11 EUCLIDEAN RAMSEY THEORY

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INTRODUCTION

Ramsey theory typically deals with problems of the following type. We are given a set S, a family $\mathcal{F}$ of subsets of S, and a positive integer r. We would like to decide whether or not for every partition of $S = C_1 \cup \cdots \cup C_r$ into r subsets, it is always true that some C_i contains some $F \in \mathcal{F}$. If so, we abbreviate this by writing $S \xrightarrow{r} \mathcal{F}$ (and we say S is r-Ramsey). If not, we write $S \not\xrightarrow{r} \mathcal{F}$. (For a comprehensive treatment of Ramsey theory, see [GRS90].)

In Euclidean Ramsey theory, S is usually taken to be the set of points in some Euclidean space $\mathbb{E}^N$, and the sets in $\mathcal{F}$ are determined by various geometric considerations. The case most studied is the one in which $\mathcal{F} = \text{Cong}(X)$ consists of all *congruent* copies of a fixed finite configuration $X \subset S = \mathbb{E}^N$. In other words, $\text{Cong}(X) = \{gX \mid g \in SO(N)\}$, where $SO(N)$ denotes the special orthogonal group acting on $\mathbb{E}^N$.

Further, we say that X is *Ramsey* if, for all r, $\mathbb{E}^N \xrightarrow{r} \text{Cong}(X)$ holds provided N is sufficiently large (depending on X and r). This we indicate by writing $\mathbb{E}^N \longrightarrow X$.

Another important case we will discuss (in Section 11.4) is that in which $\mathcal{F} = \text{Hom}(X)$ consists of all *homothetic* copies $aX + \bar{t}$ of X, where a is a positive real and $\bar{t} \in \mathbb{E}^N$. Thus, in this case $\mathcal{F}$ is just the set of all images of X under the group of positive homotheties acting on $\mathbb{E}^N$.

It is easy to see that any Ramsey (or r-Ramsey) set must be finite. A standard compactness argument shows that if $\mathbb{E}^N \xrightarrow{r} X$ then there is always a *finite* set $Y \subseteq \mathbb{E}^N$ such that $Y \xrightarrow{r} X$. Also, if X is Ramsey (or r-Ramsey) then so is any homothetic copy $aX + \bar{t}$ of X.

GLOSSARY

$\mathbb{E}^{\boldsymbol{N}} \xrightarrow{r}$ **Cong** $\boldsymbol{(X)}$**:** For any partition $\mathbb{E}^N = C_1 \cup \cdots \cup C_r$, some C_i contains a set congruent to X. We say that X is ***r*-Ramsey**. When $\text{Cong}(X)$ is understood we will usually write $\mathbb{E}^N \xrightarrow{r} X$.

$\mathbb{E}^{\boldsymbol{N}} \longrightarrow \boldsymbol{X}$**:** For every r, $\mathbb{E}^N \xrightarrow{r} \text{Cong}(X)$ holds, provided N is sufficiently large. We say in this case that X is ***Ramsey***.

11.1 *r*-RAMSEY SETS

In this section we focus on low-dimensional r-Ramsey results. We begin by stating three conjectures.

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CONJECTURE 11.1.1

For any nonequilateral triangle T (i.e., the set of 3 vertices of T),

$$\mathbb{E}^2 \xrightarrow{2} T.$$

CONJECTURE 11.1.2 (stronger)

For any partition $\mathbb{E}^2 = C_1 \cup C_2$, every triangle occurs (up to congruence) in C_1, or else the same holds for C_2, with the possible exception of a single equilateral triangle.

The partition $\mathbb{E}^2 = C_1 \cup C_2$ with

$$\begin{aligned} C_1 &= \{(x, y) \mid -\infty < x < \infty, 2m \le y < 2m + 1, m = 0, \pm 1, \pm 2, \ldots\} \\ C_2 &= \mathbb{E}^2 \setminus C_1 \end{aligned}$$

into alternating half-open strips of width 1 prevents the equilateral triangle of side $\sqrt{3}$ from occurring in a single C_i. In fact, there are other ways of 2-coloring the plane so as to avoid a monochromatic unit equilateral triangle, such as the so-called "zebra-like" colorings as described in [JKS+09]. It is also shown in [JKS+09] that if the plane is decomposed into the union of an open set and a closed set, then every equilateral triangle occurs at least one of these sets.

