13 GEOMETRIC DISCREPANCY THEORY AND UNIFORM DISTRIBUTION

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INTRODUCTION

A sequence $s_1, s_2, \ldots$ in $\mathbf{U} = [0,1)$ is said to be *uniformly distributed* if, in the limit, the number of s_j falling in any given subinterval is proportional to its length. Equivalently, $s_1, s_2, \ldots$ is uniformly distributed if the sequence of equiweighted atomic probability measures $\mu_N(s_j) = 1/N$, supported by the initial N-segments $s_1, s_2, \ldots, s_N$, converges weakly to Lebesgue measure on $\mathbf{U}$. This notion immediately generalizes to any topological space with a corresponding probability measure on the Borel sets.

Uniform distribution, as an area of study, originated from the remarkable paper of Weyl [Wey16], in which he established the fundamental result known nowadays as the Weyl criterion (see [Cas57, KN74]). This reduces a problem on uniform distribution to a study of related exponential sums, and provides a deeper understanding of certain aspects of Diophantine approximation, especially basic results such as Kronecker's density theorem. Indeed, careful analysis of the exponential sums that arise often leads to Erdős-Turán-type upper bounds, which in turn lead to quantitative statements concerning uniform distribution.

Today, the concept of uniform distribution has important applications in a number of branches of mathematics such as number theory (especially Diophantine approximation), combinatorics, ergodic theory, discrete geometry, statistics, numerical analysis, etc. In this chapter, we focus on the geometric aspects of the theory.

13.1 UNIFORM DISTRIBUTION OF SEQUENCES

GLOSSARY

Uniformly distributed: Given a sequence $(s_n)_{n\in\mathbb{N}}$, with $s_n \in \mathbf{U} = [0,1)$, let $Z_N([a,b)) = |\{j \le N \mid s_j \in [a,b)\}|$. The sequence is uniformly distributed if, for every $0 \le a < b \le 1$, $\lim_{N\to\infty} N^{-1} Z_N([a,b)) = b - a$.

Fractional part: The fractional part $\{x\}$ of a real number x is $x - \lfloor x \rfloor$.

Kronecker sequence: A sequence of points of the form $(\{N\alpha_1\}, \ldots, \{N\alpha_k\})_{N\in\mathbb{N}}$ in $\mathbf{U}^k$, where $1, \alpha_1, \ldots, \alpha_k \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$.

Discrepancy, or ***irregularity of distribution:*** The discrepancy of a sequence

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$(s_n)_{n\in\mathbb{N}}$, with $s_n \in \mathbf{U} = [0,1)$, in a subinterval $[a,b)$ of $\mathbf{U}$, is

$$\Delta_N([a,b)) = |Z_N([a,b)) - N(b-a)|.$$

More generally, the discrepancy of a sequence $(s_n)_{n\in\mathbb{N}}$, with $s_n \in \mathcal{S}$, a topological probability space, in a measurable subset $A \subset \mathcal{S}$, is $\Delta_N(A) = |Z_N(A) - N\mu(A)|$, where $Z_N(A) = |\{j \le N \mid s_j \in A\}|$.

Aligned rectangle, aligned triangle: A rectangle (resp. triangle) in $\mathbb{R}^2$ two sides of which are parallel to the coordinate axes.

Hausdorff dimension: A set S in a metric space has Hausdorff dimension m, where $0 \le m \le +\infty$, if

(i) for any $0 < k < m$, $\mu_k(S) > 0$;

(ii) for any $m < k < +\infty$, $\mu_k(S) < +\infty$.

Here, μ_k is the k-dimensional ***Hausdorff measure***, given by

$$\mu_k(S) = 2^{-k}\kappa_k \liminf_{\epsilon\to 0}\left\{ \sum_{i=1}^{\infty}(\text{diam } S_i)^k \;\middle|\; S \subset \bigcup_{i=1}^{\infty} S_i,\ \text{diam } S_i \le \epsilon \right\},$$

where κ_k is the volume of the unit ball in $\mathbb{E}^k$.

Remark. Throughout this chapter, the symbol c will always represent the generic absolute positive constant, depending only on the indicated parameters. The value generally varies from one appearance to the next.

It is not hard to prove that for any irrational number α, the sequence of fractional parts $\{N\alpha\}$ is everywhere dense in $\mathbf{U}$ (here N is the running index). Suppose that the numbers $1, \alpha_1, \ldots, \alpha_k$ are linearly independent over $\mathbb{Q}$. Then Kronecker's theorem states that the k-dimensional Kronecker sequence $(\{N\alpha_1\}, \ldots, \{N\alpha_k\})$ is dense in the unit k-cube $\mathbf{U}^k$. It is a simple consequence of the Weyl criterion that any such Kronecker sequence is uniformly distributed in $\mathbf{U}^k$, a far stronger result than the density theorem. For example, letting $k = 1$, we see that $\{N\sqrt{2}\}$ is uniformly distributed in $\mathbf{U}$.

Weyl's work led naturally to the question: How rapidly can a sequence in $\mathbf{U}$ become uniformly distributed as measured by the discrepancy $\Delta_N([a,b))$ of subintervals? Here, $\Delta_N([a,b)) = |Z_N([a,b)) - N(b-a)|$, where $Z_N([a,b))$ counts those $j \le N$ for which s_j lies in $[a,b)$. Thus we see that Δ_N measures the difference between the actual number of s_j in an interval and the expected number. The sequence is uniformly distributed if and only if $\Delta_N(I) = o(N)$ for all subintervals I. The notion of discrepancy immediately extends to any topological probability space, provided there is at hand a suitable collection of measurable sets $\mathcal{J}$ corresponding to the intervals. If A is in $\mathcal{J}$, set $\Delta_N(A) = |Z_N(A) - N\mu(A)|$.

From the works of Hardy, Littlewood, Ostrowski, and others, it became clear that the smaller the partial quotients in the continued fractions of the irrational number α are, the more uniformly distributed the sequence $\{N\alpha\}$ is. For instance, the partial quotients of quadratic irrationals are characterized by being cyclic, hence bounded. Studying the behavior of $\{N\alpha\}$ for these numbers has proved an excellent indicator of what might be optimal for general sequences in $\mathbf{U}$. Here one has $\Delta_N(I) < c(\alpha)\log N$ for all intervals I and integers $N \ge 2$. Unfortunately, one does not have anything corresponding to continued fractions in higher dimensions, and this has been an obstacle to a similar study of Kronecker sequences (see [Bec94]).

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Van der Corput gave an alternative construction of a super uniformly distributed sequence of rationals in $\mathbf{U}$ for which $\Delta_N(I) < c\log N$ for all intervals I and integers $N \geq 2$ (see [KN74, p. 127]). He also asked for the best possible estimate in this direction. In particular, he posed

PROBLEM 13.1.1 *Van der Corput Problem* [Cor35a] [Cor35b]

Can there exist a sequence for which $\Delta_N(I) < c$ for all N and I?

He conjectured, in a slightly different formulation, that such a sequence could not exist. This conjecture was affirmed by van Aardenne-Ehrenfest [A-E45], who later showed that for any sequence in $\mathbf{U}$, $\sup_I \Delta_N(I) > c\log\log N/\log\log\log N$ for infinitely many values of N [A-E49]. Her pioneering work gave the first nontrivial lower bound on the discrepancy of general sequences in $\mathbf{U}$. It is trivial to construct a sequence for which $\sup_I \Delta_N(I) \leq 1$ for infinitely many values of N.

In a classic paper, Roth showed that for any infinite sequence in $\mathbf{U}$, it must be true that $\sup_I \Delta_N(I) > c(\log N)^{1/2}$ for infinitely many N. Finally, in another classic paper, Schmidt used an entirely new method to prove the following result.

THEOREM 13.1.2 *Schmidt* [Sch72b]

The inequality $\sup_I \Delta_N(I) > c\log N$ holds for infinitely many N.

For a more detailed discussion of work arising from the van der Corput conjecture, see [BC87, pp. 3–6].

In light of van der Corput's sequence, as well as $\{N\sqrt{2}\}$, Schmidt's result is best possible. The following problem, which has been described as "excruciatingly difficult," is a major remaining open question from the classical theory.

PROBLEM 13.1.3

Extend Schmidt's result to a best possible estimate of the discrepancy for sequences in $\mathbf{U}^k$ for $k > 1$.

For a given sequence, the results above do not imply the existence of a fixed interval I in $\mathbf{U}$ for which $\sup_N \Delta_N(I) = \infty$. Let I_α denote the interval $[0, \alpha)$, where $0 < \alpha \leq 1$. Schmidt [Sch72a] showed that for any fixed sequence in $\mathbf{U}$ there are only countably many values of α for which $\Delta_N(I_\alpha)$ is bounded. The best result in this direction is due to Halász.

THEOREM 13.1.4 *Halász* [Hal81]

For any fixed sequence in $\mathbf{U}$, let A denote the set of values of α for which $\Delta_N(I_\alpha) = o(\log N)$. Then A has Hausdorff dimension 0.

For a more detailed discussion of work arising from this question, see [BC87, pp. 10–11].

The fundamental works of Roth and Schmidt opened the door to the study of discrepancy in higher dimensions, and there were surprises. In his classic paper, Roth [Rot54] transformed the heart of van der Corput's problem to a question concerning the unit square $\mathbf{U}^2$. In this new formulation, Schmidt's "$\log N$ theorem" implies that if N points are placed in $\mathbf{U}^2$, there is always an aligned rectangle $I = [\gamma_1, \alpha_1) \times [\gamma_2, \alpha_2)$ having discrepancy exceeding $c\log N$. Roth also showed that it was possible to place N points in the square $\mathbf{U}^2$ so that the discrepancy of no aligned rectangle exceeds $c\log N$. One way is to choose $p_j = ((j-1)/N, \{j\sqrt{2}\})$ for $j \leq N$. Thus, the function $c\log N$ describes the *minimax discrepancy* for aligned rectangles. However, Schmidt showed that there is always an aligned right triangle (the part of an aligned rectangle above, or below, a diagonal) with discrepancy exceeding $cN^{1/4-\epsilon}$! Later work has shown that $cN^{1/4}$ exactly describes the minimax discrepancy of aligned right triangles. This paradoxical behavior is not isolated.

