1, \ldots, v^n\} \quad \Longleftrightarrow \quad P^\Delta = \{x \in \mathbb{R}^d \mid \langle v^i, x\rangle \leq 1 \text{ for } 1 \leq i \leq n\}.$$

There are basic relations between polytopes and polytopal constructions under polarity. For example, the fact that the d-cross-polytopes C_d^Δ are the polars of the

d-cubes C_d is built into our notation. More generally, the polars of simple polytopes are simplicial, and vice versa. This can be deduced from the fact that the facets F of a polytope P correspond to the vertex figures P^{Δ}/v of its polar P^{Δ}. In fact, F and P^{Δ}/v are combinatorially polar in this situation. More generally, one has a correspondence between faces and quotients under polarity.

At a combinatorial level, all this can be derived from the fact that the face lattices $L(P)$ and $L(P^{\Delta})$ are anti-isomorphic: $L(P^{\Delta})$ may be obtained from $L(P)$ by reversing the order relations. Thus, lower intervals in $L(P)$, corresponding to faces of P, translate under polarity into upper intervals of $L(P^{\Delta})$, corresponding to quotients of P^{Δ}.

15.1.3 BASIC CONSTRUCTIONS, II

GLOSSARY

For the following constructions, let
$P \subseteq \mathbb{R}^d$ be a d-dimensional polytope with n vertices and m facets, and
$P' \subseteq \mathbb{R}^{d'}$ a d'-dimensional polytope with n' vertices and m' facets.

Scalar multiple: For $\lambda \in \mathbb{R}$, the polytope λP defined by $\lambda P := \{\lambda x \mid x \in P\}$. Here P and λP are combinatorially (in fact, affinely) equivalent for all $\lambda \neq 0$. In particular, $(-1)P = -P = \{-p \mid p \in P\}$, and $(+1)P = P$.

Minkowski sum: The polytope $P + P' := \{p + p' \mid p \in P,\ p' \in P'\}$.

It is also useful to define the difference as $P - P' = P + (-P')$. The polytopes $P + \lambda P'$ are combinatorially equivalent for all $\lambda > 0$, and similarly for $\lambda < 0$.

If $P' = \{p'\}$ is one single point, then $P - \{p'\}$ is the image of P under the translation that takes p' to the origin.

Product: The $(d+d')$-dimensional polytope $P \times P' := \{(p, p') \in \mathbb{R}^{d+d'} \mid p \in P,\ p' \in P'\}$. $P \times P'$ has $n \cdot n'$ vertices and $m + m'$ facets.

Join: The $(d+d'+1)$-polytope obtained as the convex hull $P * P'$ of $P \cup P'$, after embedding P and P' in a space where their affine hulls are skew. For example, $P * P' := \mathrm{conv}(\{(p, 0, 0) \in \mathbb{R}^{d+d'+1} \mid p \in P\} \cup \{(0, p', 1) \in \mathbb{R}^{d+d'+1} \mid p' \in P'\})$. $P * P'$ has dimension $d+d'+1$ and $n+n'$ vertices. Its k-faces are the joins of i-faces of P and $(k-i-1)$-faces of P', hence $f_k(P * P') = \sum_{i=-1}^{k} f_i(P) f_{k-i-1}(P')$.

Subdirect sum: The $(d+d')$-dimensional polytope
$P \oplus P' := \mathrm{conv}(\{(p, 0) \in \mathbb{R}^{d+d'} \mid p \in P\} \cup \{(0, p') \in \mathbb{R}^{d+d'} \mid p' \in P'\})$.
Thus the subdirect sum $P \oplus P'$ is a projection of the join $P * P'$. See McMullen [McM76].

Direct sum: If both P and P' have the origin in their interiors—this is the "usual" situation for creating subdirect sums—then $P \oplus P'$ is the direct sum of P and P'. It is $(d + d')$-dimensional, and has $n + n'$ vertices and $m \cdot m'$ facets.

Prism: The product $\mathrm{prism}(P) := P \times I$, where I denotes the real interval $I = [-1, +1] \subseteq \mathbb{R}$. It has dimension $d + 1$, $2n$ vertices and $m + 2$ facets.

Pyramid: The join $\mathrm{pyr}(P) := P * \{0\}$ of P with a point (a 0-dimensional polytope $P' = \{0\} \subseteq \mathbb{R}^0$). It has dimension $d + 1$, $n + 1$ vertices and $m + 1$ facets.

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Bipyramid: The subdirect sum $\mathrm{bipyr}(P) := P \oplus I$, where P must have the origin in its interior. It has dimension $d+1$, $n+2$ vertices and $2m$ facets.

One-point suspension, obtained by ***splitting the vertex v:*** The subdirect sum

$$\mathrm{ops}(P, v) := \mathrm{conv}(P \times \{0\} \cup \{v\} \times [-1, 1]),$$

where v is a vertex of P. It has dimension $d+1$, $n+1$ vertices and $2m - m_v$ facets, if v lies in m_v facets of P.

Lawrence extension: If $p \in \mathbb{R}^d$ is a point outside the polytope P, then the subdirect sum $(P - \{p\}) \oplus [1, 2]$ is a *Lawrence extension of P at p*. For $p \in P$ this is just a pyramid.

Wedge over a facet F of P:

$$\mathrm{wedge}(P, F) := P \times \mathbb{R} \cap \{a^T x + |x_{d+1}| \leq b\},$$

where F is a facet of P defined by $a^T x \leq b$. It has dimension $d+1$, $m+1$ facets, and $2n - n_F$ vertices, if F has n_F vertices. More generally, the wedge construction can be performed (defined by the same formula) for a face F. If F is not a facet, then the wedge will have $m+2$ facets.

In contrast to the other constructions in this section, the combinatorial type of the Minkowski sum $P + P'$ is *not* determined by the combinatorial types of its constituents P and P', and the combinatorial type of a Lawrence extension depends on the position of the extension point p with respect to P (see below).

Of course, the many constructions listed in the glossary above are not independent of each other. For instance, some of these constructions are related by polarity: for polytopes P and P' with the origin in their interiors, the product and the direct sum are related by polarity,

$$P \times P' \;=\; (P^\Delta \oplus P'^\Delta)^\Delta,$$

and this specializes to polarity relations among the pyramid, bipyramid, and prism constructions,

$$\mathrm{pyr}(P) \;=\; (\mathrm{pyr}(P^\Delta))^\Delta \qquad \text{and} \qquad \mathrm{prism}(P) \;=\; (\mathrm{bipyr}(P^\Delta))^\Delta.$$

Similarly, "cutting off a vertex" is polar to "stacking onto a facet." The wedge construction is a *subdirect product* in the sense of McMullen [McM76] and the polar dual construction to a subdirect sum.

It is interesting to study—and this has not been done systematically—how the basic polytope operations generate complicated convex polytopes from simpler ones. For example, starting from a one-dimensional polytope $I = C_1 = [-1, +1] \subset \mathbb{R}$, the direct product construction generates the cubes C_d, while direct sums generate the cross-polytopes C_d^Δ.

Even more complicated centrally symmetric polytopes, the *Hanner polytopes*, are obtained from copies of the interval I by using products and direct sums. They are interesting since they achieve with equality the conjectured bound that all centrally symmetric d-polytopes have at least 3^d nonempty faces (see Kalai [Kal89] and Sanyal, Werner and Ziegler [SWZ09]).

Every polytope can be viewed as a region of a hyperplane arrangement: For this, take as $\mathcal{A}_P$ the set of all hyperplanes of the form $\mathrm{aff}(F)$, where F is a facet of P.

For additional points, such as the points outside the polytope used for Lawrence extensions, or those used for stackings, it is often enough to know in which region, or in which lower-dimensional region, of the arrangement $\mathcal{A}_P$ they lie.

The combinatorial type of a Lawrence extension depends on the position of p in the arrangement $\mathcal{A}_P$. Thus the Lawrence extensions obtainable from P depend on the realization of P, not only on its combinatorial type.