CONJECTURE 11.1.3

For any triangle T,

$$\mathbb{E}^2 \overset{3}{\not\longrightarrow} T.$$

In the positive direction, we have [EGM+75b]:

THEOREM 11.1.4

(a) $\mathbb{E}^2 \xrightarrow{2} T$ *if T is a triangle satisfying:*

- (i) *T has a ratio between two sides equal to $2\sin\theta/2$ with $\theta = 30°$, $72°$, $90°$, or $120°$*
- (ii) *T has a $30°$, $90°$, or $150°$ angle* [Sha76]
- (iii) *T has angles $(\alpha, 2\alpha, 180° - 3\alpha)$ with $0 < \alpha < 60°$*
- (iv) *T has angles $(180° - \alpha, 180° - 2\alpha, 3\alpha - 180°)$ with $60° < \alpha < 90°$*
- (v) *T is the degenerate triangle $(a, 2a, 3a)$*
- (vi) *T has sides (a, b, c) satisfying*

$$a^6 - 2a^4b^2 + a^2b^4 - 3a^2b^2c^2 + b^2c^2 = 0$$

or

$$a^4c^2 + b^4a^2 + c^4b^2 - 5a^2b^2c^2 = 0$$

- (vii) *T has sides (a, b, c) satisfying*

$$c^2 = a^2 + 2b^2 \quad \text{with} \quad a < 2b \qquad \text{[Sha76]}$$

- (viii) *T has sides (a, b, c) satisfying*

$$a^2 + c^2 = 4b^2 \ \text{with} \ \ 3b^2 < 2a^2 < 5b^2 \qquad \text{[Sha76]}$$

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(ix) *T has sides equal in length to the sides and circumradius of an isosceles triangle*

(b) $\mathbb{E}^3 \xrightarrow{2} T$ *for any nondegenerate triangle T*

(c) $\mathbb{E}^3 \xrightarrow{3} T$ *for any nondegenerate right triangle T* [BT96]

(d) $\mathbb{E}^3 \overset{12}{\not\rightarrow} T$, *a triangle with angles* $(30°, 60°, 90°)$ [Bón93]

(e) $\mathbb{E}^2 \overset{2}{\not\rightarrow} Q^2$ *(4 points forming a square)*

(f) $\mathbb{E}^4 \overset{2}{\not\rightarrow} Q^2$ [Can96a]

(g) $\mathbb{E}^5 \xrightarrow{2} R^2$, *any rectangle* [Tót96]

(h) $\mathbb{E}^n \overset{4}{\not\rightarrow} \circ\!\overset{1}{\text{——}}\!\circ\!\overset{1}{\text{——}}\!\circ$ *for any n (a degenerate $(1,1,2)$ triangle)*

(i) $\mathbb{E}^n \overset{16}{\not\rightarrow} \circ\!\overset{a}{\text{——}}\!\circ\!\overset{b}{\text{——}}\!\circ$ *for any n (a degenerate $(a, b, a+b)$ triangle).*

It is not known whether the 4 in (h) or the 16 in (i) can be replaced by smaller values. Other results of this type can be found in [EGM+73], [EGM+75a], [EGM+75b], [Sha76], and [CFG91].

The 2-point set X_2 consisting of two points a unit distance apart is the simplest set about which such questions can be asked, and has a particularly interesting history (see [Soi91] for details). It is clear that

$$\mathbb{E}^1 \overset{2}{\not\rightarrow} X_2 \quad \text{and} \quad \mathbb{E}^2 \xrightarrow{2} X_2.$$

To see that $\mathbb{E}^2 \xrightarrow{3} X_2$, consider the 7-point Moser graph shown in Figure 11.1.1. All edges have length 1. On the other hand, $\mathbb{E}^2 \overset{7}{\not\rightarrow} X_2$, which can be seen by an appropriate periodic 7-coloring (= partition into 7 parts) of a tiling of $\mathbb{E}^2$ by regular hexagons of diameter 0.9 (see Figure 1.3.1).

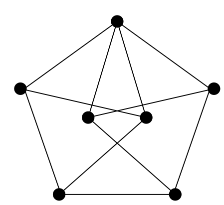

FIGURE 11.1.1
The Moser graph.

Definition: The ***chromatic number*** of $\mathbb{E}^n$, denoted by $\chi(\mathbb{E}^n)$, is the least m such that $\mathbb{E}^n \overset{m}{\not\rightarrow} X_2$.

By the above remarks,

$$4 \le \chi(\mathbb{E}^2) \le 7.$$

These bounds have remained unchanged for over 50 years.

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Some evidence that $\chi(\mathbb{E}^2) \geq 5$ (in the author's opinion) is given by the following result of O'Donnell:

THEOREM 11.1.5 [O'D00a], [O'D00b]

For any $g > 0$, there is 4-chromatic unit distance graph in $\mathbb{E}^2$ with girth greater than g.