Generally, if one studies a collection $\mathcal{J}$ of "nice" sets such as disks, aligned boxes, rotated cubes, etc., in $\mathbf{U}^k$ or some other convex region, it turns out that the minimax discrepancy is either bounded above by $c(\log N)^r$ or bounded below by cN^s, with nothing halfway. In $\mathbf{U}^k$, typically $s = (k-1)/2k$. Thus, there tends to be a logarithmic version of the Vapnik-Chervonenkis principle in operation (see Chapter 40 of this Handbook for a related discussion). Later, we shall see how certain geometric properties place $\mathcal{J}$ in one or the other of these two classes.

13.2 THE GENERAL FREE PLACEMENT PROBLEM FOR *N* POINTS

One can ask for bounds on the discrepancy of N variable points $\mathcal{P} = \{p_1, p_2, \ldots, p_N\}$ that are freely placed in a domain $\mathbf{K}$ in Euclidean t-space $\mathbb{E}^t$. By contrast, when one considers the discrepancy of a *sequence* in $\mathbf{K}$, the initial n-segment of $p_1, \ldots, p_n, \ldots, p_N$ remains fixed for $n \leq N$ as new points appear with increasing N. For a given $\mathbf{K}$, as the unit interval $\mathbf{U}$ demonstrates, estimates for these two problems are quite different as functions of N. The freely placed points in $\mathbf{U}$ need never have discrepancy exceeding 1.

With Roth's reformulation (discussed in Section 13.3), the classical problem is easier to state and, more importantly, it generalizes in a natural manner to a wide class of problems. The bulk of geometric discrepancy problems are now posed as free placement problems. In practically all situations, the domain $\mathbf{K}$ has a very simple description as a cube, disk, sphere, etc., and standard notation is used in the specific situations.

PROBABILITY MEASURES AND DISCREPANCY

In a free placement problem there are two probability measures in play. First, there is the atomic measure μ^+ that assigns weight $1/N$ to each p_j. Second, there is a probability measure μ^- on the Borel sets of $\mathbf{K}$. The measure μ^- is generally the restriction of a natural uniform measure, such as scaled Lebesgue measure. An example would be given by $\mu^- = \sigma/4\pi$ on the unit sphere $\mathbf{S}^2$, where σ is the usual surface measure. It is convenient to define the signed measure $\mu = \mu^+ - \mu^-$ (in the previous section μ^- was denoted by μ). The discrepancy of a Borel set A is, as before, given by $\Delta(A) = |Z(A) - N\mu^-(A)| = N|\mu(A)|$.

The function Δ is always restricted to a very special collection $\mathcal{J}$ of sets, and the challenge lies in obtaining estimates concerning the restricted Δ. It is the central importance of the collection $\mathcal{J}$ that gives the study of discrepancy its distinct character. In a given problem it is sometimes possible to reduce the size of $\mathcal{J}$. Taking the unit interval $\mathbf{U}$ as an example, letting $\mathcal{J}$ be the collection of intervals $[\gamma, \alpha)$ seems to be the obvious choice. But a moment's reflection shows that only intervals of the form $I_\alpha = [0, \alpha)$ need be considered for estimates of discrepancy. At most a factor of 2 is introduced in any estimate of bounds.

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NOTIONS OF DISCREPANCY

In most interesting problems $\mathcal{J}$ itself carries a measure ν in the sense of integral geometry, and this adds much more structure. While there is artistic latitude in the choice of ν, more often than not there is a natural measure on $\mathcal{J}$. In the example of $\mathbf{U}$, by identifying $I_\alpha = [0, \alpha)$ with its right endpoint, it is clear that Lebesgue measure on $\mathbf{U}$ is the natural choice for ν.

Given that the measure ν exists, for $0 < W < \infty$ define

$$\|\Delta(\mathcal{P}, \mathcal{J})\|_W = \left(\int_{\mathcal{J}} (\Delta(A))^W d\nu \right)^{1/W} \qquad \text{and} \qquad \|\Delta(\mathcal{P}, \mathcal{J})\|_\infty = \sup_{\mathcal{J}} \Delta(A),$$

and for $0 < W \leq \infty$ define

$$D(\mathbf{K}, \mathcal{J}, W, N) = \inf_{|\mathcal{P}|=N} \{\|\Delta(\mathcal{P}, \mathcal{J})\|_W\}. \tag{13.2.1}$$

The determination of the "minimax" $D(\mathbf{K}, \mathcal{J}, \infty, N)$ is generally the most important as well as the most difficult problem in the study. It should be noted that the function $D(\mathbf{K}, \mathcal{J}, \infty, N)$ is defined even if the measure ν is not. The term $D(\mathbf{K}, \mathcal{J}, 2, N)$ has been shown to be intimately related to problems in numerical integration in some special cases, and is of increasing importance. These various functions $D(\mathbf{K}, \mathcal{J}, W, N)$ measure how well the continuous distribution μ^- can be approximated by N freely placed atoms.

For $W \geq 1$, the inequality

$$\nu(\mathcal{J})^{-1/W} \|\Delta(\mathcal{P}, \mathcal{J})\|_W \leq \|\Delta(\mathcal{P}, \mathcal{J})\|_\infty \tag{13.2.2}$$

provides a general approach for obtaining a lower bound for $D(\mathbf{K}, \mathcal{J}, \infty, N)$. The choice $W = 2$ has especially fruitful, but good estimates of $D(\mathbf{K}, \mathcal{J}, W, N)$ for any W are of independent interest.

An upper bound on $D(\mathbf{K}, \mathcal{J}, \infty, N)$ generally is obtained by showing the existence of a favorable example. This may be done either by a direct construction, often extremely difficult to verify, or by a probabilistic argument showing such an example does exist without giving it explicitly. These comments would apply as well to upper bounds for any $D(\mathbf{K}, \mathcal{J}, W, N)$.

13.3 ALIGNED RECTANGLES IN THE UNIT SQUARE

The unit square $\mathbf{U}^2 = [0, 1) \times [0, 1)$ is by far the most thoroughly studied 2-dimensional object. The main reason for this is Roth's reformulation of the van der Corput problem. Many of the interesting questions that arose have been answered, and we give a summary of the highlights.

For $\mathbf{U}^2$ one wishes to study the discrepancy of rectangles of the type $I = [\gamma_1, \alpha_1) \times [\gamma_2, \alpha_2)$. It is a trivial observation that only those I for which $\gamma_1 = \gamma_2 = 0$ need be considered, and this restricted family, denoted by $\mathcal{B}^2$, is the choice for $\mathcal{J}$. By considering this smaller collection one introduces at most a factor of 4 on bounds. There is a natural measure ν on $\mathcal{B}^2$, which may be identified with Lebesgue measure on $\mathbf{U}^2$ via the upper right corner points (α_1, α_2). In the same spirit, let $\mathcal{B}^1$ denote the previously introduced collection of intervals $I_\alpha = [0, \alpha)$ in $\mathbf{U}$.

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THEOREM 13.3.1 *Roth's Equivalence* [Rot54] [BC87, pp. 6–7]

Let f be a positive increasing function tending to infinity. Then the following two statements are equivalent:

(i) *There is an absolute positive constant c_1 such that for any finite sequence $s_1, s_2, \ldots, s_N$ in $\mathbf{U}$, there always exists a positive integer $n \leq N$ such that $\|\Delta(\mathcal{P}_n, \mathcal{B}^1)\|_\infty > c_1 f(N)$. Here, $\mathcal{P}_n$ is the initial n-segment.*

(ii) *There is an absolute positive constant c_2 such that for all positive integers N, $D(\mathbf{U}^2, \mathcal{B}^2, \infty, N) > c_2 f(N)$.*

The equivalence shows that the central question of bounds for the van der Corput problem can be replaced by an elegant problem concerning the free placement of N points in the unit square $\mathbf{U}^2$. The mapping $s_j \to ((j-1)/N, s_j)$ plays a role in the proof of this equivalence. If one takes as $\mathcal{P}_N$ the image in $\mathbf{U}^2$ under the mapping of the initial N-segment of the van der Corput sequence, the following upper bound theorem may be proved.

THEOREM 13.3.2 *Lerch* [BC87, Theorem 4, $K = 2$]

For $N \geq 2$,

$$D(\mathbf{U}^2, \mathcal{B}^2, \infty, N) < c \log N. \tag{13.3.1}$$

The corresponding lower bound is established by the important "$\log N$ theorem" of Schmidt.

THEOREM 13.3.3 *Schmidt* [Sch72b] [BC87, Theorem 3B]

One has

$$D(\mathbf{U}^2, \mathcal{B}^2, \infty, N) > c \log N. \tag{13.3.2}$$

By an explicit lattice construction, Davenport [Dav56] gave the best possible upper bound estimate for $W = 2$. His analysis shows that if the irrational number α has continued fractions with bounded partial quotients, then the $N = 2M$ points in $\mathbf{U}^2$ given by

$$p_j^{\pm} = ((j-1)/M, \{\pm j\alpha\}), \qquad j \leq M,$$

can be taken as $\mathcal{P}$ in proving the following theorem. Other proofs have been given by Vilenkin [Vil67], Halton and Zaremba [HZ69], and Roth [Rot76].