The Lawrence extension may seem like quite a simple little construction. However, it has the amazing property that it can encode crucial information about the position of a point *outside* a d-polytope into the boundary structure of a $(d{+}1)$-polytope, and thus is an essential ingredient in some remarkable constructions, such as universality results (see e.g., Ziegler [Zie08], Richter-Gebert [Ric96], and Adiprasito, Padrol and Theran [APT15]), and high-dimensional projectively unique polytopes (Adiprasito and Ziegler [AZ15], Adiprasito and Padrol [AP16]).

15.1.4 MORE EXAMPLES

There are many interesting classes of polytopes arising from diverse areas of mathematics (as well as physics, optimization, crystallography, etc.). Some of these are discussed below. More classes of examples appear in other chapters of this Handbook. For example, regular and semiregular polytopes are discussed in Chapter 18, while polytopes that arise as Voronoi cells of lattices appear in Chapters 3, 7, and 64.

GLOSSARY

Graph of a polytope: The graph $G(P) = (V(P), E(P))$ with vertex set $V(P) = \mathcal{F}_0(P)$ and edge set $E(P) = \{\{v^1, v^2\} \subseteq \binom{V}{2} \mid \mathrm{conv}\{v^1, v^2\} \in \mathcal{F}_1(P)\}$.

Zonotope: Any d-polytope Z that can be represented as the image of an n-dimensional cube C_n $(n \geq d)$ under an affine map; equivalently, any polytope that can be written as a Minkowski sum of n line segments (1-dimensional polytopes). The smallest n such that Z is an image of C_n is the ***number of zones*** of Z.

Moment curve: The curve γ in $\mathbb{R}^d$ defined by $\gamma : \mathbb{R} \longrightarrow \mathbb{R}^d$, $t \longmapsto (t, t^2, \ldots, t^d)^T$.

Cyclic polytope: The convex hull of a finite set of points on a moment curve, or any polytope combinatorially equivalent to it.

k-neighborly polytope: A polytope such that each subset of at most k vertices forms the vertex set of a face. Thus every polytope is 1-neighborly, and a polytope is 2-neighborly if and only if its graph is complete.

Neighborly polytope: A d-dimensional polytope that is $\lfloor d/2 \rfloor$-neighborly.

(0,1)*-polytope:* A polytope all of whose vertex coordinates are 0 or 1, that is, whose vertex set is a subset of the vertex set $\{0,1\}^d$ of the unit cube.

ZONOTOPES

Zonotopes appear in quite different guises. They can equivalently be defined as the Minkowski sums of finite sets of line segments (1-dimensional polytopes), as the affine projections of d-cubes, or as polytopes all of whose faces (equivalently, all 2-faces) exhibit central symmetry. Thus a 2-dimensional polytope is a zonotope if and only if it is centrally symmetric. By a classical result of McMullen, a d-dimensional polytope is a zonotope if and only if its 2-faces are centrally symmetric.

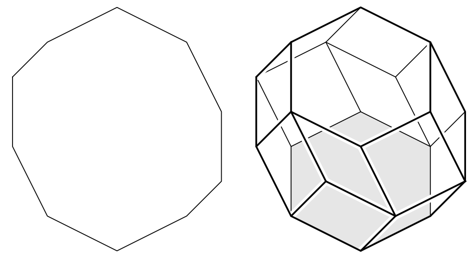

FIGURE 15.1.4
A 2-dimensional and a 3-dimensional zonotope, each with 5 zones, and thus obtainable as affine projections of a 5-dimensional cube. The 2-dimensional one is a projection of the 3-dimensional one. Every projection of a zonotope is a zonotope.

Among the most prominent zonotopes are the permutohedra: The ***permutohedron*** Π_{d-1} is constructed by taking the convex hull of all d-vectors whose coordinates are $\{1, 2, \ldots, d\}$, in any order. The permutohedron Π_{d-1} is a $(d-1)$-dimensional polytope (contained in the hyperplane $\{x \in \mathbb{R}^d \mid \sum_{i=1}^{d} x_i = d(d+1)/2\}$) with $d!$ vertices and $2^d - 2$ facets.

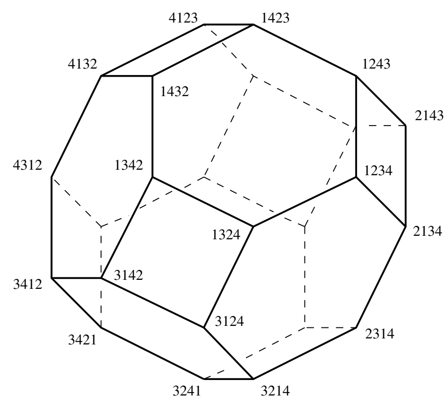

FIGURE 15.1.5
The 3-dimensional permutohedron Π_3. The vertices are labeled by the permutations that, when applied to the coordinate vector in $\mathbb{R}^4$, yield $(1,2,3,4)^T$.
Note that the coordinates of each vertex are given by reading the inverses *(!) of the permutation that is used to label the vertex.*

One unusual feature of permutohedra is that they are simple zonotopes: These are rare in general, and the (unsolved) problem of classifying them is equivalent to the problem of classifying all simplicial arrangements of hyperplanes (see Section 6.3.3).

Zonotopes are important because their theory is equivalent to the theories of vector configurations (realizable oriented matroids) and of hyperplane arrangements. In fact, the system of line segments that generates a zonotope can be considered as a vector configuration, and the hyperplanes that are orthogonal to the line segments provide the associated hyperplane arrangement. We refer to [BLS$^+$93, Section 2.2] and [Zie95, Lecture 7].

Finally, we mention in passing a surprising bijective correspondence between the tilings of a zonotope with smaller zonotopes and oriented matroid liftings (realizable or not) of the oriented matroid of a zonotope. This correspondence is known as the *Bohne–Dress theorem*; we refer to Richter-Gebert and Ziegler [RZ94].

CYCLIC POLYTOPES

Cyclic polytopes can be constructed by taking the convex hull of $n > d$ points on the moment curve in $\mathbb{R}^d$. The "standard construction" is to define a cyclic polytope $C_d(n)$ as the convex hull of n integer points on this curve, such as

$$C_d(n) \ := \ \text{conv}\{\gamma(1), \gamma(2), \ldots, \gamma(n)\}.$$

However, the combinatorial type of $C_d(n)$ is given by the—entirely combinatorial—***Gale evenness criterion***: If $C_d(n) = \text{conv}\{\gamma(t_1), \ldots, \gamma(t_n)\}$, with $t_1 < \cdots < t_n$, then $\gamma(t_{i_1}), \ldots, \gamma(t_{i_d})$ determine a facet if and only if the number of indices in $\{i_1, ..., i_d\}$ lying between any two indices *not* in that set is even. Thus, the combinatorial type does not depend on the specific choice of points on the moment curve [Zie95, Example 0.6; Theorem 0.7].

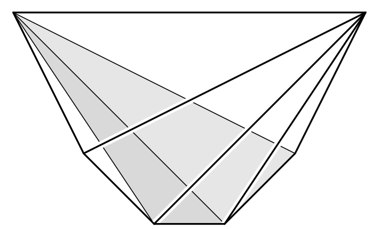

FIGURE 15.1.6
A 3-dimensional cyclic polytope $C_3(6)$ with 6 vertices. (In a projection of γ to the x_1x_2-plane, the curve γ and hence the vertices of $C_3(6)$ lie on the parabola $x_2 = x_1^2$.)

The first property of cyclic polytopes to notice is that they are simplicial. The second, more surprising, property is that they are neighborly. This implies that among all d-polytopes P with n vertices, the cyclic polytopes maximize the number $f_i(P)$ of i-dimensional faces for $i < \lfloor d/2 \rfloor$. The same fact holds for all i: This is part of McMullen's upper bound theorem (see below). In particular, cyclic polytopes have a very large number of facets,

$$f_{d-1}\big(C_d(n)\big) \ = \ \binom{n - \lceil \frac{d}{2} \rceil}{\lfloor \frac{d}{2} \rfloor} \ + \ \binom{n - 1 - \lceil \frac{d-1}{2} \rceil}{\lfloor \frac{d-1}{2} \rfloor}.$$

For example, any cyclic 4-polytope $C_4(n)$ has $n(n-3)/2$ facets. Thus $C_4(8)$ has 8 vertices, any two of them adjacent, and 20 facets. This is more than the 16 facets of the 4-cross-polytope, which also has 8 vertices!