Note that the Moser graph has girth 3.

PROBLEM 11.1.6

Determine the exact value of $\chi(\mathbb{E}^2)$.

The best bounds currently known for $\mathbb{E}^n$ are:

$$(1.239 + o(1))^n < \chi(\mathbb{E}^n) < (3 + o(1))^n$$

(see [FW81], [CFG91],[Rai00], [BMP05]).

A "near miss" for showing $\chi(\mathbb{E}^2) < 7$ was found by Soifer [Soi92]. He shows that there exists a partition $\mathbb{E}^2 = C_1 \cup \cdots \cup C_7$ where C_i contains no pair of points at distance 1 for $1 \leq i \leq 6$, while C_7 has no pair at distance $1/\sqrt{5}$.

The best bounds known for $\chi(\mathbb{E}^3)$ are:

$$6 \leq \chi(\mathbb{E}^3) \leq 15.$$

The lower bound is due to Nechushtan [Nec00] and the upper bound is due (independently to Coulson [Col02], and R. Radoičić and G. Tóth [RT02], improving earlier results of Székely and Wormald [SW89] and Bóna/Tóth [BT96]).

See Section 1.3 for more details.

An interesting phenomenon, first pointed out by Székely [Szé84], suggests that the true value of $\chi(\mathbb{E}^2)$ may depend on which axioms for set theory are being used. In ZFC, the standard Zermelo/Fraenkel axioms together with the Axiom of Choice, non-(Lebesgue)-measurable sets exist and can be used to prevent monochromatic configurations from occurring. Indeed, it was shown by Falconer [Fal81] that if the plane is decomposed into four *Lebesgue measurable* sets, then one of the sets must contain a unit distance. In other words, the "measurable" chromatic number of the plane is at least 5. On the other hand, if the Axiom of Choice is replaced by the axiom LM which asserts that every set of reals is Lebesgue measurable, then such constructions are not possible and the chromatic number of the plane may be 4 in these systems. (It is known by a result of Solovay [Sol70] that ZFC and ZF + LM are equally consistent). Further results of this type are given in the papers of Shelah and Soifer [SS03, SS04, Soi05]. Sets for which the chromatic number depends on whether or not the color classes are required to be measurable, are said to have an "ambiguous" chromatic number. In [Pay09], Payne constructs a number of interesting examples of unit-distance graphs in $\mathbb{R}^n$ which have ambiguous chromatic number. This is further evidence that the chromatic number of various configurations in $\mathbb{R}^n$ may depend on the flavor of set theory you prefer!

11.2 RAMSEY SETS

Recall that X is Ramsey (written $\mathbb{E}^N \longrightarrow X$) if, for all r, if $\mathbb{E}^N = C_1 \cup \cdots \cup C_r$ then some C_i must contain a congruent copy of X, provided only that $N \geq N_0(X, r)$.

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GLOSSARY

Spherical: X is spherical if it lies on the surface of some sphere.

Rectangular: X is rectangular if it is a subset of the vertices of a rectangular parallelepiped.

Simplex: X is a simplex if it spans $\mathbb{E}^{|X|-1}$.

THEOREM 11.2.1 [EGM+73]

If X and Y are Ramsey then so is $X \times Y$.

Thus, since any 2-point set is Ramsey (for any r, consider the unit simplex S_{2r+1} in $\mathbb{E}^{2r}$ scaled appropriately), then so is any rectangular parallelepiped. This implies:

THEOREM 11.2.2

Any rectangular set is Ramsey.

Frankl and Rödl strengthen this significantly in the following way.

Definition: A set $A \subset \mathbb{E}^n$ is called ***super-Ramsey*** if there exist positive constants c and ϵ and subsets $X = X(N) \subset \mathbb{E}^N$ for every $N \geq N_0(X)$ such that:

(i) $|X| < c^n$;

(ii) $|Y| < |X|/(1+\epsilon)^n$ holds for all subsets $Y \subset X$ containing no congruent copy of A.

THEOREM 11.2.3 [FR90]

(i) *All two-element sets are super-Ramsey.*

(ii) *If A and B are super-Ramsey then so is $A \times B$.*

COROLLARY 11.2.4

If X is rectangular then X is super-Ramsey.

In the other direction we have

THEOREM 11.2.5

Any Ramsey set is spherical.

The simplest nonspherical set is the degenerate $(1, 1, 2)$ triangle.