THEOREM 13.3.4 *Davenport* [Dav56] [BC87, Theorem 2A]

For $N \geq 2$,

$$D(\mathbf{U}^2, \mathcal{B}^2, 2, N) < c(\log N)^{1/2}. \tag{13.3.3}$$

This complements the following lower bound obtained by Roth in his classic paper.

THEOREM 13.3.5 *Roth* [Rot54] [BC87, Theorem 1A, $K = 2$]

One has

$$D(\mathbf{U}^2, \mathcal{B}^2, 2, N) > c(\log N)^{1/2}. \tag{13.3.4}$$

For $W = 1$, an upper bound $D(\mathbf{U}^2, \mathcal{B}^2, 1, N) < c(\log N)^{1/2}$ follows at once from Davenport's bound (13.3.3) by the monotonicity of $D(\mathbf{U}^2, \mathcal{B}^2, W, N)$ as a function of W. The corresponding lower bound was obtained by Halász.

THEOREM 13.3.6 *Halász* [Hal81] [BC87, Theorem 1C, $K = 2$]

One has

$$D(\mathbf{U}^2, \mathcal{B}^2, 1, N) > c(\log N)^{1/2}. \tag{13.3.5}$$

Halász (see [BC87, Theorem 3C]) deduced that there is always an aligned square of discrepancy larger than $c \log N$. Of course, the square generally will not be a member of the special collection $\mathcal{B}^2$. Ruzsa [Ruz93] has given a clever elementary proof that the existence of such a square follows directly from inequality (13.3.2) above; see also [Mat99].

The ideas developed in the study of discrepancy can be applied to approximations of integrals. We briefly mention two examples, both restricted to 2 dimensions for the sake of simplicity.

A function ψ is termed ***M-simple*** if $\psi(x) = \sum_{j=1}^{M} m_j \chi_{B_j}(x)$, where χ_{B_j} is the characteristic function of the aligned rectangle B_j. In this theorem, the lower bounds are nontrivial because of the logarithmic factors coming from discrepancy theory on $\mathbf{U}^2$.

THEOREM 13.3.7 *Chen* [Che85] [Che87] [BC87, Theorems 5A, 5C]

Let the function f be defined on $\mathbf{U}^2$ by $f(x) = C + \int_{B(x)} g(y)dy$ where C is a constant, g is nonzero on a set of positive measure in $\mathbf{U}^2$, and $B(x_1, x_2) = [0, x_1) \times [0, x_2)$. Then, for any M-simple function ψ,

$$\|f - \psi\|_W > c(f, W) M^{-1} (\log M)^{1/2}, \qquad 1 \le W < \infty;$$
$$\|f - \psi\|_\infty > c(f) M^{-1} \log M.$$

Let $\mathcal{C}$ be the class of all continuous real valued functions on $\mathbf{U}^2$, endowed with the Wiener sheet measure ω. For every function $f \in \mathcal{C}$ and every set $\mathcal{P}$ of N points in $\mathbf{U}^2$, let

$$I(f) = \int_{\mathbf{U}^2} f(x)dx \qquad \text{and} \qquad U(\mathcal{P}, f) = \frac{1}{N} \sum_{p \in \mathcal{P}} f(p).$$

THEOREM 13.3.8 *Woźniakowski* [Woź91]

One has

$$\inf_{|\mathcal{P}|=N} \left(\int_{\mathcal{C}} |U(\mathcal{P}, f) - I(f)|^2 d\omega \right)^{1/2} = \frac{D(\mathbf{U}^2, \mathcal{B}^2, 2, N)}{N}.$$

13.4 ALIGNED BOXES IN A UNIT k-CUBE

The van der Corput problem led to the study of $D(\mathbf{U}^2, \mathcal{B}^2, W, N)$, which in turn led to the study of $D(\mathbf{U}^k, \mathcal{B}^k, W, N)$ for $W > 0$ and general positive integers k. Here, $\mathcal{B}^k$ denotes the collection of boxes $I = [0, \alpha_1) \times \ldots \times [0, \alpha_k)$, and the measure ν is identified with Lebesgue measure on $\mathbf{U}^k$ via the corner points $(\alpha_1, \ldots, \alpha_k)$.

The principle of Roth's equivalence extends so that the discrepancy problem for sequences in $\mathbf{U}^k$ reformulates as a free placement problem in $\mathbf{U}^{k+1}$, so that we discuss only the latter version. Inequalities (13.3.1)–(13.3.5) give the exact order of magnitude of $D(\mathbf{U}^2, \mathcal{B}^2, W, N)$ for the most natural values of W, namely

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$1 \le W \le 2$ and $W = \infty$, with the latter being top prize. While much is known, knowledge of $D(\mathbf{U}^k, \mathcal{B}^k, W, N)$ is incomplete, especially for $W = \infty$, while there is ongoing work on the case $W = 1$ which may lead to its complete solution. It should be remarked that if k and N are fixed, then $D(\mathbf{U}^k, \mathcal{B}^k, W, N)$ is a nondecreasing function of the positive real number W.

As was indicated earlier, upper bound methods generally fall into two classes, explicit constructions and probabilistic existence arguments. In practice, careful constructions are made prior to a probabilistic averaging process. Chen's proof of the following upper bound theorem involved extensive combinatorial and number-theoretic constructions as well as probabilistic considerations.

THEOREM 13.4.1 *Chen* [Che80] [BC87, Theorem 2D]

For positive real numbers W, and integers $k \ge 2$ and $N \ge 2$,

$$D(\mathbf{U}^k, \mathcal{B}^k, W, N) < c(W, k)(\log N)^{(k-1)/2}. \tag{13.4.1}$$

A second proof was given by Chen [Che83] (see also [BC87, Section 3.5]), where the idea of digit shifts was first used in the subject. Earlier, Roth [Rot80] (see also [BC87, Theorem 2C]) treated the case $W = 2$. The inequality (13.4.1) highlights one of the truly baffling aspects of the theory, namely the apparent jump discontinuity in the asymptotic behavior of $D(\mathbf{U}^k, \mathcal{B}^k, W, N)$ at $W = \infty$. This discontinuity is most dramatically established for $k = 2$, but is known to occur for any $k \ge 3$ (see (13.4.4) below).

Explicit multidimensional sequences greatly generalizing the van der Corput sequence also have been used to obtain upper bounds for $D(\mathbf{U}^k, \mathcal{B}^k, \infty, N)$. Halton constructed explicit point sets in $\mathbf{U}^k$ in order to prove the next theorem. Faure (see [BC87, Section 3.2]) gave a different proof of the same result.

THEOREM 13.4.2 *Halton* [Hal60] [BC87, Theorem 4]

For integers $k \ge 2$ and $N \ge 2$,

$$D(\mathbf{U}^k, \mathcal{B}^k, \infty, N) < c(k)(\log N)^{k-1}. \tag{13.4.2}$$

In order to prove (13.3.3), Davenport used properties of special lattices in 2 dimensions. However, it took many years before we had success with lattices in higher dimensions. Skriganov has established some most interesting results, which imply the following theorem. Given a region, a lattice is termed ***admissible*** if the region contains no member of the lattice except possibly the origin (see [Cas59]). Examples for the following theorem are given by lattices arising from algebraic integers in totally real algebraic number fields.

THEOREM 13.4.3 *Skriganov* [Skr94]

Suppose Γ is a fixed k-dimensional lattice admissible for the region $|x_1 x_2 \ldots x_k| < 1$.

(i) *Halton's upper bound inequality* (13.4.2) *holds if the N points are obtained by intersecting $\mathbf{U}^k$ with $t\Gamma$, where $t > 0$ is a suitably chosen real scalar.*

(ii) *With the same choice of t as in part* (i)*, there exists $x \in \mathbb{E}^k$ such that Chen's upper bound inequality* (13.4.1) *holds if the N points are obtained by intersecting $\mathbf{U}^k$ with $t\Gamma + x$.*

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Later, using p-adic Fourier-Walsh analysis together with ideas originating from coding theory, Chen and Skriganov [CS02] have obtained explicit constructions that give (13.4.1) in the special case $W = 2$, with an explicitly given constant $c(2, k)$. The extension to arbitrary positive real numbers W was given by Skriganov [Skr06]. Alternative proofs of these results, using dyadic Fourier-Walsh analysis, were given by Dick and Pillichshammer [DP14] in the special case $W = 2$, and by Dick [Dic14] for arbitrary positive real numbers W.

Moving to lower bound estimates, the following theorem of Schmidt is complemented by Chen's result (13.4.1). For $W \geq 2$ this lower bound is due to Roth, since D is monotone in W.

THEOREM 13.4.4 *Schmidt* [Sch77a] [BC87, Theorem 1B]

For $W > 1$ and integers $k \geq 2$,

$$D(\mathbf{U}^k, \mathcal{B}^k, W, N) > c(W, k)(\log N)^{(k-1)/2}. \tag{13.4.3}$$

Concerning $W = 1$, there is the result of Halász, which is probably not optimal. It is reasonably conjectured that $(k-1)/2$ is the correct exponent.

THEOREM 13.4.5 *Halász* [Hal81] [BC87, Theorem 1C]

For integers $k \geq 2$,

$$D(\mathbf{U}^k, \mathcal{B}^k, 1, N) > c(k)(\log N)^{1/2}.$$

The next lower bound estimate, although probably not best possible, firmly establishes a discontinuity in asymptotic behavior at $W = \infty$ for all $k \geq 3$.