NEIGHBORLY POLYTOPES

Here are a few observations about neighborly polytopes. For more information, see [BLS$^+$93, Section 9.4] and the references quoted there.

The first observation is that if a polytope is k-neighborly for some $k > \lfloor d/2 \rfloor$, then it is a simplex. Thus, if one ignores the simplices, then $\lfloor d/2 \rfloor$-neighborly polytopes form the extreme case, which motivates calling them simply "neighborly." However, only in even dimensions $d = 2m$ do the neighborly polytopes have very special structure. For example, one can show that even-dimensional neighborly polytopes are necessarily simplicial, but this is not true in general. For the latter, note that, for example, all 3-dimensional polytopes are neighborly by definition, and that if P is a neighborly polytope of dimension $d = 2m$, then $\mathrm{pyr}(P)$ is neighborly of dimension $2m{+}1$.

All simplicial neighborly d-polytopes with n vertices have the same number of facets (in fact, the same f-vector $(f_0, f_1, \ldots, f_{d-1})$) as $C_d(n)$. They constitute the class of polytopes with the maximal number of i-faces for all i: This is the statement of McMullen's upper bound theorem. We refer to Chapter 17 for a thorough discussion of f-vector theory.

Every even-dimensional neighborly polytope with $n \leq d + 3$ vertices is combinatorially equivalent to a cyclic polytope. This covers, for instance, the polar of the product of two triangles, $(\Delta_2 \times \Delta_2)^\Delta$, which is easily seen to be a 4-dimensional neighborly polytope with 6 vertices; see Figure 15.1.9. The first example of an even-dimensional neighborly polytope that is not cyclic appears for $d = 4$ and $n = 8$. It can easily be described in terms of its affine Gale diagram; see below.

Neighborly polytopes may at first glance seem to be very peculiar and rare objects, but there are several indications that they are not quite as unusual as they seem. In fact, the class of neighborly polytopes is believed to be very rich. Thus, Shemer [She82] has shown that for fixed even d the number of nonisomorphic neighborly d-polytopes with n vertices grows superexponentially with n (see also [Pad13]). Also, many of the (0,1)-polytopes studied in combinatorial optimization turn out to be at least 2-neighborly. Both these effects illustrate that "neighborliness" is not an isolated phenomenon.

OPEN PROBLEMS

1. Can every neighborly d-polytope $P \subseteq \mathbb{R}^d$ with n vertices be extended by a new vertex $v \in \mathbb{R}^d$ to a neighborly polytope $P' := \mathrm{conv}(P \cup \{v\})$ with $n{+}1$ vertices? This has been asked by Shemer [She82, p. 314]. It has been verified only recently for small parameters d and n by Miyata and Padrol [MP15].

2. It is a classic problem of Perles whether every simplicial polytope is a quotient of a neighborly polytope. For polytopes with at most $d{+}4$ vertices this was confirmed by Kortenkamp [Kor97]. Adiprasito and Padrol [AP16] disproved a related conjecture, that every polytope is a subpolytope of a stacked polytope.

3. Some computer experiments with random polytopes suggest that

   - one obtains a neighborly polytope with high probability (which increases rapidly with the dimension of the space),
   - the most probable combinatorial type is a cyclic polytope,
   - but still this probability of a cyclic polytope tends to zero.

   However, none of this has been proved. See Bokowski and Sturmfels [BS89, p. 101], Bokowski, Richter-Gebert, and Schindler [BRS92], Vershik and Sporychev [VS92], and Donoho and Tanner [DT09].

(0,1)-POLYTOPES

There is a $(0, 1)$-polytope (given in terms of a $\mathcal{V}$-presentation) associated with every finite set system $\mathcal{S} \subseteq 2^E$ (where E is a finite set, and 2^E denotes the collection of all of its subsets), via

$$P[\mathcal{S}] \;:=\; \mathrm{conv}\Big\{\sum_{i\in F} e^i \;\Big|\; F\in\mathcal{S}\Big\} \;\subseteq\; \mathbb{R}^E.$$

The combinatorial optimization contains a multitude of extensive studies on (partial) $\mathcal{H}$-descriptions of special $(0,1)$-polytopes, such as for example

- the ***traveling salesman polytopes*** T^n, where E is the edge set of a complete graph K_n, and $\mathcal{F}$ is the set of all $(n-1)!$ Hamilton cycles (simple circuits through all the vertices) in E (see Grötschel and Padberg [GP85]);
- the ***cut*** and ***equicut polytopes***, where E is the edge set of—for example—a complete graph, and $\mathcal{S}$ represents the family of all cuts, or all equicuts (given by a partition of the vertex set into two blocks of equal size) of the graph (see Deza and Laurent [DL97]).

Besides their importance for combinatorial optimization, there is a great deal of interesting polytope theory associated with such polytopes. It turns out that some of these polytopes are so complicated that a complete $\mathcal{H}$-description or any other "full understanding" will remain out of reach. For example, Billera and Sarangarajan [BS96] showed that *every* $(0,1)$-polytope appears as a face of a traveling salesman polytope. Cut polytopes seem to have particularly many facets, though that has not been proven. Equicut polytopes were used by Kahn and Kalai [KK93] in their striking disproof of Borsuk's conjecture (see also [AZ98]).

Despite the detailed structure theory for the "special" $(0,1)$-polytopes of combinatorial optimization, there is very little known about "general" $(0,1)$-polytopes. For example, what is the "typical," or the maximal, number of facets of a $(0,1)$-polytope? Based on a random construction Bárány and Pór [BP01] proved the existence of d-dimensional $(0,1)$-polytopes with $(c\,d/\log d)^{d/4}$ facets, where c is a universal constant. This lower bound has been improved to $(c\,d/\log d)^{d/2}$ by Gatzouras et al. [GGM05]. The best known upper bounds are of order $(d-2)!$. Another question, which is not only intrinsically interesting but might also provide new clues for basic questions of linear and combinatorial optimization, is: What is the maximal number of faces in a 2-dimensional projection of a $(0,1)$-polytope? For a survey on $(0,1)$-polytopes see [Zie00].

15.1.5 THREE-DIMENSIONAL POLYTOPES AND PLANAR GRAPHS

GLOSSARY

d-connected graph: A connected graph that remains connected if any $d-1$ vertices are deleted.

Drawing of a graph: A representation in the plane where the vertices are represented by distinct points, and simple Jordan arcs (typically: polygonal, or at least piecewise smooth) are drawn between the pairs of adjacent vertices.

Planar graph: A graph that can be drawn in the plane with Jordan arcs that are disjoint except for their endpoints.

Realization space: The set of all coordinatizations of a combinatorial structure, modulo affine coordinate transformations; see Section 6.3.2.

Isotopy property: A combinatorial structure (such as a combinatorial type of polytope) has the isotopy property if any two realizations with the same orientation can be deformed into each other by a continuous deformation that maintains the combinatorial type. Equivalently, the isotopy property holds for a combinatorial structure if and only if its realization space is connected.

THEOREM 15.1.3 *Steinitz's Theorem* [Ste22, Satz 43, S. 77] [SR34]

For every 3*-dimensional polytope* P*, the graph* $G(P)$ *is a planar,* 3*-connected graph. Conversely, for every planar* 3*-connected graph with at least* 4 *vertices, there is a unique combinatorial type of* 3*-polytope* P *with* $G(P) \cong G$*.*

Furthermore, the realization space $\mathcal{R}(P)$ *of a combinatorial type of* 3*-polytope is homeomorphic to* $\mathbb{R}^{f_1(P)-6}$*, and contains rational points. In particular,* 3*-polytopes have the isotopy property, and they can be realized with integer vertex coordinates.*

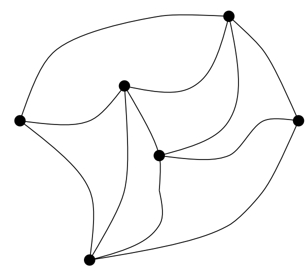

FIGURE 15.1.7
A (planar drawing of a) 3*-connected, planar, unnamed graph. The formidable task of any proof of Steinitz's theorem is to construct a* 3*-polytope with this graph.*

There are three essentially different strategies known that yield proofs of Steinitz's theorem. The first approach, due to Steinitz, provides a construction sequence for any type of 3-polytope, starting from a tetrahedron, and using only local operations such as cutting off vertices and polarity. See [Zie95, Lecture 4]. The second "Tutte type" of proof realizes any combinatorial type by a global minimization argument, which as an intermediate step provides a special planar representation of the graph by a framework with a positive self-stress; see Richter-Gebert [Ric96]. The third approach, spear-headed by Thurston, derives Steinitz's Theorem from the Koebe–Andreev–Thurston circle packing theorem. It yields that every 3-polytope has an (essentially unique!) realization with its edges tangent to the unit sphere. This proof can be derived from a variational principle obtained by Bobenko and Springborn [BS04]. For an exposition see Ziegler [Zie07, Lecture 1].