Concerning simplices, we have the result of Frankl and Rödl:

THEOREM 11.2.6 [FR90]

Every simplex is Ramsey.

In fact, they show that for any simplex X, there is a constant $c = c(X)$ such that for all r,

$$\mathbb{E}^{c \log r} \xrightarrow{r} X,$$

which follows from their result:

THEOREM 11.2.7

Every simplex is super-Ramsey.

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It was an open problem for more than 20 years as to whether the set of vertices of a regular pentagon was Ramsey. This was finally settled by Křiž [Kři91] who proved the following two fundamental results:

THEOREM 11.2.8 [Kři91]

Suppose $X \subseteq \mathbb{E}^N$ has a transitive solvable group of isometries. Then X is Ramsey.

COROLLARY 11.2.9

Any set of vertices of a regular polygon is Ramsey.

THEOREM 11.2.10 [Kři91]

Suppose $X \subseteq \mathbb{E}^N$ has a transitive group of isometries that has a solvable subgroup with at most two orbits. Then X is Ramsey.

COROLLARY 11.2.11

The vertex sets of the Platonic solids are Ramsey.

CONJECTURE 11.2.12

Any 4-point subset of a circle is Ramsey.

Křiž [Kři92] has shown this holds if a pair of opposite sides of the 4-point set are parallel (i.e., form a trapezoid).

Certainly, the outstanding open problem in Euclidean Ramsey theory is to determine the Ramsey sets. The author (bravely?) makes the following:

CONJECTURE 11.2.13 ($1000)

Any spherical set is Ramsey.

If true then this would imply that the Ramsey sets are exactly the spherical sets.

Recently, an alternative conjecture has been suggested by Leader, Russell and Walters [LRW12]. Let us call a finite configuration **C** in Euclidean space *transitive* if it has a transitive group of symmetries. Further, let us say that **C** is *subtransitive* if it is a subset of a transitive configuration.

CONJECTURE 11.2.14 *[LRW12]*

Any Ramsey set is subtransitive.

These authors have also shown [LRW11] that almost all 4-points subsets of a unit circle are *not* subtransitive. Thus, the question as to whether 4-point cyclic subsets are Ramsey sharply separates these two conjectures!

We point out that a result of Spencer [Spe79] shows that any finite configuration C in $\mathbb{E}^n$ is arbitrarily close to a Ramsey set. Let us say that C' is ϵ-close to C if C' can be obtained by moving each point of C by a distance of at most ϵ.

THEOREM 11.2.15 [Spe79]

For every finite configuration $C \subset \mathbb{E}^n$ and every $\epsilon > 0$, there is any ϵ-close configuration C' which is a Ramsey set.

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11.3 SPHERE-RAMSEY SETS

Since spherical sets play a special role in Euclidean Ramsey theory, it is natural that the following concept arises.

GLOSSARY

$\boldsymbol{S^N(\rho)}$**:** A sphere in $\mathbb{E}^N$ with radius ρ.

Sphere-Ramsey: X is sphere-Ramsey if, for all r, there exist $N = N(X,r)$ and $\rho = \rho(X,r)$ such that

$$S^N(\rho) \xrightarrow{r} X.$$

In this case we write $S^N(\rho) \longrightarrow X$.

For a spherical set X, let $\rho(X)$ denote its circumradius, i.e., the radius of the smallest sphere containing X as a subset.

Remark. If X and Y are sphere-Ramsey then so is $X \times Y$.

THEOREM 11.3.1 [Gra83]

If X is rectangular then X is sphere-Ramsey.

In [Gra83], it was conjectured that in fact if X is rectangular and $\rho(X) = 1$ then $S^N(1+\epsilon) \longrightarrow X$ should hold. This was proved by Frankl and Rödl [FR90] in a much stronger "super-Ramsey" form.

Concerning simplices, Matousěk and Rödl proved the following spherical analogue of simplices being Ramsey:

THEOREM 11.3.2 [MR95]

For any simplex X with $\rho(X) = 1$, any r, and any $\epsilon > 0$, there exists $N = N(X, r, \epsilon)$ such that

$$S^N(1+\epsilon) \xrightarrow{r} X.$$

The proof uses an interesting mix of techniques from combinatorics, linear algebra, and Banach space theory.

The following results show that the "blowup factor" of $1+\epsilon$ is really needed.