THEOREM 13.4.6 *Bilyk, Lacey, Vagharshakyan* [BLV08]

For integers $k \geq 3$, there exist constants $\delta_k \in (0, 1/2)$ such that

$$D(\mathbf{U}^k, \mathcal{B}^k, \infty, N) > c(k)(\log N)^{(k-1)/2+\delta_k}. \tag{13.4.4}$$

Earlier, Beck [Bec89] had established a weaker lower bound for the case $k = 3$, of the form

$$D(\mathbf{U}^k, \mathcal{B}^k, \infty, N) > c(3)(\log N)(\log \log N)^{c_3},$$

where c_3 can be taken to be any positive real number less than $1/8$. These bounds represent the first improvements of Roth's lower bound

$$D(\mathbf{U}^k, \mathcal{B}^k, \infty, N) > c(k)(\log N)^{(k-1)/2},$$

established over 60 years ago.

Can the factor $1/2$ be removed from the exponent? This is the "great open problem." Recently, there has been evidence that suggests that perhaps

$$D(\mathbf{U}^k, \mathcal{B}^k, \infty, N) > c(k)(\log N)^{k/2}, \tag{13.4.5}$$

but no more. To discuss this, we need to modify the definition (13.2.1) in light of the idea of digit shifts introduced by Chen [Che83]. Let $\mathcal{S}$ be a finite set of dyadic digit shifts. For every point set $\mathcal{P}$ and every dyadic shift $S \in \mathcal{S}$, let $\mathcal{P}(S)$ denote the image of $\mathcal{P}$ under S. Corresponding to (13.2.1), let

$$E(\mathbf{K}, \mathcal{J}, W, N, \mathcal{S}) = \inf_{|\mathcal{P}|=N} \sup_{S \in \mathcal{S}} \{\|\Delta(\mathcal{P}(S), \mathcal{J})\|_W\}.$$

In Skriganov [Skr16], it is shown that for every N, there exists a finite set $\mathcal{S}$ of dyadic digit shifts, depending only on N and k, such that

$$E(\mathbf{U}^k, \mathcal{B}^k, \infty, N, \mathcal{S}) > c(k)(\log N)^{k/2}, \tag{13.4.6}$$

an estimate consistent with the suggestion (13.4.5). In the same paper, it is also shown that for every fixed $W > 0$ and for every N, there exists a finite set $\mathcal{S}$ of dyadic digit shifts, depending only on N, k and W, such that

$$E(\mathbf{U}^k, \mathcal{B}^k, W, N, \mathcal{S}) > c(W, k)(\log N)^{(k-1)/2}, \tag{13.4.7}$$

somewhat extending (13.4.3).

Beck has also refined Roth's estimate in a geometric direction.

THEOREM 13.4.7 *Beck* [BC87, Theorem 19A]

Let $\mathcal{J}$ be the collection of aligned cubes contained in $\mathbf{U}^k$. Then

$$D(\mathbf{U}^k, \mathcal{J}, \infty, N) > c(k)(\log N)^{(k-1)/2}. \tag{13.4.8}$$

Actually, Beck's method shows $D(\mathbf{U}^k, \mathcal{J}, 2, N) > c(k)(\log N)^{(k-1)/2}$, with respect to a natural measure ν on sets of aligned cubes. This improves Roth's inequality $D(\mathbf{U}^k, \mathcal{B}^k, 2, N) > c(k)(\log N)^{(k-1)/2}$. So far, it has not been possible to extend Ruzsa's ideas to higher dimensions in order to show that the previous theorem follows directly from Roth's estimate. However, Drmota [Drm96] has published a new proof that $D(\mathbf{U}^k, \mathcal{J}, 2, N) > c(k)D(\mathbf{U}^k, \mathcal{B}^k, 2, N)$, and this does imply (13.4.8).

13.5 MOTION-INVARIANT PROBLEMS

In this section and the next three, we discuss collections $\mathcal{J}$ of convex sets having the property that any set in $\mathcal{J}$ may be moved by a direct (orientation preserving) motion of $\mathbb{E}^k$ and yet remain in $\mathcal{J}$. Motion-invariant problems were first extensively studied by Schmidt, and many of his estimates, obtained by a difficult technique using integral equations, were close to best possible. The book [BC87] contains an account of Schmidt's methods. Later, the Fourier transform method of Beck has achieved results that in general surpass those obtained by Schmidt. For a broad class of problems, Beck's Fourier method gives nearly best possible estimates for $D(\mathbf{K}, \mathcal{J}, 2, N)$.

The pleasant surprise is that if $\mathcal{J}$ is motion-invariant, then the bounds on $D(\mathbf{K}, \mathcal{J}, \infty, N)$ turn out to be very close to those for $D(\mathbf{K}, \mathcal{J}, 2, N)$. This is shown by a probabilistic upper bound method, which generally pins $D(\mathbf{K}, \mathcal{J}, \infty, N)$ between bounds differing at most by a factor of $c(k)(\log N)^{1/2}$.

The simplest motion-invariant example is given by letting $\mathcal{J}$ be the collection of all directly congruent copies of a given convex set A. In this situation, $\mathcal{J}$ carries a natural measure ν, which may be identified with Haar measure on the motion group on $\mathbb{E}^k$. A broader choice would be to let $\mathcal{J}$ be all sets in $\mathbb{E}^k$ directly similar to A. Again, there is a natural measure ν on $\mathcal{J}$. However, for the results stated in the next two sections, the various measures ν on the choices for $\mathcal{J}$ will not be discussed in great detail. In most situations, such measures do play an active role in the proofs through inequality (13.2.2) with $W = 2$. A complete exposition of integration in the context of integral geometry, Haar measure, etc., may be found in the book by Santaló [San76].

For any domain $\mathbf{K}$ in $\mathbb{E}^t$ and each collection $\mathcal{J}$, it is helpful to define three auxiliary collections:

Definition:

(i) $\mathcal{J}_{tor}$ consists of those subsets of $\mathbf{K}$ obtained by reducing elements of $\mathcal{J}$ modulo $\mathbb{Z}^k$. To avoid messiness, let us always suppose that $\mathcal{J}$ has been restricted so that this reduction is 1–1 on each member of $\mathcal{J}$. For example, one might consider only those members of $\mathcal{J}$ having diameter less than 1.

(ii) $\mathcal{J}_c$ consists of those subsets of $\mathbf{K}$ that are members of $\mathcal{J}$.

(iii) $\mathcal{J}_i$ consists of those subsets of $\mathbf{K}$ obtained by intersecting $\mathbf{K}$ with members of $\mathcal{J}$.

Note that $\mathcal{J}_c$ and $\mathcal{J}_i$ are well defined for any domain $\mathbf{K}$. However, $\mathcal{J}_{tor}$ essentially applies only to $\mathbf{U}^k$. If viewed as a flat torus, then $\mathbf{U}^k$ is the proper domain for Kronecker sequences and Weyl's exponential sums. There are several general inequalities for discrepancy results involving $\mathcal{J}_{tor}$, $\mathcal{J}_c$, and $\mathcal{J}_i$. For example, we have $D(\mathbf{U}^k, \mathcal{J}_c, \infty, N) \leq D(\mathbf{U}^k, \mathcal{J}_{tor}, \infty, N)$ because $\mathcal{J}_c$ is contained in $\mathcal{J}_{tor}$. Also, if the members of $\mathcal{J}$ have diameters less than 1, then we have $D(\mathbf{U}^k, \mathcal{J}_{tor}, \infty, N) \leq 2^k D(\mathbf{U}^k, \mathcal{J}_i, \infty, N)$, since any set in $\mathcal{J}_{tor}$ is the union of at most 2^k sets in $\mathcal{J}_i$.

13.6 SIMILAR OBJECTS IN THE UNIT k-CUBE

GLOSSARY

If A is a compact convex set in $\mathbb{E}^k$, let $d(A)$ denote the diameter of A, $r(A)$ denote the radius of the largest k-ball contained in A, and $\sigma(\partial A)$ denote the surface content of ∂A. The collection $\mathcal{J}$ is said to be ***ds-generated*** by A if $\mathcal{J}$ consists of all directly similar images of A having diameters not exceeding $d(A)$.

We state two pivotal theorems of Beck. As usual, if $\mathcal{S}$ is a discrete set, $Z(B)$ denotes the cardinality of $B \cap \mathcal{S}$.

THEOREM 13.6.1 *Beck* [Bec87] [BC87, Theorem 17A]

Let $\mathcal{S}$ be an arbitrary infinite discrete set in $\mathbb{E}^k$, A be a compact convex set with $r(A) \geq 1$, and $\mathcal{J}$ be ds-generated by A. Then there is a set B in $\mathcal{J}$ such that

$$|Z(B) - \operatorname{vol} B| > c(k)(\sigma(\partial A))^{1/2}. \tag{13.6.1}$$

COROLLARY 13.6.2 *Beck* [BC87, Corollary 17B]

Let A be a compact convex body in $\mathbb{E}^k$ with $r(A) \geq N^{-1/k}$, and let $\mathcal{J}$ be ds-generated by A. Then

$$D(\mathbf{U}^k, \mathcal{J}_{tor}, \infty, N) > c(A)N^{(k-1)/2k}. \tag{13.6.2}$$

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The deduction of Corollary 13.6.2 from Theorem 13.6.1 involves a simple rescaling argument. Another important aspect of Beck's work is the introduction of upper bound methods based on probabilistic considerations. The following result shows that Theorem 13.6.1 is very nearly best possible.