OPEN PROBLEMS

Because of Steinitz's theorem and its extensions and corollaries, the theory of 3-dimensional polytopes is quite complete and satisfactory. Nevertheless, some basic open problems remain.

1. It can be shown that every combinatorial type of 3-polytope with n vertices and a triangular facet can be realized with integer coordinates belonging to $\{1, 2, \ldots, 29^n\}^3$ (Ribó Mor, Rote and Schulz [RRS11], improving upon previous bounds by Onn and Sturmfels [OS94] and Richter-Gebert [Ric96, Sect. 13.2]), but it is not clear whether this can be replaced by a polynomial upper bound. No nontrivial lower bounds seem to be available.

2. If P has a nontrivial group G of symmetries, then it also has a symmetric realization (Mani [Man71]). However, it is not clear whether for all 3-polytopes the space of all G-symmetric realizations $\mathcal{R}^{\mathrm{G}}(P)$ is still homeomorphic to some $\mathbb{R}^k$. (It does not contain rational points in general, e.g., for the regular icosahedron!)

15.1.6 FOUR-DIMENSIONAL POLYTOPES AND SCHLEGEL DIAGRAMS

GLOSSARY

Subdivision of a polytope P: A collection of polytopes $P_1, \dots, P_\ell \subseteq \mathbb{R}^d$ such that $P = P_1 \cup \cdots \cup P_\ell$, and for $i \neq j$ we have that $P_i \cap P_j$ is a proper face of P_i and P_j (possibly empty). In this case we write $P = \uplus P_i$.

Triangulation of a polytope: A subdivision into simplices. (See Chapter 16.)

Schlegel diagram: A $(d{-}1)$-dimensional representation $\mathcal{D}(P,F)$ of a d-dimensional polytope P given by a subdivision of a facet F, obtained as follows: Take a point of view outside of P but very close to a relative interior point of the facet F, and then let $\mathcal{D}(P,F)$ be the decomposition of F given by all the other facets of P, as seen from this point of view in the "window" F.

$(d{-}1)$-diagram: A subdivision $\mathcal{D}$ of a $(d{-}1)$-polytope F such that the intersection of any polytope in $\mathcal{D}$ with the boundary of F is a face of F (which may be empty).

Basic primary semialgebraic set defined over $\mathbb{Z}$: The solution set $S \subseteq \mathbb{R}^k$ of a finite set of equations and strict inequalities of the form $f_i(x) = 0$ or $g_j(x) > 0$, where the f_i and g_j are polynomials in k variables with integer coefficients.

Stable equivalence: Equivalence relation between semialgebraic sets generated by rational changes of coordinates and certain types of "stable" projections with contractible fibers. See Richter-Gebert [Ric96, Section 2.5].

In particular, if two sets are stably equivalent, then they have the same homotopy type, and they have the same arithmetic properties with respect to subfields of $\mathbb{R}$; e.g., either both or neither of them contain a rational point.

The situation for 4-polytopes is fundamentally different from that for 3-dimensional polytopes. One reason is that there is no similar reduction of 4-polytope theory to a combinatorial (graph) problem.

The main results about graphs of d-polytopes are that they are d-connected (Balinski [Bal61]), and that each contains a subdivision of the complete graph on $d{+}1$ vertices, $K_{d+1} = G(T_d)$ (Grünbaum [Grü67, p. 200]). In particular, all graphs of 4-polytopes are 4-connected, and none of them is planar; see also Chapter 19.

Schlegel diagrams provide a reasonably efficient tool for the visualization of 4-polytopes: We have a fighting chance to understand some important properties in terms of the 3-dimensional (!) geometry of Schlegel diagrams.

A $(d{-}1)$-diagram is a polytopal complex that "looks like" a Schlegel diagram, although there are diagrams (even 2-diagrams) that are not Schlegel diagrams.

The situation is somewhat nicer for *simple* polytopes. The combinatorial structure of a simple polytope is entirely determined by the abstract graph: This is due

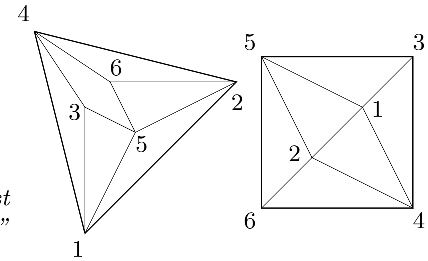

FIGURE 15.1.8
Two Schlegel diagrams of our unnamed 3-polytope, the first based on a triangle facet, the second on the "bottom square."

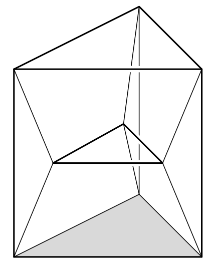

FIGURE 15.1.9
A Schlegel diagram of the product of two triangles. (This is a 4-dimensional polytope with 6 triangular prisms as facets, any two of them adjacent!)

to Blind and Mani-Levitska [BM87], with a simple proof by Kalai [Kal88] and an efficient (polynomial-time) reconstruction algorithm by Friedman [Fri09]. Moreover, the geometry of higher-dimensional simple d-polytopes can be understood in terms of $(d-1)$-diagrams: For $d \geq 4$ all simple $(d-1)$-diagrams "are Schlegel," that is, they represent genuine d-dimensional polytopes (Whiteley, see Rybnikov [Ryb99]).

The fundamental difference between the theories for polytopes in dimensions 3 and 4 is most apparent in the contrast between Steinitz's theorem and the following result, which states simply that all the "nice" properties of 3-polytopes established in Steinitz's theorem fail dramatically for 4-dimensional polytopes. Indeed, Richter-Gebert showed that the realization spaces of 4-polytopes exhibit the same type of "universality" that was established by Mnëv for the realization spaces of planar point configurations/line arrangements, as well as for d-polytopes with $d+4$ vertices, as discussed in Chapter 6 (see Thm. 6.3.3):

THEOREM 15.1.4 *Universality for 4-Polytopes* [Ric96]

The realization space of a 4-dimensional polytope can be "arbitrarily wild": For every basic primary semialgebraic set S defined over $\mathbb{Z}$ there is a 4-dimensional polytope $P[S]$ whose realization space $\mathcal{R}(P[S])$ is stably equivalent to S.

In particular, this implies the following.

- *The isotopy property fails for 4-dimensional polytopes.*
- *There are nonrational 4-polytopes: combinatorial types that cannot be realized with rational vertex coordinates.*
- *The coordinates needed to represent all combinatorial types of rational 4-polytopes with integer vertices grow doubly exponentially with $f_0(P)$.*

The complete proof of this universality theorem is given in [Ric96]. One key component of the proof corresponds to another failure of a 3-dimensional phenomenon in dimension 4: For any facet (2-face) F of a 3-dimensional polytope P, the shape of F can be arbitrarily prescribed; in other words, the canonical map of realization spaces $\mathcal{R}(P) \longrightarrow \mathcal{R}(F)$ is always surjective. Richter-Gebert shows that a similar statement fails in dimension 4, even if F is a 2-dimensional pentagonal face: See Figure 15.1.10 for the case of a hexagon.

Preliminary version (August 10, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

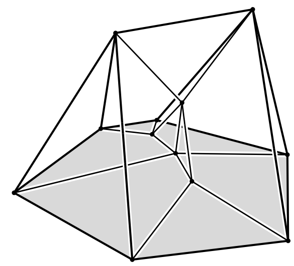

FIGURE 15.1.10
Schlegel diagram of a 4-dimensional polytope with 8 facets and 12 vertices, for which the shape of the base hexagon cannot be prescribed arbitrarily.