THEOREM 11.3.3 [Gra83]

Let $X = \{x_1, \dots, x_m\} \subset \mathbb{E}^N$ such that:

(i) *for some nonempty $I \subseteq \{1, 2, \dots, m\}$, there exist nonzero a_i, $i \in I$, with*

$$\sum_{i\in I} a_i x_i = 0 \in \mathbb{E}^N,$$

(ii) *for all nonempty $J \subseteq I$,*

$$\sum_{j\in J} a_j \neq 0.$$

Then X is not sphere-Ramsey.

This implies that $X \subset S^N(1)$ is not sphere-Ramsey if the convex hull of X contains the center of $S^N(1)$.

Definition: A simplex $X \subset \mathbb{E}^N$ is called ***exceptional*** if there is a subset $A \subseteq X$, $|A| \geq 2$, such that the affine hull of A translated to the origin has a nontrivial intersection with the linear span of the points of $X \setminus A$ regarded as vectors.

THEOREM 11.3.4 [MR95]

If X is a simplex with $\rho(X) = 1$ and $S^N(1) \longrightarrow X$ then X must be exceptional.

It is not known whether it is true for exceptional X that $S^N(1) \longrightarrow X$. The simplest nontrivial case is for the set of three points $\{a, b, c\}$ lying on some great circle of $S^N(1)$ (with center o) so that the line joining a and b is parallel to the line joining o and c. We close with a fundamental conjecture:

CONJECTURE 11.3.5

If X is Ramsey, then X is sphere-Ramsey.

11.4 EDGE-RAMSEY SETS

In this variant (introduced in [EGM$^+$75b], we color all the line segments $[a, b]$ in $\mathbb{E}^n$ rather than coloring the points. Analogously to our earlier definition, we will say that a configuration E of line segments is ***edge-Ramsey*** if for any r, there is an $N = N(r)$ such any r-coloring of the line segments in $\mathbb{E}^N$ contains a monochromatic congruent copy of E (up to some Euclidean motion). The main results known for edge-Ramsey configurations are the following:

THEOREM 11.4.1 [EGM$^+$75b]

If E is edge-Ramsey then all edges of E must have the same length.

THEOREM 11.4.2 [Gra83]

If E is edge-Ramsey then the endpoints of the edges of E must lie on two spheres.

THEOREM 11.4.3 [Gra83]

If the endpoints of E do not lie on a sphere and the graph formed by E is not bipartite then E is not edge-Ramsey.

It is clear that the edge set of an n-dimensional simplex is edge-Ramsey. Less obvious (but equally true) are the following.

THEOREM 11.4.4 [Can96b]

The edge set of an n-cube is edge-Ramsey.

THEOREM 11.4.5 [Can96b]

The edge set of an n-dimensional cross polytope is edge-Ramsey.

This set, a generalization of the octahedron, has as its edges all $2n(n-1)$ line segments of the form $[(0, 0, ..., \pm 1, ..., 0), (0, 0, ..., 0, \pm 1, ..., 0)]$ where the two ± 1's occur in different positions.

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THEOREM 11.4.6 [Can96b]

The edge set of a regular n-gon is not edge-Ramsey if $n = 5$ or $n \geq 7$.

Since regular n-gons are edge-Ramsey for $n = 2$, 3, and 4, the only undecided value is $n = 6$.

PROBLEM 11.4.7 *Is the edge set of a regular hexagon edge-Ramsey?*

The situation is not as simple as one might hope since as pointed out by Cantwell [Can96b]:

(i) If AB is a line segment with C as its midpoint, then the set E_1 consisting of the line segments AC and CB is not edge-Ramsey, even though its graph is bipartite and A, B, C lie on two spheres.

(ii) There exist nonspherical sets that are edge-Ramsey.

PROBLEM 11.4.8 *Characterize edge-Ramsey configurations.*

It is not clear at this point what a reasonable conjecture might be. For more results on these topics, see [Can96b] or [Gra83].

11.5 HOMOTHETIC RAMSEY SETS AND DENSITY THEOREMS

In this section we will survey various results of the type $\mathbb{E}^N \xrightarrow{r} \mathrm{Hom}(X)$, the set of positive homothetic images $aX + \bar{t}$ of a given set X. Thus, we are allowed to dilate and translate X but we cannot rotate it. The classic result of this type is van der Waerden's theorem, which asserts the following:

THEOREM 11.5.1 [Wae27]

If $X = \{1, 2, \ldots, m\}$ then $\mathbb{E} \xrightarrow{r} \mathrm{Hom}(X)$.

(Note that $\mathrm{Hom}(X)$ is just the set of m-term arithmetic progressions.)