THEOREM 13.6.3 *Beck* [BC87, Theorem 18A]

Let A be a compact convex body in $\mathbb{E}^k$ with $r(A) \geq 1$, and let $\mathcal{J}$ be ds-generated by A. Then there exists an infinite discrete set $\mathcal{S}_0$ such that for every set B in $\mathcal{J}$,

$$|Z(B) - \operatorname{vol} B| < c(k)(\sigma(\partial A))^{1/2}(\log \sigma(\partial A))^{1/2}. \tag{13.6.3}$$

COROLLARY 13.6.4 *Beck* [BC87, Corollary 18C]

Let A be a compact convex body in $\mathbb{E}^k$, and $\mathcal{J}$ be ds-generated by A. Then

$$D(\mathbf{U}^k, \mathcal{J}_{tor}, \infty, N) < c(A)N^{(k-1)/2k}(\log N)^{1/2}. \tag{13.6.4}$$

Beck (see [BC87, pp. 129–130]) deduced several related corollaries from Theorem 13.6.3. The example sets $\mathcal{P}_N$ for Corollary 13.6.4 can be taken as the initial segments of a certain fixed sequence whose choice definitely depends on A. If $d(A) = \lambda$ and A is either a disk (solid sphere) or a cube, then the right side of (13.6.2) takes the form $c(k)(\lambda^k N)^{(k-1)/2k}$. Montgomery [Mon89] has obtained a similar lower bound for cubes and disks.

The problem of estimating discrepancy for $\mathcal{J}_c$ is even more challenging because of "boundary effects." We state, as an example, a theorem for disks. The right inequality follows from (13.6.4).

THEOREM 13.6.5 *Beck* [Bec87] [BC87, Theorem 16A]

Let $\mathcal{J}$ be ds-generated by a k-disk. Then for every $\epsilon > 0$,

$$c_1(k,\epsilon)N^{(k-1)/2k-\epsilon} < D(\mathbf{U}^k, \mathcal{J}_c, \infty, N) < c_2(k)N^{(k-1)/2k}(\log N)^{1/2}. \tag{13.6.5}$$

Because all the lower bounds above come from $\mathbf{L}^2$ estimates, these various results (13.6.1) – (13.6.5) allow us to make the general statement that for W in the range $2 \leq W \leq \infty$, the magnitude of $D(\mathbf{U}^k, \mathcal{J}, W, N)$ is controlled by $N^{(k-1)/2k}$. Thus there is no extreme discontinuity in asymptotic behavior at $W = \infty$. However, work by Beck and Chen proves that there is a discontinuity at some W satisfying $1 \leq W \leq 2$, and the following results indicate that $W = 1$ is a likely candidate.

THEOREM 13.6.6 *Beck, Chen* [BC93b]

Let $\mathcal{J}$ be ds-generated by a convex polygon A with $d(A) < 1$. Then

$$\begin{aligned} D(\mathbf{U}^2, \mathcal{J}_{tor}, W, N) &< c(A,W)N^{(W-1)/2W}, \qquad 1 < W \leq 2; \\ D(\mathbf{U}^2, \mathcal{J}_{tor}, 1, N) &< c(A)(\log N)^2. \end{aligned} \tag{13.6.6}$$

In fact, Theorem 13.6.6 is motivated by the study of discrepancy with respect to halfplanes, and is established by ideas used to establish Theorem 13.9.8 below. Note the similarities of the inequalities (13.6.6) and (13.9.5). After all, a convex polygon is the intersection of a finite number of halfplanes, and so the proof of Theorem 13.6.6 involves carrying out the idea of the proof of Theorem 13.9.8 a finite number of times.

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The next theorem shows that powers of N other than $N^{(k-1)/2k}$ may appear for $2 \le W \le \infty$. It deals with what has been termed the ***isotropic discrepancy*** in $\mathbf{U}^k$.

THEOREM 13.6.7 *Schmidt* [Sch75] [BC87, Theorem 15]
Let $\mathcal{J}$ be the collection of all convex sets in $\mathbb{E}^k$. Then

$$D(\mathbf{U}^k, \mathcal{J}_i, \infty, N) > c(k) N^{(k-1)/(k+1)}. \tag{13.6.7}$$

The function $N^{(k-1)/(k+1)}$ dominates $N^{(k-1)/2k}$, so that this largest possible choice for $\mathcal{J}$ does in fact yield a larger discrepancy. Beck has shown by probabilistic techniques that the inequality (13.6.7), excepting a possible logarithmic factor, is best possible for $k = 2$.

The following result shows that for certain rotation-invariant $\mathcal{J}$ the discrepancy of Kronecker sequences (defined in Section 13.1) will not behave as $cN^{(k-1)/2k}$, but as the square of this quantity.

THEOREM 13.6.8 *Larcher* [Lar91]
Let the sequence of point sets $\mathcal{P}_N$ be the initial segments of a Kronecker sequence in $\mathbf{U}^k$, and let $\mathcal{J}$ be ds-generated by a cube of edge length $\lambda < 1$. Then, for each N,

$$\|\Delta(\mathcal{P}_N, \mathcal{J}_i)\|_\infty > c(k) \lambda^{k-1} N^{(k-1)/k}.$$

Furthermore, the exponent $(k-1)/k$ cannot be increased.

13.7 CONGRUENT OBJECTS IN THE UNIT k-CUBE

GLOSSARY

If $\mathcal{J}$ consists of all directly congruent copies of a convex set A, we say that A ***dm-generates*** $\mathcal{J}$. Simple examples are given by the collection of all k-disks of a fixed radius r or by the collection of all k-cubes of a fixed edge length λ.

Given a convex set A, there is some evidence for the conjecture that the discrepancy for the dm-generated collection will be essentially as large as that for the ds-generated collection. However, this is generally very difficult to establish, even in very specific situations. There are the following results in this direction. The upper bound inequalities all come from Corollary 13.6.4 above.

THEOREM 13.7.1 *Beck* [BC87, Theorem 22A]
Let $\mathcal{J}$ be dm-generated by a square of edge length λ. Then

$$c_1(\lambda) N^{1/8} < D(\mathbf{U}^2, \mathcal{J}_{tor}, \infty, N) < c_2(\lambda) N^{1/4} (\log N)^{1/2}.$$

It is felt that $N^{1/4}$ gives the proper lower bound, and for $\mathcal{J}_i$ this is definitely true. The lower bound in the next result follows at once from the work of Alexander [Ale91] described in Section 13.9.

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THEOREM 13.7.2 *Alexander, Beck*
Let $\mathcal{J}$ be dm-generated by a k-cube of edge length λ. Then

$$c_1(\lambda, k)N^{(k-1)/2k} < D(\mathbf{U}^k, \mathcal{J}_i, \infty, N) < c_2(\lambda, k)N^{(k-1)/2k}(\log N)^{1/2}.$$

A similar result probably holds for k-disks, but this has been established only for $k = 2$.

THEOREM 13.7.3 *Beck* [BC87, Theorem 22B]
Let $\mathcal{J}$ be dm-generated by a 2-disk of radius r. Then

$$c_1(r)N^{1/4} < D(\mathbf{U}^2, \mathcal{J}_i, \infty, N) < c_2(r)N^{1/4}(\log N)^{1/2}.$$

13.8 WORK OF MONTGOMERY

It should be reported that Montgomery [Mon89] has independently developed a lower bound method which, as does Beck's method, uses techniques from harmonic analysis. Montgomery's method, especially in dimension 2, obtains for a number of special classes $\mathcal{J}$ estimates comparable to those obtained by Beck's method. In particular, Montgomery has considered $\mathcal{J}$ that are ds-generated by a region whose boundary is a piecewise smooth simple closed curve.

13.9 HALFSPACES AND RELATED OBJECTS

GLOSSARY

Segment: Given a compact subset $\mathbf{K}$ and a closed halfspace H in $\mathbb{E}^k$, $K \cap H$ is called a segment of $\mathbf{K}$.

Slab: The region between two parallel hyperplanes.

Spherical slice: The intersection of two open hemispheres on a sphere.

Let H be a closed halfspace in $\mathbb{E}^k$. Then the collection $\mathcal{H}^k$ of all closed halfspaces is dm-generated by H, and if we associate H with the oriented hyperplane ∂H, there is a well-known invariant measure ν on $\mathcal{H}^k$. Further information concerning this and related measures may be found in Chapter 12 of Santaló [San76]. For a compact domain $\mathbf{K}$ in $\mathbb{E}^k$, it is clear that only the collection $\mathcal{H}_i^k$, the *segments* of $\mathbf{K}$, are proper for study, since $\mathcal{H}_c^k$ is empty and $\mathcal{H}_{tor}^k$ is unsuitable.

In this section, it is necessary for the domain $\mathbf{K}$ to be somewhat more general; hence we make only the following broad assumptions:

(i) $\mathbf{K}$ lies on the boundary of a fixed convex set $\mathbf{M}$ in $\mathbb{E}^{k+1}$;

(ii) $\sigma(\mathbf{K}) = 1$, where σ is the usual k-measure on $\partial\mathbf{M}$.

Since $\mathbb{E}^k$ is the boundary of a halfspace in $\mathbb{E}^{k+1}$, any set in $\mathbb{E}^k$ of unit Lebesgue k-measure satisfies these assumptions. The normalization of assumption (ii) is for convenience, and, by rescaling, the inequalities of this section may be applied to any uniform probability measure on a domain $\mathbf{K}$ in $\mathbb{E}^{k+1}$. Such rescaling only affects dimensional constants; for standard domains, such as the unit k-sphere $\mathbf{S}^k$ and the unit k-disk $\mathbf{D}^k$, this will be done without comment.

Although in applications $\mathbf{K}$ will have a simple geometric description, the next theorem treats the general situation and obtains the essentially exact magnitude of $D(\mathbf{K}, \mathcal{H}_i^{k+1}, 2, N)$. If $\mathbf{K}$ lies in $\mathbb{E}^k$, then $\mathcal{H}^{k+1}$ may be replaced by $\mathcal{H}^k$. If ν is properly normalized, this change invokes no rescaling.