A problem that is left open is the structure of the realization spaces of simplicial 4-polytopes. All that is available now is a universality theorem for simplicial polytopes without a dimension bound, which had been claimed by Mnëv and by others since the 1980s, but for which a proof was provided only recently by Adiprasito and Padrol [AP17], and a single example of a simplicial 4-polytope that violates the isotopy property, by Bokowski et al. [BEK84] (see [Bok06, p. 142] [Fir15b, Sect. 1.3.4] for correct coordinates, as a rational inscribed polytope).

15.1.7 POLYTOPES WITH FEW VERTICES AND GALE DIAGRAMS

GLOSSARY

Polytope with few vertices: A polytope that has only a few more vertices than its dimension; usually a d-polytope with at most $d{+}4$ vertices.

(Affine) Gale diagram: A configuration of n not necessarily pairwise distinct, signed/bicolored ("positive"/"black" and "negative"/"white") points in affine space $\mathbb{R}^{n-d-2}$, which encodes a d-polytope with n vertices uniquely up to projective transformations.

The computation of a Gale diagram involves only simple linear algebra. For this, let $V \in \mathbb{R}^{d\times n}$ be a matrix whose columns consist of coordinates for the vertices of a d-polytope. For simplicity, we assume that P is not a pyramid, and that the vertices $\{v^1, \ldots, v^{d+1}\}$ affinely span $\mathbb{R}^d$. Let $\widetilde{V} \in \mathbb{R}^{(d+1)\times n}$ be obtained from V by adding an extra (terminal) row of ones. The vector configuration given by the columns of $\widetilde{V}$ represents the *oriented matroid* of P; see Chapter 6.

Now perform row operations on the matrix $\widetilde{V}$ to get it into the form $\widetilde{V} \sim (I_{d+1}|A)$, where I_{d+1} denotes a unit matrix, and $A \in \mathbb{R}^{(d+1)\times(n-d-1)}$. (The row operations do not change the oriented matroid.) The columns of the matrix $\widetilde{V}^* := (-A^T|I_{n-d-1}) \in \mathbb{R}^{(n-d-1)\times n}$ then represent the dual oriented matroid. We find a vector $a \in \mathbb{R}^{n-d-1}$ that has nonzero scalar product with all the columns of $\widetilde{V}^*$, divide each column w^* of $\widetilde{V}^*$ by the value $\langle a, w^*\rangle$, and delete from the resulting matrix any row that affinely depends on the others, thus obtaining a matrix $W \in \mathbb{R}^{(n-d-2)\times n}$. The columns of W give a bicolored point configuration in $\mathbb{R}^{n-d-2}$, where *black* points are used for the columns where $\langle a, w^*\rangle > 0$, and *white* points for the others. This bicolored point configuration represents an affine Gale diagram of P.

An affine configuration of bicolored points (consisting of n points that affinely span $\mathbb{R}^e$) represents a polytope (with n vertices, of dimension $n-e-2$) if and only

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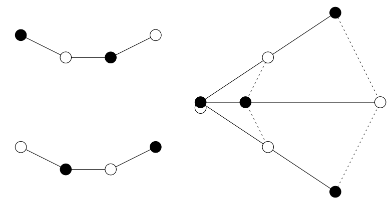

FIGURE 15.1.11
Two affine Gale diagrams of 4-dimensional polytopes: for a noncyclic neighborly polytope with 8 vertices, and for the polar (with 8 vertices) of the polytope with 8 facets from Figure 15.1.10, for which the shape of a hexagonal face cannot be prescribed arbitrarily.

if for any hyperplane spanned by some of the points, and for each side of it, the number of black points on this side, plus the number of white points on the other side, is at least 2.

The final information one needs is how to read off properties of a polytope from its affine Gale diagram: A set of points represents a face if and only if the bicolored points *not* in the set support an affine dependency, with positive coefficients on the black points, and with negative coefficients on the white points. Equivalently, the convex hull of all the black points not in our set, and the convex hull of all the white points not in the set, intersect in their relative interiors.

Affine Gale diagrams have been *very* successfully used to study and classify polytopes with few vertices.

d+1 vertices: The only d-polytopes with d+1 vertices are the d-simplices.

d+2 vertices: This corresponds to the situation of 0-dimensional affine Gale diagrams. There are exactly $\lfloor d^2/4 \rfloor$ combinatorial types of d-polytopes with d+2 vertices: They are of the form $\Delta_{a-1} * (\Delta_b \oplus \Delta_c)$ with $d = a+b+c$, $a \geq 0$, $b \geq c \geq 0$, that is, multiple (a-fold) pyramids over simplicial polytopes that are direct sums of simplices $\Delta_b \oplus \Delta_c$. Among these, the $\lfloor d/2 \rfloor$ types with $a = 0$ are the simplicial ones.

d+3 vertices: All d-polytopes with d+3 vertices are realizable with (small) integral coordinates and satisfy the isotopy property: All this can be easily analyzed in terms of 1-dimensional affine Gale diagrams. In addition, formulas for the numbers of

- polytopes [Fus06, Thm. 1],
- simplicial polytopes [Grü67, Sect. 6.2, Thm. 6.3.2, p. 113 and p. 424],
- neighborly polytopes [McM74], and
- simplicial neighborly polytopes [AM73]

have been produced using Gale diagrams. This leads to subtle enumeration problems, some of them connected to colored necklace counting problems.

d+4 vertices: Here anything can go wrong: The universality theorem for oriented matroids of rank 3 yields a universality theorem for d-polytopes with d+4 vertices. See Section 6.3.4.

We refer to [Zie95, Lecture 6] for a detailed introduction to affine Gale diagrams.

15.2 METRIC PROPERTIES

The combinatorial data of a polytope—vertices, edges, . . . , facets—have their counterparts in genuine geometric data, such as face volumes, surface areas, quermassintegrals, and the like. In this second half of the chapter, we give a brief sketch of some key geometric concepts related to polytopes.

However, the topics of combinatorial and of geometric invariants are not disjoint at all: Much of the beauty of the theory stems from the subtle interplay between the two sides. Thus, the computation of volumes inevitably leads to the construction of triangulations (explicitly or implicitly), mixed volumes lead to mixed subdivisions of Minkowski sums (one "hot topic" for current research in the area), quermassintegrals relate to face enumeration, and so on.

A concrete and striking example of the interplay is related to Kalai's 3^d conjecture on the face numbers of centrally symmetric polytopes $P \subset \mathbb{R}^d$ mentioned before: Using convex geometric methods, Figiel, Lindenstrauss and Milman [FLM77] proved that $\ln f_0(P) \ln f_{d-1}(P) \geq \frac{1}{16}d$.

Furthermore, the study of polytopes yields a powerful approach to the theory of convex bodies: Sometimes one can extend properties of polytopes to arbitrary convex bodies by approximation [Sch14]. However, there are also properties valid for polytopes that fail for convex bodies in general. This bug/feature is designed to keep the game interesting.

15.2.1 VOLUME AND SURFACE AREA

GLOSSARY

Volume of a d-simplex T: $V(T) = \frac{1}{d!}\left|\det\begin{pmatrix} v^0 & \cdots & v^d \\ 1 & \cdots & 1\end{pmatrix}\right|$, where $T = \operatorname{conv}\{v^0,\dots,v^d\}$ with $v^0,\dots,v^d \in \mathbb{R}^d$.

Volume of a d-polytope: $V(P) := \sum_{T\in\Delta(P)} V(T)$, where $\Delta(P)$ is any triangulation of P.

k-volume $V^k(P)$ of a k-polytope $P \subseteq \mathbb{R}^d$: The volume of P, computed with respect to the k-dimensional Euclidean measure induced on aff(P).

Surface area of a d-polytope P: $F(P) := \sum_{T\in\Delta(P),\,F\in\mathcal{F}_{d-1}(P)} V^{d-1}(T\cap F)$, where $\Delta(P)$ is a triangulation of P.