By the compactness theorem mentioned in the Introduction there exists, for each m, a minimum value $W(m)$ such that

$$\{1, 2, \ldots, W(m)\} \xrightarrow{2} \mathrm{Hom}(X).$$

The determination or even estimation of $W(m)$ seems to be extremely difficult. The known values are:

m	1	2	3	4	5	6
$W(m)$	1	3	9	35	178	1132

The best general result from below (due to Berlekamp—see [GRS90]) is

$$W(p + 1) \geq p \cdot 2^p, \quad p \text{ prime}.$$

The best upper bound known follows from a spectacular result of Gowers [Gow01]:

$$W(m) < 2^{2^{2^{2^{2^{m+9}}}}}$$

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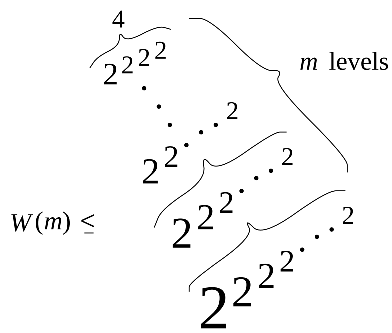
This settled a long-standing \$1000 conjecture of the author. This result is a corollary of Gowers's new quantitative form of Szemerédi's theorem mentioned in the next section. It improves on the earlier bound of Shelah: [She88]:



The following conjecture of the author has been open for more than 30 years:

CONJECTURE 11.5.2 (\$1000)

For all m,

$$W(m) \leq 2^{m^2}$$

The generalization to $\mathbb{E}^N$ is due independently to Gallai and Witt (see [GRS90]).

THEOREM 11.5.3

For any finite set $X \subset \mathbb{E}^n$,

$$\mathbb{E}^N \longrightarrow \mathrm{Hom}(X).$$

We remark here that a number of results in (Euclidean) Ramsey theory have stronger so-called *density* versions. As an example, we state the well-known theorem of Szemerédi.

GLOSSARY

$\mathbb{N}$: The set of natural numbers $\{1, 2, 3, \ldots\}$.

$\overline{\boldsymbol{\delta}}(\boldsymbol{A})$: The ***upper density*** of a set $A \subseteq \mathbb{N}$ is defined by:

$$\overline{\delta}(A) = \limsup_{n \longrightarrow \infty} \frac{|A \cap \{1, 2, \ldots, n\}|}{n} .$$

THEOREM 11.5.4 (Szemerédi [Sze75])

If $A \subseteq \mathbb{N}$ has $\overline{\delta}(A) > 0$ then A contains arbitrarily long arithmetic progressions.

That is, $A \cap \mathrm{Hom}\{1, 2, \ldots, m\} \neq \emptyset$ for all m. This clearly implies van der Waerden's theorem since $\mathbb{N} = C_1 \cup \cdots \cup C_r \Rightarrow \max_i \overline{\delta}(C_i) \geq 1/r$.

Furstenberg [Fur77] has given a quite different proof of Szemerédi's theorem, using tools from ergodic theory and topological dynamics. This approach has proved to be very powerful, allowing Furstenberg, Katznelson, and others to prove density versions of the Hales-Jewett theorem (see [FK91]), the Gallai-Witt theorem, and many others. Gowers has proved the following strong quantitative version of Szemerédi's theorem:

THEOREM 11.5.5 [Gow01]

For every $k > 0$, any subset of $1, 2, ..., N$ of size at least $N(\log\log N)^{-c(k)}$ contains a k-term arithmetic progression, where $c(k) = 2^{-2^{k+9}}$.

Recently, the Polymath project, initiated by Gowers, has resulted in several new proofs of the density Hales-Jewett theorem (see [Tao09, Poly12, DKT14]).

There are other ways of expressing the fact that A is relatively dense in $\mathbb{N}$ besides the condition that $\overline{\delta}(A) > 0$. One would expect that these could also be used as a basis for a density version of van der Waerden or Gallai-Witt. Very little is currently known in this direction, however. We conclude this section with several conjectures of this type.

CONJECTURE 11.5.6 (Erdős)

If $A \subseteq \mathbb{N}$ satisfies $\sum_{a\in A} 1/a = \infty$ then A contains arbitrarily long arithmetic progressions.

CONJECTURE 11.5.7 (Graham)

If $A \subseteq \mathbb{N}\times\mathbb{N}$ with $\sum_{(x,y)\in A} 1/(x^2+y^2) = \infty$ then A contains the 4 vertices of an axis-aligned square.

More generally, I expect that A will always contain a homothetic image of $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, m\}$ for all m. Of course, if we assume A has positive upper density, then this result follows from the density Hales-Jewett theorem [FK91]. A nice combinatorial proof by Solymosi for the square appears in [Sol04].