THEOREM 13.9.1 *Alexander* [Ale91]

Let $\mathcal{K}$ be the collection of all $\mathbf{K}$ satisfying assumptions (i) *and* (ii) *above. Then*

$$c_1(k)N^{(k-1)/2k} < \inf_{K\in\mathcal{K}} D(\mathbf{K}, \mathcal{H}_i^{k+1}, 2, N) < c_2(\mathbf{M})N^{(k-1)/2k}. \tag{13.9.1}$$

The upper bound of (13.9.1) can be proved by an indirect probabilistic method introduced by Alexander [Ale72] for $\mathbf{K} = \mathbf{S}^2$, but the method of Beck and Chen [BC90] also may be applied for standard choices of $\mathbf{K}$ such as $\mathbf{U}^k$ and $\mathbf{D}^k$. When $\mathbf{M} = \mathbf{K} = \mathbf{S}^k$, the segments are the spherical caps. For this important special case the upper bound is due to Stolarsky [Sto73], while the lower bound is due to Beck [Bec84] (see also [BC87, Theorem 24B]).

Since the ν-measure of the halfspaces that separate $\mathbf{M}$ is less than $c(k)d(\mathbf{M})$, inequality (13.2.2) may be applied to obtain a lower bound for $D(\mathbf{K}, \mathcal{H}_i^{k+1}, \infty, N)$. The upper bound in the following theorem should be taken in the context of actual applications such as $\mathbf{M}$ being a k-sphere $\mathbf{S}^k$, a compact convex body in $\mathbb{E}^k$, or more generally, a compact convex hypersurface in $\mathbb{E}^{k+1}$.

THEOREM 13.9.2 *Alexander, Beck*

Let $\mathcal{K}$ be the collection of $\mathbf{K}$ satisfying assumptions (i) *and* (ii) *above. Furthermore, suppose that $\mathbf{M}$ is of finite diameter. Then*

$$c_3(k)(d(\mathbf{M}))^{-1/2}N^{(k-1)/2k} < \inf_{K\in\mathcal{K}} D(\mathbf{K}, \mathcal{H}_i^{k+1}, \infty, N) < c_4(\mathbf{M})N^{(k-1)/2k}(\log N)^{1/2}. \tag{13.9.2}$$

For $\mathbf{M} = \mathbf{K} = \mathbf{S}^k$, inequalities (13.9.2) are due to Beck, improving a slightly weaker lower bound by Schmidt [Sch69]. Consideration of $\mathbf{K} = \mathbf{U}^2$ makes it obvious that there exists an aligned right triangle with discrepancy at least $cN^{1/4}$, as stated in Section 13.1. For the case $\mathbf{M} = \mathbf{K} = \mathbf{D}^2$, a unit 2-disk (Roth's *disk-segment problem*), Beck [Bec83] (see also [BC87, Theorem 23A]) obtained inequalities (13.9.2), excepting a factor $(\log N)^{-7/2}$ in the lower bound. Later, Alexander [Ale90] improved the lower bound, and Matoušek [Mat95] obtained essentially the same upper bound. Matoušek's work on $\mathbf{D}^2$ makes it seem likely that Beck's factor $(\log N)^{1/2}$ in his general upper bound theorem might be removable in many specific situations, but this is very challenging.

THEOREM 13.9.3 *Alexander, Matoušek*

For Roth's disk-segment problem,

$$c_1N^{1/4} < D(\mathbf{D}^2, \mathcal{H}_i^2, \infty, N) < c_2N^{1/4}. \tag{13.9.3}$$

Alexander's lower bound method, by the nature of the convolutions employed, gives information on the discrepancy of slabs. This is especially apparent in the

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work of Chazelle, Matoušek, and Sharir, who have developed a more direct and geometrically transparent version of Alexander's method. The following theorem on the discrepancy of thin slabs is a corollary to their technique. It is clear that if a slab has discrepancy Δ, then one of the two bounding halfspaces has discrepancy at least $\Delta/2$.

THEOREM 13.9.4 *Chazelle, Matoušek, Sharir* [CMS95]
Let N points lie in the unit cube $\mathbf{U}^k$. Then there exists a slab $\mathbf{T}$ of width $c_1(k)N^{-1/k}$ such that $\Delta(\mathbf{T}) > c_2(k)N^{(k-1)/2k}$.

Alexander [Ale94] has investigated the effect of the dimension k on the discrepancy of halfspaces, and obtained somewhat complicated inequalities that imply the following result.

THEOREM 13.9.5 *Alexander*
For the lower bounds in inequalities (13.9.1) *and* (13.9.2) *above, there is an absolute positive constant c such that one may choose $c_1(k) > ck^{-3/4}$ and $c_3(k) > ck^{-1}$.*

Schmidt [Sch69] studied the discrepancy of spherical slices (the intersection of two open hemispheres) on $\mathbf{S}^k$. Associating a hemisphere with its pole, Schmidt identified ν with the normalized product measure on $\mathbf{S}^k \times \mathbf{S}^k$. Blümlinger [Blü91] demonstrated a surprising relationship between halfspace (spherical cap) and slice discrepancy for $\mathbf{S}^k$. However, his definition for ν in terms of Haar measure on $SO(k+1)$ differed somewhat from Schmidt's.

THEOREM 13.9.6 *Blümlinger*
Let $\mathcal{S}^k$ be the collection of slices of $\mathbf{S}^k$. Then

$$c(k)D(\mathbf{S}^k, \mathcal{H}_i^{k+1}, 2, N) < D(\mathbf{S}^k, \mathcal{S}^k, 2, N). \tag{13.9.4}$$

For the next result, the left inequality follows from inequalities (13.2.2), (13.9.1), and (13.9.4). Blümlinger uses a version of Beck's probabilistic method to establish the right inequality.

THEOREM 13.9.7 *Blümlinger*
For slice discrepancy on $\mathbf{S}^k$,

$$c_1(k)N^{(k-1)/2k} < D(\mathbf{S}^k, \mathcal{S}^k, \infty, N) < c_2(k)N^{(k-1)/2k}(\log N)^{1/2}.$$

Grabner [Gra91] has given an Erdős-Turán type upper bound on spherical cap discrepancy in terms of spherical harmonics. This adds to the considerable body of results extending inequalities for exponential sums to other sets of orthonormal functions, and thereby extends the Weyl theory.

All of the results so far in this section treat $2 \leq W \leq \infty$. For W in the range $1 \leq W < 2$ there is mystery, but we do have the following result, related to inequality (13.6.6), showing that a dramatic change in asymptotic behavior occurs in the range $1 \leq W \leq 2$. For $\mathbf{U}^2$, Beck and Chen show that regular grid points will work for the upper bound example for $W = 1$, and they are able to modify their method to apply to any bounded convex domain in $\mathbb{E}^2$.

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THEOREM 13.9.8 *Beck, Chen* [BC93a]

Let $\mathbf{K}$ *be a bounded convex domain in* $\mathbb{E}^2$. *Then*

$$\begin{aligned} D(\mathbf{K}, \mathcal{H}_i^2, W, N) &< c(\mathbf{K}, W) N^{(W-1)/2W}, \qquad 1 < W \leq 2; \\ D(\mathbf{K}, \mathcal{H}_i^2, 1, N) &< c(\mathbf{K})(\log N)^2. \end{aligned} \tag{13.9.5}$$

13.10 BOUNDARIES OF GENERATORS FOR HOMOTHETICALLY INVARIANT J

We have already noted several factors that play a role in determining whether $D(\mathbf{K}, \mathcal{J}, W, N)$ behaves like N^r as opposed to $(\log N)^s$. Beck's work shows that if $\mathcal{J}$ is dm-generated, $D(\mathbf{K}, \mathcal{J}, \infty, N)$ behaves as N^r. However, the work of Beck and Chen clearly shows that if W is sufficiently small, then even for motion-invariant $\mathcal{J}$, it may be that $D(\mathbf{K}, \mathcal{J}, W, N)$ is bounded above by $(\log N)^s$.

Beck [Bec88] has extensively studied $D(\mathbf{U}^2, \mathcal{J}_{tor}, \infty, N)$ under the assumption that $\mathcal{J}$ is homothetically invariant, and in this section we shall record some of the results obtained.

It turns out that the boundary shape of a generator is the critical element in determining to which, if either, class $\mathcal{J}$ belongs. Remarkably, for the "typical" homothetically invariant class $\mathcal{J}$, $D(\mathbf{U}^2, \mathcal{J}_{tor}, \infty, N)$ oscillates infinitely often to be larger than $N^{1/4-\epsilon}$ and smaller than $(\log N)^{4+\epsilon}$.

GLOSSARY

The convex set A ***h-generates*** $\mathcal{J}$ if $\mathcal{J}$ consists of all homothetic images B of A with $d(B) \leq d(A)$.

Blaschke-Hausdorff metric: The metric on the space CONV(2) of all compact convex sets in $\mathbb{E}^2$ in which the distance between two sets is the minimum distance from any point of one set to the other.

A set is of ***first category*** if it is a countable union of nowhere dense sets.

If one considers the two examples of $\mathcal{J}$ being h-generated by an aligned square and by a disk, previously stated results make it very likely that shape strongly affects discrepancy for homothetically invariant $\mathcal{J}$. The first two theorems quantify this phenomenon for two very standard boundary shapes, first polygons, then smooth closed curves.

THEOREM 13.10.1 *Beck* [Bec88] [BC87, Corollary 20D]

Let $\mathcal{J}$ *be h-generated by a convex polygon* A. *Then, for any* $\epsilon > 0$,

$$D(\mathbf{U}^2, \mathcal{J}_{tor}, \infty, N) = o((\log N)^{4+\epsilon}). \tag{13.10.1}$$

Beck and Chen [BC89] have given a less complicated argument that obtains $o((\log N)^{5+\epsilon})$ on the right side of (13.10.1).