The volume $V(P)$ (i.e., the d-dimensional Lebesgue measure) and the surface area $F(P)$ of a d-polytope $P \subseteq \mathbb{R}^d$ can be derived from any triangulation of P, since volumes of simplices are easy to compute. The crux for this is in the (efficient?) generation of a triangulation, a topic on which Chapters 16 and 29 of this Handbook have more to say.

The following recursive approach only implicitly generates a triangulation, but derives explicit volume formulas. Let $P \subseteq \mathbb{R}^d$ ($P \neq \emptyset$) be a polytope. If $d = 0$ then we set $V(P) = 1$. Otherwise we set $\mathcal{S}_{d-1}(P) := \{u \in S^{d-1} \mid \dim(H(P,u)\cap P) = d-1\}$, and use this to define the volume of P as

$$V(P) := \frac{1}{d}\sum_{u\in\mathcal{S}_{d-1}(P)} h(P,u)\cdot V^{d-1}(H(P,u)\cap P).$$

Thus, for any d-polytope the volume is a sum of its facet volumes, each weighted by $1/d$ times its signed distance from the origin. This can be interpreted geometrically as follows: Assume for simplicity that the origin is in the interior of P.

Then the collection $\{\mathrm{conv}(F \cup \{0\}) \mid F \in \mathcal{F}_{d-1}(P)\}$ is a subdivision of P into d-dimensional pyramids, where the base of $\mathrm{conv}(F \cup \{0\})$ has $(d-1)$-dimensional volume $V^{d-1}(F)$—to be computed recursively, the height of the pyramid is $h(P, u^F)$, and thus its volume is $\frac{1}{d} h(P, u^F) \cdot V^{d-1}(F)$; compare to Figure 15.2.1. The formula remains valid even if the origin is outside P or on its boundary.

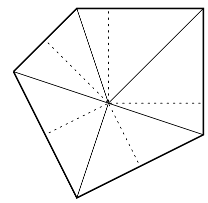

FIGURE 15.2.1
This pentagon, with the origin in its interior, is decomposed into five pyramids (triangles), each with one of the pentagon facets (edges) F_i as its base. For each pyramid, the height, of length $h(P, u^{F_i})$, is drawn as a dotted line.

A classical and beautiful result of Minkowski states that the volumes of the facets $V^{d-1}(F)$ along with their normals (directions) u^F determine a polytope as uniquely as possible:

THEOREM 15.2.1 *Minkowski* (cf. [Sch14, pp. 455])

Let $u^1, \ldots, u^n \in S^{d-1}$ be pairwise distinct unit vectors linearly spanning $\mathbb{R}^d$ and let $f_1, \ldots, f_n > 0$ be positive real numbers. There exists a d-dimensional polytope P with n facets $F_1, \ldots, F_n$ such that $F_i = H(P, u^i) \cap P$ and $V^{d-1}(F_i) = f_i$, $1 \le i \le n$, if and only if

$$\sum_{i=1}^{n} f_i \, u^i = 0.$$

Moreover, P is uniquely determined up to translations.

This statement is a particular instance of the modern and "hot" L_p-Minkowski problem for which we refer to [BLYZ13] and the references within.

Note that $V(P) \ge 0$. This holds with strict inequality if and only if the polytope P has full dimension d. The surface area $F(P)$ can also be expressed as

$$F(P) \;=\; \sum_{u \in \mathcal{S}_{d-1}(P)} V^{d-1}(H(P, u) \cap P).$$

Thus for a d-polytope the surface area is the sum of the $(d-1)$-volumes of its facets. If $\dim(P) = d-1$, then $F(P)$ is twice the $(d-1)$-volume of P. One has $F(P) = 0$ if and only if $\dim(P) < d-1$.

TABLE 15.2.1

POLYTOPE	$f_k(\cdot)$	VOLUME	SURFACE AREA
C_d	$2^{d-k}\binom{d}{k}$	2^d	$2d \cdot 2^{d-1}$
C_d^{Δ}	$2^{k+1}\binom{d}{k+1}$	$\frac{2^d}{d!}$	$2^d \dfrac{\sqrt{d}}{(d-1)!}$
T_d	$\binom{d+1}{k+1}$	$\dfrac{\sqrt{d+1}}{d!}$	$(d+1) \cdot \dfrac{\sqrt{d}}{(d-1)!}$

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Both the volume and the surface area are continuous with respect to the Hausdorff metric (as defined in Chap. 2). They are monotone and invariant with respect to rigid motions. The volume is homogeneous of degree d, i.e., $V(\mu P) = \mu^d V(P)$ for $\mu \geq 0$, whereas the surface area is homogeneous of degree $d-1$. For further properties of the functionals $V(\cdot)$ and $F(\cdot)$ see [Had57] and [Sch14].

Table 15.2.1 gives the numbers of k-faces, the volume, and the surface area of the d-cube C_d (with edge length 2), of the cross-polytope C_d^Δ with edge length $\sqrt{2}$, and of the regular simplex T_d with edge length $\sqrt{2}$.

15.2.2 MIXED VOLUMES

GLOSSARY

Volume polynomial: The volume of the Minkowski sum $\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_r P_r$, which is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_r \geq 0$. Here the P_i may be convex polytopes of any dimension, or more general (closed, bounded) convex sets.

Mixed volumes: The suitably normalized coefficients of the volume polynomial of $P_1, \ldots, P_r$.

Normal cone: The normal cone $N(F, P)$ of a face F of a polytope P is the set of all vectors $v \in \mathbb{R}^d$ such that the supporting hyperplane $H(P, v)$ contains F, i.e.,

$$N(F, P) = \left\{ v \in \mathbb{R}^d \;\middle|\; F \subseteq H(P, v) \cap P \right\}.$$

THEOREM 15.2.2 *Mixed Volumes* (cf. [Sch14, pp. 275])

Let $P_1, \ldots, P_r \subseteq \mathbb{R}^d$ be polytopes, $r \geq 1$, and $\lambda_1, \ldots, \lambda_r \geq 0$. The volume of $\lambda_1 P_1 + \cdots + \lambda_r P_r$ is a homogeneous polynomial in $\lambda_1, \ldots, \lambda_r$ of degree d. Thus it can be written in the form

$$V(\lambda_1 P_1 + \cdots + \lambda_r P_r) = \sum_{(i(1),\ldots,i(d)) \in \{1,2,\ldots,r\}^d} \lambda_{i(1)} \cdots \lambda_{i(d)} \cdot V(P_{i(1)}, \ldots, P_{i(d)}),$$

*where the coefficients in this expansion are chosen to be symmetric in their indices. Furthermore, the coefficient $V(P_{i(1)}, \ldots, P_{i(d)})$ depends only on $P_{i(1)}, \ldots, P_{i(d)}$. It is called the **mixed volume** of the polytopes $P_{i(1)}, \ldots, P_{i(d)}$.*

With the abbreviation

$$V(P_1, k_1; \ldots; P_r, k_r) \;:=\; V(\underbrace{P_1, \ldots, P_1}_{k_1 \text{ times}}, \ldots, \underbrace{P_r, \ldots, P_r}_{k_r \text{ times}}),$$

the polynomial becomes

$$V(\lambda_1 P_1 + \cdots + \lambda_r P_r) \;=\; \sum_{\substack{k_1,\ldots,k_r \geq 0 \\ k_1+\cdots+k_r = d}} \binom{d}{k_1, \ldots, k_r} \lambda_1^{k_1} \cdots \lambda_r^{k_r} V(P_1, k_1; \ldots; P_r, k_r).$$

In particular, the volume of the polytope P_i is given by the mixed volume $V(P_1, 0; \ldots; P_i, d; \ldots; P_r, 0)$. The theorem is also valid for arbitrary convex bodies: This is a good example of a result where the general case can be derived from the

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polytope case by approximation. For more about the properties of mixed volumes from different points of view see Schneider [Sch14], Sangwine-Yager [San93], and McMullen [McM93].

The definition of the mixed volumes as coefficients of a polynomial is somewhat unsatisfactory. Schneider gave the following explicit rule, which generalizes an earlier result of Betke [Bet92] for the case $r = 2$. It uses information about the normal cones at certain faces. For this, note that $N(F, P)$ is a finitely generated cone, which can be written explicitly as the sum of the orthogonal complement of $\mathrm{aff}(P)$ and the positive hull of those unit vectors u that are both parallel to $\mathrm{aff}(P)$ and induce supporting hyperplanes $H(P, u)$ that contain a facet of P including F. Thus, for $P \subseteq \mathbb{R}^d$ the dimension of $N(F, P)$ is $d - \dim(F)$.