Finally, we mention a direction in which the group $SO(n)$ is enlarged to allow dilations as well.

Definition: For a set $W \subseteq \mathbb{E}^k$, define the ***upper density*** $\overline{\delta}(W)$ of W by

$$\overline{\delta}(W) := \limsup_{R\longrightarrow\infty} \frac{m(B(o,R)\cap W)}{m(B(o,R))},$$

where $B(o,R)$ denotes the k-ball $\left\{(x_1,\ldots,x_k)\in\mathbb{E}^k \;\middle|\; \sum_{i=1}^{k} x_i^2 \le R^2\right\}$ centered at the origin, and m denotes Lebesgue measure.

THEOREM 11.5.8 (Bourgain [Bou86])

Let $X \subseteq \mathbb{E}^k$ be a simplex. If $W \subseteq \mathbb{E}^k$ with $\overline{\delta}(W) > 0$ then there exists t_0 such that for all $t > t_0$, W contains a congruent copy of tX.

Some restrictions on X are necessary as the following result shows.

THEOREM 11.5.9 (Graham [Gra94])

Let $X \subseteq \mathbb{E}^k$ be nonspherical. Then for any N there exist a set $W \subseteq \mathbb{E}^N$ with $\overline{\delta}(W) > 0$ and a set $T \subseteq \mathbb{R}$ with $\underline{\delta}(T) > 0$ such that W contains no congruent copy of tX for any $t \in T$.

Here $\underline{\delta}$ denotes ***lower density***, defined similarly to $\overline{\delta}$ but with lim inf replacing lim sup.

It is clear that much remains to be done here.

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11.6 VARIATIONS

There are quite a few variants of the preceding topics that have received attention in the literature (e.g., see [Sch93]). We mention some of the more interesting ones.

ASYMMETRIC RAMSEY THEOREMS

Typical results of this type assert that for given sets X_1 and X_2 (for example), for every partition of $\mathbb{E}^N = C_1 \cup C_2$, either C_1 contains a congruent copy of X_1, or C_2 contains a congruent copy of X_2. We can denote this by

$$\mathbb{E}^N \xrightarrow{2} (X_1, X_2).$$

Here is a sampling of results of this type (more of which can be found in [EGM$^+$73], [EGM$^+$75a], [EGM$^+$75b]).

(i) $\mathbb{E}^2 \xrightarrow{2} (T_2, T_3)$ where T_i is any subset of $\mathbb{E}^2$ with i points, $i = 2, 3$.

(ii) $\mathbb{E}^2 \xrightarrow{2} (P_2, P_4)$ where P_2 is a set of two points at a distance 1, and P_4 is a set of four collinear points with distance 1 between consecutive points.

(iii) $\mathbb{E}^3 \xrightarrow{2} (T, Q^2)$ where T is an isosceles right triangle and Q^2 is a square.

(iv) $\mathbb{E}^2 \xrightarrow{2} (P_2, T_4)$ where P_2 is as in (ii) and T_4 is any set of four points [Juh79].

(v) There is a set T_8 of 8 points such that

$$\mathbb{E}^2 \overset{2}{\not\longrightarrow} (P_2, T_8) \quad \text{[CT94]}.$$

This strengthens an earlier result of Juhász [Juh79], which proved this for a certain set of 12 points.

POLYCHROMATIC RAMSEY THEOREMS

Here, instead of asking for a copy of the target set X in a single C_i, we require only that it be contained in the union of a small number of C_i, say at most m of the C_i.

Let us indicate this by writing $\mathbb{E}^N \underset{m}{\to} X$.

(i) If $\mathbb{E}^N \underset{m}{\to} X$ then X must be embeddable on the union of m concentric spheres [EGM$^+$73].

(ii) Suppose X_i is finite and $\mathbb{E}^N \underset{m_i}{\longrightarrow} X_i$, $1 \le i \le t$. Then

$$\mathbb{E}^N \underset{m_1 m_2 \cdots m_t}{\longrightarrow} X_1 \times X_2 \times \cdots \times X_t \qquad \text{[ERS83]}.$$

(iii) If X_6 is the 6-point set formed by taking the four vertices of a square together with the midpoints of two adjacent sides then $\mathbb{E}^2 \not\to X_6$ but $\mathbb{E}^2 \underset{2}{\longrightarrow} X_6$.