THEOREM 13.10.2 *Beck* [Bec88] [BC87, Corollary 19F]

Let A *be a compact convex set in* $\mathbb{E}^2$ *with a twice continuously differentiable boundary curve having strictly positive curvature. If* A *h-generates* $\mathcal{J}$, *then for* $N \geq 2$,

$$D(\mathbf{U}^2, \mathcal{J}_{tor}, \infty, N) > c(A) N^{1/4} (\log N)^{-1/2}. \tag{13.10.2}$$

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For sufficiently smooth positively curved bodies, Drmota [Drm93] has extended (13.10.2) into higher dimensions and also removed the logarithmic factor. Thus, he obtains a lower bound of the form $c(A)N^{(k-1)/2k}$, along with the standard upper bound obtained by Beck's probabilistic method.

Let CONV(2) denote the usual locally compact space of all compact convex sets in $\mathbb{E}^2$ endowed with the Blaschke-Hausdorff metric. There is the following surprising result, which quantifies the oscillatory behavior mentioned above.

THEOREM 13.10.3 *Beck* [BC87, Theorem 21]

Let $\epsilon > 0$ be given. For all A in CONV(2)*, excepting a set of first category, if $\mathcal{J}$ is h-generated by A, then each of the following two inequalities is satisfied infinitely often:*

(i) $D(\mathbf{U}^2, \mathcal{J}_{tor}, \infty, N) < (\log N)^{4+\epsilon}$.

(ii) $D(\mathbf{U}^2, \mathcal{J}_{tor}, \infty, N) > N^{1/4}(\log N)^{-(1+\epsilon)/2}$.

In fact, the final theorem of this section will say more about the rationale of such estimates.

The next theorem gives the best lower bound estimate known if it is assumed only that the generator A has nonempty interior, certainly a minimal hypothesis.

THEOREM 13.10.4 *Beck* [Bec88] [BC87, Corollary 19G]

If $\mathcal{J}$ is h-generated by a compact convex set A having positive area, then

$$D(\mathbf{U}^2, \mathcal{J}_{tor}, \infty, N) > c(A)(\log N)^{1/2}.$$

Possibly the right side should be $c(A)\log N$, which would be best possible as the example of aligned squares demonstrates. Lastly, we discuss the important theorem underlying most of these results about h-generated $\mathcal{J}$. Let A be a member of CONV(2) with nonempty interior, and for each integer $l \geq 3$ let A_l be an inscribed l-gon of maximal area. The Nth ***approximability number*** $\xi_N(A)$ is defined as the smallest integer l such that the area of $A \setminus A_l$ is less than l^2/N.

THEOREM 13.10.5 *Beck* [Bec88] [BC87, Corollary 19H, Theorem 20C]

Let A be a member of CONV(2) *with nonempty interior. Then if $\mathcal{J}$ is h-generated by A, we have*

$$c_1(A)(\xi_N(A))^{1/2}(\log N)^{-1/4} < D(\mathbf{U}^2, \mathcal{J}_{tor}, \infty, N) < c_2(A,\epsilon)\xi_N(A)(\log N)^{4+\epsilon}. \tag{13.10.3}$$

The proof of the preceding fundamental theorem, which is in fact the join of two major theorems, is long, but the import is clear; namely, that for h-generated $\mathcal{J}$, if one understands $\xi_N(A)$, then one essentially understands $D(\mathbf{U}^2, \mathcal{J}_{tor}, \infty, N)$. If $\xi_N(A)$ remains nearly constant for long intervals, then A acts like a polygon and D will drift below $(\log N)^{4+2\epsilon}$. If, at some stage, ∂A behaves as if it consists of circular arcs, then $\xi_N(A)$ will begin to grow as $cN^{1/2}$.

For still more information concerning the material in this section, along with the proofs, see [BC87, Chapter 7]. Károlyi [Kár95a, Kár95b] has extended the idea of approximability number to higher dimensions and obtained upper bounds analogous to those in (13.10.3).

13.11 $D(\mathsf{K},J,2,N)$ IN LIGHT OF DISTANCE GEOMETRY

Although knowledge of $D(\mathbf{K},\mathcal{J},\infty,N)$ is our highest aim, in the great majority of problems this is achieved by first obtaining bounds on $D(\mathbf{K},\mathcal{J},2,N)$. In this section, we briefly show how this function fits nicely into the theory of metric spaces of negative type. In our situation, the distance between points will be given by a Crofton formula with respect to the measure ν on $\mathcal{J}$. This approach evolved from a paper written in 1971 by Alexander and Stolarsky investigating extremal problems in distance geometry, and has been developed in a number of subsequent papers by both authors studying special cases. However, we reverse history and leap immediately to a formulation suitable for our present purposes. We avoid mention of certain technical assumptions concerning $\mathcal{J}$ and ν which cause no difficulty in practice.

Assume that $\mathbf{K}$ is a compact convex set in $\mathbb{E}^k$ and that $\mathcal{J} = \mathcal{J}_c$. This latter assumption causes no loss of generality since one can always just redefine $\mathcal{J}$. Let ν, as usual, be a measure on $\mathcal{J}$, with the further assumption that $\nu(\mathcal{J}) < \infty$.

Definition: If p and q are points in $\mathbf{K}$, the set A in $\mathcal{J}$ is said to ***separate*** p and q if A contains exactly one of these two points. The ***distance function*** ρ on $\mathbf{K}$ is defined by the Crofton formula $\rho(p,q) = (1/2)\,\nu\{J \mid J \text{ separates } p \text{ and } q\}$, and if μ is any signed measure on $\mathbf{K}$ having finite positive and negative parts, one defines the functional $\mathbf{I}(\mu)$ by

$$\mathbf{I}(\mu) = \iint \rho(p,q)d\mu(p)d\mu(q).$$

With these definitions one obtains the following representation for $\mathbf{I}(\mu)$.

THEOREM 13.11.1 *Alexander* [Ale91]

One has

$$\mathbf{I}(\mu) = \int_{\mathcal{J}} \mu(A)\mu(\mathbf{K}\setminus A)d\nu(A). \tag{13.11.1}$$

For μ satisfying the condition of total mass zero, $\int_K d\mu = 0$, the integrand in (13.11.1) becomes $-(\mu(A))^2$. The signed measures $\mu = \mu^+ - \mu^-$ that we are considering, with μ^- being a uniform probability measure on $\mathbf{K}$ and μ^+ consisting of N atoms of equal weight $1/N$, certainly have total mass zero. Here one has $\Delta(A) = N\mu(A)$. Hence there is the following corollary.

COROLLARY 13.11.2

For the signed measures μ presently considered, if $\mathcal{P}$ denotes the N points supporting μ^+, then

$$-N^2\mathbf{I}(\mu) = \int_{\mathcal{J}} (\Delta(A))^2 d\nu(A) = (\|\Delta(\mathcal{P},\mathcal{J})\|_2)^2. \tag{13.11.2}$$

Thus if one studies the metric ρ, it may be possible to prove that $-\mathbf{I}(\mu) > f(N)$, whence it follows that $(D(\mathbf{K},\mathcal{J},2,N))^2 > N^2 f(N)$. If $\mathcal{J}$ consists of the halfspaces of $\mathbb{E}^k$, then ρ is the Euclidean metric. In this important special case, Alexander [Ale91] was able to make good estimates. Chazelle, Matoušek, and Sharir [CMS95] and A.D. Rogers [Rog94] contributed still more techniques for treating the halfspace problem.

If μ_1 and μ_2 are any two signed measures of total mass 1 on $\mathbf{K}$, then one can define the ***relative discrepancy*** $\Delta(A) = N(\mu_1(A) - \mu_2(A))$. The first equality of (13.11.2) still holds if $\mu = \mu_1 - \mu_2$. A signed measure μ_0 of total mass 1 is termed ***optimal*** if it solves the integral equation $\int_K \rho(x,y)d\mu(y) = \lambda$ for some positive number λ. If an optimal measure μ_0 exists, then $\mathbf{I}(\mu_0) = \lambda$ maximizes $\mathbf{I}$ on the class of all signed Borel measures of total mass 1 on $\mathbf{K}$. In the presence of an optimal measure, one has the following very pretty identity.

THEOREM 13.11.3 *Generalized Stolarsky Identity*

Suppose that the measure μ_0 is optimal on $\mathbf{K}$, and that μ is any signed measure of total mass 1 on $\mathbf{K}$. If Δ is the relative discrepancy with respect to μ_0 and μ, then

$$N^2\mathbf{I}(\mu) + \int_{\mathcal{J}} (\Delta(A))^2 d\nu(A) = N^2\mathbf{I}(\mu_0). \tag{13.11.3}$$

The first important example of this formula is due to Stolarsky [Sto73] where he treated the sphere $\mathbf{S}^k$, taking as μ the uniform atomic measure supported by N variable points. For $\mathbf{S}^k$ it is clear that the uniform probability measure μ_0 is optimal. His integrals involving the spherical caps are equivalent, up to a scale factor, to integrals with respect to the measure on the halfspaces of $\mathbb{E}^k$ for which ρ is the Euclidean metric. Stolarsky's tying of a geometric extremal problem to Schmidt's work on the discrepancy of spherical caps was a major step forward in the study of discrepancy and of distance geometry.

Very little has been done to investigate the deeper nature of the individual metrics ρ determined by classes $\mathcal{J}$ other than halfspaces. They are all metrics of ***negative type***, which essentially means that $\mathbf{I}(\mu) \leq 0$ if μ has total mass 0. There is a certain amount of general theory, begun by Schoenberg and developed by a number of others, but it does not apply directly to the problem of estimating discrepancy.