THEOREM 15.2.3 *Schneider's Summation Formula* [Sch94]

Let $P_1, \ldots, P_r \subseteq \mathbb{R}^d$ be polytopes, $r \geq 2$. Let $x^1, \ldots, x^r \in \mathbb{R}^d$ with $x^1 + \cdots + x^r = 0$, $(x^1, \ldots, x^r) \neq (0, \ldots, 0)$, and

$$\bigcap_{i=1}^{r} \left(\mathrm{relint} N(F_i, P_i) - x^i\right) \;=\; \emptyset$$

whenever F_i is a face of P_i and $\dim(F_1) + \cdots + \dim(F_r) > d$. Then

$$\binom{d}{k_1, \ldots, k_r} V(P_1, k_1; \ldots; P_r, k_r) = \sum_{(F_1, \ldots, F_r)} V(F_1 + \cdots + F_r),$$

where the summation extends over the r-tuples $(F_1, \ldots, F_r)$ of k_i-faces F_i of P_i with $\dim(F_1 + \cdots + F_r) = d$ and $\bigcap_{i=1}^{r} \left(N(F_i, P_i) - x^i\right) \;\neq\; \emptyset$.

The choice of the vectors $x^1, \ldots, x^r$ implies that the selected k_i-faces $F_i \subseteq P_i$ of a summand $F_1 + \cdots + F_r$ are contained in complementary subspaces. Hence one may also write

$$\binom{d}{k_1, \ldots, k_r} V(P_1, k_1; \ldots; P_r, k_r) = \sum_{(F_1, \ldots, F_r)} [F_1, \ldots, F_r] \cdot V^{k_1}(F_1) \cdots V^{k_r}(F_r),$$

where $[F_1, \ldots, F_r]$ denotes the volume of the parallelepiped that is the sum of unit cubes in the affine hulls of $F_1, \ldots, F_r$.

Finally, we remark that the selected sums of faces in the formula of the theorem form a subdivision of the polytope $P_1 + \cdots + P_r$, i.e.,

$$P_1 + \cdots + P_r \;=\; \biguplus_{(F_1, \ldots, F_r)} \left(F_1 + \cdots + F_r\right).$$

See Figure 15.2.2 for an example.

VOLUMES OF ZONOTOPES

If all summands in a Minkowski sum $Z = P_1 + \cdots + P_r$ are line segments, say $P_i = p^i + [0, 1]z^i = \mathrm{conv}\{p^i, p^i + z^i\}$ with $p^i, z^i \in \mathbb{R}^d$ for $1 \leq i \leq r$, then the resulting polytope Z is a zonotope. In this case the summation rule immediately gives $V(P_1, k_1; \ldots; P_r, k_r) = 0$ if the vectors

$$\underbrace{z^1, \ldots, z^1}_{k_1 \text{ times}}, \ldots, \underbrace{z^r, \ldots, z^r}_{k_r \text{ times}}$$

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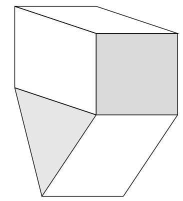
FIGURE 15.2.2
Here the Minkowski sum of a square P_1 and a triangle P_2 is decomposed into translates of P_1 and of P_2 (this corresponds to two summands with $F_1 = P_1$ and $F_2 = P_2$, respectively), together with three "mixed" faces that arise as sums $F_1 + F_2$, where F_1 and F_2 are faces of P_1 and P_2 (corresponding to summands with $\dim(F_1) = \dim(F_2) = 1$*).*



are linearly dependent. This can also be seen directly from dimension considerations. Otherwise, for $k_{i(1)} = k_{i(2)} = \cdots = k_{i(d)} = 1$, say,

$$V(P_1, k_1; \ldots; P_r, k_r) = \frac{1}{d!} \left|\det\left(z^{i(1)}, z^{i(2)}, \ldots, z^{i(d)}\right)\right|.$$

Therefore, one obtains McMullen's formula for the volume of the zonotope Z (cf. Shephard [She74]):

$$V(Z) = \sum_{1 \le i(1) < i(2) < \cdots < i(d) \le r} \left|\det(z^{i(1)}, \ldots, z^{i(d)})\right|.$$

15.2.3 QUERMASSINTEGRALS AND INTRINSIC VOLUMES

GLOSSARY

***i*th quermassintegral $W_i(P)$:** The mixed volume $V(P, d-i; B_d, i)$ of a polytope P and the d-dimensional unit ball B_d.

κ_d: The volume of B_d. Hence $\kappa_0 = 1$, $\kappa_1 = 2$, $\kappa_2 = \pi$, etc.

***i*th intrinsic volume $V_i(P)$:** The $(d-i)$th quermassintegral, scaled by the constant $\binom{d}{i}/\kappa_{d-i}$.

Outer parallel body of P at distance λ: The convex body $P + \lambda B_d$ for some $\lambda > 0$.

External angle $\gamma(F,P)$ at a face F of a polytope P: The volume of the intersection $\big(\mathrm{lin}(F - x^F) + N(F,P)\big) \cap B_d$ divided by κ_d, for $x^F \in \mathrm{relint}(F)$, where $\mathrm{lin}(\cdot)$ denotes the linear hull. Thus $\gamma(F,P)$ is the "fraction of $\mathbb{R}^d$ taken up by $\mathrm{lin}(F - x^F) + N(F,P)$." Equivalently, the external angle at a k-face F is the fraction of the spherical volume of S covered by $N(F,P) \cap S$, where S denotes the $(d-k-1)$-dimensional unit sphere in $\mathrm{lin}(N(F,P))$.

Internal angle $\beta(F,G)$ for faces $F \subseteq G$: The "fraction" of $\mathrm{lin}\{G - x^F\}$ taken up by the cone $\mathrm{pos}\{x - x^F \mid x \in G\}$, for $x^F \in \mathrm{relint}(F)$. A detailed discussion of relations between external and internal angles can be found in McMullen [McM75].

The quermassintegrals are generalizations of both the volume and the surface area of P. In fact, they can also be seen as the continuous convex geometry analogs of face numbers.

For a polytope $P \subseteq \mathbb{R}^d$ and the d-dimensional unit ball B_d, the mixed volume formula, applied to the outer parallel body $P + \lambda B_d$, gives

$$V(P + \lambda B_d) = \sum_{i=0}^{d} \binom{d}{i} \lambda^i W_i(P),$$

with the convention $W_i(P) = V(P, d-i; B_d, i)$. This formula is known as the ***Steiner polynomial***. The mixed volume $W_i(P)$, the ith quermassintegral of P, is an important quantity and of significant geometric interest [Had57, Sch14]. As special cases, $W_0(P) = V(P)$ is the volume, $dW_1(P) = F(P)$ is the surface area, and $W_d(P) = \kappa_d$.

For the geometric interpretation of $W_i(P)$ for polytopes, we use a normalization of the quermassintegrals due to McMullen [McM75]: For $0 \le i \le d$, the ith intrinsic volume of P is defined by

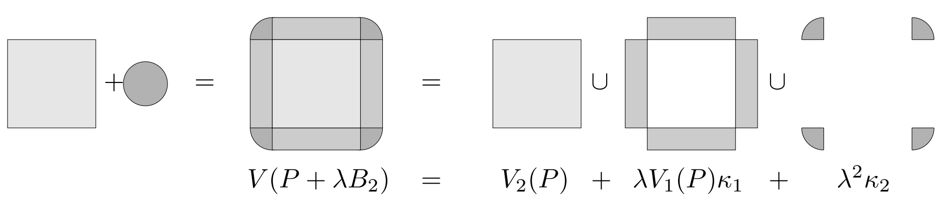
$$V_i(P) \ := \ \frac{\binom{d}{i}}{\kappa_{d-i}} W_{d-i}(P).$$

With this notation the Steiner polynomial can be written as

$$V(P + \lambda B_d) = \sum_{i=0}^{d} \lambda^{d-i} \kappa_{d-i} V_i(P).$$

See Figure 15.2.3 for an example. $V_d(P)$ is the volume of P, $V_{d-1}(P)$ is half the surface area, and $V_0(P) = 1$. One advantage of this normalization is that the intrinsic volumes are unchanged if P is embedded in some Euclidean space of different dimension. Thus, for $\dim(P) = k \le d$, $V_k(P)$ is the ordinary k-volume of P with respect to the Euclidean structure induced in $\operatorname{aff}(P)$.