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(iv) If X is the set of vertices of a regular simplex in $\mathbb{E}^N$ together with the trisection points of each of its edges then

$$\mathbb{E}^2 \not\rightarrow X_6 \quad \text{but} \quad \mathbb{E}^2 \xrightarrow[3]{} X_6.$$

It is not known if $\mathbb{E}^2 \xrightarrow[2]{} X_6$. Many other results of this type can be found in [ERS83].

PARTITIONS OF $\mathbb{E}^n$ WITH ARBITRARILY MANY PARTS

Since $\mathbb{E}^2 \overset{7}{\not\rightarrow} P_2$, where P_2 is a set of two points with unit distance, one might ask whether there is any nontrivial result of the type $\mathbb{E}^2 \xrightarrow{m} \mathcal{F}$ when m is allowed to go to infinity. Of course, if $\mathcal{F}$ is sufficiently large, then there certainly are. There are some interesting geometric examples for which $\mathcal{F}$ is not too large.

THEOREM 11.6.1 [Gra80a]

For any partition of $\mathbb{E}^n$ into finitely many parts, some part contains, for all $\alpha > 0$ and all sets of lines $L_1, \ldots, L_n$ that span $\mathbb{E}^n$, a simplex having volume α and edges through one vertex parallel to the L_i.

Many other theorems of this type are possible (see [Gra80a]).

PARTITIONS WITH INFINITELY MANY PARTS

Results of this type tend to have a strong set-theoretic flavor. For example: $\mathbb{E}^2 \overset{\aleph_0}{\not\rightarrow} T_3$ where T_3 is an equilateral triangle [Ced69]. In other words, $\mathbb{E}^2$ can be partitioned into countably many parts so that no part contains the vertices of an equilateral triangle. In fact, this was recently strengthened by Schmerl [Sch94b] who showed that for all N,

$$\mathbb{E}^N \overset{\aleph_0}{\not\rightarrow} T_3.$$

In fact, this result holds for *any* fixed triangle T in place of T_3 [Sch94b]. Schmerl also has shown [Sch94a] that there is a partition of $\mathbb{E}^N$ into countably many parts such that no part contains the vertices of *any* isosceles triangle.

Another result of this type is this:

THEOREM 11.6.2 [Kun]

Assuming the Continuum Hypothesis, it is possible to partition $\mathbb{E}^2$ into countably many parts, none of which contains the vertices of a triangle with **rational** *area.*

We also note the interesting result of Erdős and Komjath:

THEOREM 11.6.3 [EK90]

The existence of a partition of $\mathbb{E}^2$ into countably many sets, none of which contains the vertices of a **right** *triangle is equivalent to the Continuum Hypothesis.*

The reader can consult Komjath [Kom97] for more results of this type.

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COMPLEXITY ISSUES

S. Burr [Bur82] has shown that the algorithmic question of deciding if a given set $X \subset \mathbb{N} \times \mathbb{N}$ can be partitioned $X = C_1 \cup C_2 \cup C_3$ so that $x, y \in C_i$ implies distance$(x, y) \geq 6$, for $i = 1, 2, 3$, is NP-complete. (Also, he shows that a certain infinite version of this is undecidable.)

Finally, we make a few remarks about the celebrated problem of Esther Klein (who became Mrs. Szekeres), which, in some sense, initiated this whole area (see [Sze73] for a charming history).

THEOREM 11.6.4 [ES35]

There is a minimum function $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that any set of $f(n)$ points in $\mathbb{E}^2$ in general position contains the vertices of a convex n-gon.

This result of Erdős and George Szekeres actually spawned an independent genesis of Ramsey theory. The best bounds currently known for $f(n)$ are:

$$2^{n-2} + 1 \leq f(n) \leq 2^{n+4n^{4/5}}.$$

The lower bound appears in [ES35]. The upper bound is a striking new result of Andrew Suk [Suk17]. It applies for n sufficiently large and is the first significant improvement of the original upper bound $\binom{2n-4}{n-2+1}$ of Erdős and Szekeres.

CONJECTURE 11.6.5

Prove (or disprove) that $f(n) = 2^{n-2} + 1$, $n \geq 3$.

(See Chapter 1 of this Handbook for more details.)

11.7 SOURCES AND RELATED MATERIAL

SURVEYS

The principal surveys for results in Euclidean Ramsey theory are [GRS90], [Gra80b], [Gra85], and [Gra94]. The first of these is a monograph on Ramsey theory in general, with a section devoted to Euclidean Ramsey theory, while the last three are specifically about the topics discussed in the present chapter.

RELATED CHAPTERS

Chapter 1: Finite point configurations
Chapter 13: Geometric discrepancy theory and uniform distribution

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