13.12 UNIFORM PLACEMENT OF POINTS ON SPHERES

As demonstrated by Stolarsky, formula (13.11.3) shows that if one places N points on $\mathbf{S}^k$ so that the sum of all distances is maximized, then $D(\mathbf{S}^k, \mathcal{H}_i^k, 2, N)$ is achieved by this arrangement. Berman and Hanes [BH77] have given a pretty algorithm that searches for optimal configurations. For $k = 2$, while the exact configurations are not known for $N \geq 5$, this algorithm appears to be successful for $N \leq 50$. For such an N surprisingly few rival configurations will be found. Lubotzky, Phillips, and Sarnak [LPS86] have given an algorithm, based on iterations of a specially chosen element in $SO(3)$, which can be used to place many thousands of reasonably well distributed points on $\mathbf{S}^2$. Difficult analysis shows that these points are well placed, but not optimally placed, relative to $\mathcal{H}_i^2$. On the other hand, it is shown that these points are essentially optimally placed with respect to a nongeometric operator discrepancy. Data concerning applications to numerical integration are also included in the paper. More recently, Rakhmanov, Saff, and Zhou [RSZ94] have studied the problem of placing points uniformly on a sphere relative to optimizing certain functionals, and they state a number of interesting conjectures.

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In yet another theoretical direction, the existence of very well distributed point sets on $\mathbf{S}^k$ allows the sphere, after difficult analysis, to be closely approximated by equi-edged zonotopes (sums of line segments). The papers of Wagner [Wag93] and of Bourgain and Lindenstrauss [BL93] treat this problem.

13.13 COMBINATORIAL DISCREPANCY

GLOSSARY

A ***2-coloring*** of $\mathbf{X}$ is a mapping $\chi : \mathbf{X} \to \{-1, 1\}$. For each such χ there is a natural integer-valued set function μ_χ on the finite subsets of $\mathbf{X}$ defined by $\mu_\chi(A) = \sum_{x \in A} \chi(x)$, and if $\mathcal{J}$ is a given family of finite subsets of $\mathbf{X}$ we define

$$D(\mathbf{X}, \mathcal{J}) = \min_{\chi} \max_{A \in \mathcal{J}} |\mu_\chi(A)|.$$

Degree: If $\mathcal{J}$ is a collection of subsets of a finite set X, $\deg \mathcal{J} = \max\{|\mathcal{J}(x)| \bigm| x \in \mathbf{X}\}$, where $\mathcal{J}(x)$ is the subcollection consisting of those members of $\mathcal{J}$ that contain x.

The collection $\mathcal{J}$ ***shatters*** a set $S \subset X$ if, for any given subset $B \subset S$, there exists A in $\mathcal{J}$ such that $B = A \cap S$. The ***VC-dimension*** of $\mathcal{J}$ is defined by $\dim_{vc} \mathcal{J} = \max\{|S| \bigm| S \subset \mathbf{X},\ \mathcal{J} \text{ shatters } S\}$. For $m \leq |\mathbf{X}|$, the ***primal shatter function*** $\pi_{\mathcal{J}}$ is defined by

$$\pi_{\mathcal{J}}(m) = \max_{\substack{Y \subset \mathbf{X} \\ |Y| \leq m}} |\{Y \cap A \mid A \in \mathcal{J}\}|.$$

The ***dual shatter function*** is defined by $\pi^*_{\mathcal{J}}(m) = \pi_{\mathcal{J}^*}(m)$, where $\mathbf{X}^* = \mathcal{J}$, and $\mathcal{J}^* = \{\mathcal{J}(x) \mid x \in \mathbf{X}\}$.

Techniques in combinatorial discrepancy theory have proved very powerful in this geometric setting. Here one 2-colors a discrete set and studies the discrepancy of a special class $\mathcal{J}$ of subsets as measured by $|\#\text{red} - \#\text{blue}|$. If one 2-colors the first N positive integers, then the beautiful "1/4 theorem" of Roth [Rot64] says that there will always be an arithmetic progression having discrepancy at least $cN^{1/4}$. This result should be compared to van der Waerden's theorem, which says that there is a long monochromatic progression, whose discrepancy obviously will be its length. However, it is known that this length need not be more than $\log N$, and the minimax might be as small as $\log\log\ldots\log N$ (here the number of iterated logarithms may be arbitrarily large). Moreover, general results concerning combinatorial discrepancy, for example, those that use the Vapnik-Chervonenkis dimension, are very useful in computational geometry; cf. Chapter 47.

Combinatorial discrepancy theory involves discrepancy estimates arising from 2-colorings of a set $\mathbf{X}$. Upper bound estimates of combinatorial discrepancy have proved to be very helpful in obtaining upper bound estimates of geometric discrepancy. In this final section we briefly discuss various properties of the collection $\mathcal{J}$ that lead to useful upper bound estimates of combinatorial discrepancy.

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The simplest property of the collection $\mathcal{J}$ is its cardinality $|\mathcal{J}|$. Here, Spencer obtained a fine result.

THEOREM 13.13.1 *Spencer* [AS93]
Let $\mathbf{X}$ *be a finite set. If* $|\mathcal{J}| \geq |\mathbf{X}|$, *then*

$$D(\mathbf{X}, \mathcal{J}) \leq c \left(|\mathbf{X}| \log \left(1 + \frac{|\mathcal{J}|}{|\mathbf{X}|} \right) \right)^{1/2}.$$

Applications and extensions of the following theorem may be found in [BC87, Chapter 8].

THEOREM 13.13.2 *Beck, Fiala* [BF81] [BC87, Lemma 8.5.]
Let $\mathbf{X}$ *be a finite set. Then*

$$D(\mathbf{X}, \mathcal{J}) \leq 2 \deg \mathcal{J} - 1.$$

Since $\pi_{\mathcal{J}}(m) = 2^m$ if and only if $\dim_{vc} \mathcal{J} \geq m$, the function $\pi_{\mathcal{J}}$ contains much more information than does VC-dimension alone. If $\dim_{vc} \mathcal{J} = d$, then $\pi_{\mathcal{J}}(m)$ is polynomially bounded by cm^d. However, in many geometric situations this bound on the shatter function can be improved, leading to better discrepancy bounds. Detailed discussions may be found in the papers by Haussler and Welzl [HW87] and by Chazelle and Welzl [CW89].

Dual objects are defined in the usual manner (see Glossary). We state several results.

THEOREM 13.13.3 *Matoušek, Welzl, Wernisch* [MWW93]
Suppose that $(\mathbf{X}, \mathcal{J})$ *is a finite set system with* $|\mathbf{X}| = n$. *If* $\pi_{\mathcal{J}}(m) \leq c_1 m^d$ *for* $m \leq n$, *then*

$$\begin{aligned} D(\mathbf{X}, \mathcal{J}) &\leq c_2 n^{(d-1)/2d} (\log n)^{1+1/2d}, && d > 1, \\ D(\mathbf{X}, \mathcal{J}) &\leq c_3 (\log n)^{5/2}, && d = 1. \end{aligned} \tag{13.13.1}$$

If $\pi^*_{\mathcal{J}}(m) \leq c_4 m^d$ *for* $m \leq |\mathcal{J}|$, *then*

$$\begin{aligned} D(\mathbf{X}, \mathcal{J}) &\leq c_5 n^{(d-1)/2d} \log n, && d > 1, \\ D(\mathbf{X}, \mathcal{J}) &\leq c_6 (\log n)^{3/2}, && d = 1. \end{aligned} \tag{13.13.2}$$

Matoušek [Mat95] has shown that the factor $(\log n)^{1+1/2d}$ may be dropped from inequality (13.13.1) for $d > 1$, and has applied this result to halfspaces with great effect (see inequality (13.9.3)). One part of Matoušek's argument depends on combinatorial results of Haussler [Hau95].

13.14 RECENT NEW DIRECTIONS

We mention three recent developments of great interest.

Beck [Bec14] has identified some super irregularity phenomena in long and narrow hyperbolic regions, where the discrepancy is found to be of comparable size to the expectation.

Khinchin's conjecture on strong uniformity is false, although for a long time many believed it to be true. Motivated by continuous versions of the conjecture, Beck [Bec15] has made interesting and deep studies into super uniform motions.

Matoušek and Nikolov [MN15] has recently obtained a very strong lower bound of $c(\log n)^{d-1}$ for the combinatorial discrepancy of n-point sets in the d-dimensional unit cube with respect to axis-parallel boxes. This is the combinatorial analogue of the problem studied in Theorem 13.4.6.

13.15 SOURCES AND RELATED MATERIAL

FURTHER READING

There are a few principal surveys on discrepancy theory. The first survey [Sch77b] covers the early development. The Cambridge tract [BC87] is a comprehensive account up to the mid-1980s. The account [DT97] contains a comprehensive list of results and references but few detailed proofs. The exquisite account [Mat99] is most suitable for beginners, as the exposition is very down to earth. The recent volume [CST14] is a collection of essays by experts in various areas.

Among related texts, [KN74] deals mostly with uniform distribution, [Cha00] deals with the discrepancy method and is geared towards computer science, while [DP10] concerns mostly numerical integration.

Auxiliary texts relating to this chapter include [Cas57], [Cas59], and [San76] and [AS93].

RELATED CHAPTERS

Chapter 1: Finite point configurations
Chapter 2: Packing and covering
Chapter 11: Euclidean Ramsey theory
Chapter 12: Discrete aspects of stochastic geometry
Chapter 40: Range searching
Chapter 44: Randomization and derandomization
Chapter 47: Epsilon-nets and epsilon-approximations
Chapter 52: Computer graphics

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