FIGURE 15.2.3
The Minkowski sum of a square P with a ball λB_2 yields the outer parallel body. This outer parallel body can be decomposed into pieces, whose volumes, $V(P)$, $\lambda V_1(P)\kappa_1$, and $\lambda^2\kappa_2$, correspond to the three terms in the Steiner polynomial.



For a $(\dim(P) - 2)$-face F, the concept of external angle (see the glossary) reduces to the "usual" concept: then the external angle is given by $\frac{1}{2\pi} \arccos\langle u^{F_1}, u^{F_2}\rangle$ for unit normal vectors $u^{F_1}, u^{F_2} \in S^{d-1}$ to the facets F_1, F_2 with $F_1 \cap F_2 = F$. One has $\gamma(P, P) = 1$ for the polytope itself and $\gamma(F, P) = 1/2$ for each facet F. Using this concept, we get

$$V_k(P) = \sum_{F \in \mathcal{F}_k(P)} \gamma(F, P) \cdot V^k(F).$$

Internal and external angles are also useful tools in order to express combinatorial properties of polytopes (see the application below). One classical example is ***Gram's equation*** [Gra74] [Grü67, Sect. 14.1].

$$\sum_{k=0}^{d-1}(-1)^k \sum_{F\in\mathcal{F}_k(P)} \beta(F,P) = (-1)^{d-1}.$$

This formula is quite similar to the Euler relation for the face numbers of a polytope (see Chapter 17). It was discovered by Shephard and by Welzl that Gram's equation follows directly from Euler's relation applied to a random projection [Grü67, p. 315a].

SOME COMPUTATIONS

In principle, one can use the external angle formula to determine the intrinsic volumes of a given polytope, but in general it is hard to calculate external angles. Indeed, for the computation of spherical volumes there are explicit formulas only in small dimensions.

In what follows, we give formulas for the intrinsic volumes of the polytopes C_d, C_d^Δ, and T_d. For this, we identify the k-faces of C_d with the k-cube C_k and the k-faces of C_d^Δ and of T_d with T_k, for $0 \le k < d$.

The case of the cube C_d is rather trivial. Since $\gamma(C_k, C_d) = 2^{-(d-k)}$ one gets (see Table 15.2.1)

$$V_k(C_d) = 2^k\binom{d}{k}.$$

For the regular simplex T_d we have

$$V_k(T_d) = \binom{d+1}{k+1}\cdot\frac{\sqrt{k+1}}{k!}\cdot\gamma(T_k, T_d).$$

An explicit formula for the external angles of a regular simplex by Ruben [Rub60] [Had79] is:

$$\gamma(T_k, T_d) = \sqrt{\frac{k+1}{\pi}}\int_{-\infty}^{\infty} e^{-(k+1)x^2}\left(\frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} e^{-y^2}dy\right)^{d-k} dx.$$

For the regular cross-polytope we find for $k \le d-1$ that

$$V_k(C_d^\Delta) = 2^{k+1}\binom{d}{k+1}\cdot\frac{\sqrt{k+1}}{k!}\cdot\gamma(T_k, C_d^\Delta).$$

For this, the external angles of C_d^Δ were determined by Betke and Henk [BH93]:

$$\gamma(T_k, C_d^\Delta) = \sqrt{\frac{k+1}{\pi}}\int_{0}^{\infty} e^{-(k+1)x^2}\left(\frac{2}{\sqrt{\pi}}\int_{0}^{x} e^{-y^2}dy\right)^{d-k-1} dx.$$

AN APPLICATION

External angles and internal angles play a crucial role in work by Affentranger and Schneider [AS92] (see also [BV94]), who computed the expected number of

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k-faces of the orthogonal projection of a polytope $P \subseteq \mathbb{R}^d$ onto a randomly chosen isotropic subspace of dimension n. Let $E[f_k(P;n)]$ be that number. Then for $0 \le k < n \le d-1$ it was shown that

$$E[f_k(P;n)] \;=\; 2\sum_{m\ge 0}\;\sum_{F\in\mathcal{F}_k(P)}\;\sum_{\substack{G\in\mathcal{F}_{n-1-2m}(P)\\ F\subseteq G}} \beta(F,G)\gamma(G,P),$$

where $\beta(F,G)$ is the internal angle of the face F with respect to a face $G \supseteq F$.

In the sequel we apply the above formula to the polytopes C_d, C_d^{Δ}, and T_d. For the cubes one has $\beta(C_k,C_l) = (1/2)^{l-k}$, while the number of l-faces of C_d containing any given k-face is equal to $\binom{d-k}{l-k}$. Hence

$$E[f_k(C_d;n)] = 2\binom{d}{k}\sum_{m\ge 0}\binom{d-k}{n-1-k-2m}.$$

In particular, $E[f_k(C_d;d-1)] = (2^{d-k}-2)\binom{d}{k}$.

For the cross-polytope C_d^{Δ} the number of l-faces that contain a k-face is equal to $2^{l-k}\binom{d-k-1}{l-k}$. Thus

$$E[f_k(C_d^{\Delta};n)] = \\ 2\binom{d}{k+1}\sum_{m\ge 0} 2^{n-2m}\binom{d-k-1}{n-1-k-2m}\beta(T_k,T_{n-1-2m})\gamma(T_{n-1-2m},C_d^{\Delta}).$$

In the same way one obtains for T_d

$$E[f_k(T_d;n)] = \\ 2\binom{d+1}{k+1}\sum_{m\ge 0}\binom{d-k}{n-1-k-2m}\beta(T_k,T_{n-1-2m})\gamma(T_{n-1-2m},T_d).$$

For the last two formulas one needs the internal angles $\beta(T_k,T_l)$ of the regular simplex T_d, for $0 \le k \le l \le d$, For this, one has the following complex integral [BH99]:

$$\beta(T_k,T_l) = \frac{(k{+}1{+}l)^{1/2}(k{+}1)^{(l-1)/2}}{\pi^{(l+1)/2}}\int_{-\infty}^{\infty} e^{-w^2}\left(\int_0^{\infty} e^{-(k+1)y^2+2iwy}dy\right)^l dw.$$

Using this formula one can determine the asymptotic behavior of $E[f_k(C_d^{\Delta};n)]$ and $E[f_k(T_d;n)]$ as n tends to infinity [BH99].

15.3 SOURCES AND RELATED MATERIAL

FURTHER READING

The classic account of the combinatorial theory of convex polytopes was given by Grünbaum in 1967 [Grü67]. It inspired and guided a great part of the subsequent research in the field. Besides the related chapters of this Handbook, we refer to [Zie95] and the handbook surveys by Klee and Kleinschmidt [KK95] and by Bayer and Lee [BL93] for further reading.

For the geometric theory of convex bodies we refer to the Handbook of Convex Geometry [GW93], to Schneider [Sch14] for an excellent monograph, and as an introduction to modern convex geometry we recommend [Bal97]. For high-dimensional/asymptotic methods and results see also the recent book [AGM15].

As for the algorithmic aspects of computing volumes, etc., we refer to Chapter 36 of this Handbook, on Computational Convexity, and to the additional references given there. For further and recent aspects related to random polytopes we refer to Chapter 12 "Discrete Aspects of Stochastic Geometry" of this Handbook.

RELATED CHAPTERS

Chapter 6: Oriented matroids
Chapter 7: Lattice points and lattice polytopes
Chapter 16: Subdivisions and triangulations of polytopes
Chapter 17: Face numbers of polytopes and complexes
Chapter 18: Symmetry of polytopes and polyhedra
Chapter 19: Polytope skeletons and paths
Chapter 26: Convex hull computations
Chapter 29: Triangulations and mesh generation
Chapter 36: Computational convexity
Chapter 64: Crystals, periodic and aperiodic
Chapter 67: Software

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