10 GEOMETRIC GRAPH THEORY

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INTRODUCTION

In the traditional areas of graph theory (Ramsey theory, extremal graph theory, random graphs, etc.), graphs are regarded as abstract binary relations. The relevant methods are often incapable of providing satisfactory answers to questions arising in geometric applications. Geometric graph theory focuses on combinatorial and geometric properties of graphs drawn in the plane by straight-line edges (or, more generally, by edges represented by simple Jordan arcs). It is a fairly new discipline abounding in open problems, but it has already yielded some striking results that have proved instrumental in the solution of several basic problems in combinatorial and computational geometry (including the k-set problem and metric questions discussed in Sections 1.1 and 1.2, respectively, of this Handbook). This chapter is partitioned into extremal problems (Section 10.1), crossing numbers (Section 10.2), and generalizations (Section 10.3).

10.1 EXTREMAL PROBLEMS

Turán's classical theorem [Tur54] determines the maximum number of edges that an abstract graph with n vertices can have without containing, as a subgraph, a complete graph with k vertices. In the spirit of this result, one can raise the following general question. Given a class $\mathcal{H}$ of so-called *forbidden geometric subgraphs*, what is the maximum number of edges that a geometric graph of n vertices can have without containing a geometric subgraph belonging to $\mathcal{H}$? Similarly, Ramsey's theorem [Ram30] for abstract graphs has some natural analogues for geometric graphs. In this section we will be concerned mainly with problems of these two types.

GLOSSARY

Geometric graph: A graph drawn in the plane by (possibly crossing) straight-line segments; i.e., a pair $(V(G), E(G))$, where $V(G)$ is a set of points ('vertices'), no three of which are collinear, and $E(G)$ is a set of segments ('edges') whose endpoints belong to $V(G)$.

Convex geometric graph: A geometric graph whose vertices are in *convex position*; i.e., they form the vertex set of a convex polygon.

Cyclic chromatic number of a convex geometric graph: The minimum number $\chi_c(G)$ of colors needed to color all vertices of G so that each color class consists of consecutive vertices along the boundary of the convex hull of the vertex set.

Convex matching: A convex geometric graph consisting of disjoint edges, each of which belongs to the boundary of the convex hull of its vertex set.

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Parallel matching: A convex geometric graph consisting of disjoint edges, the convex hull of whose vertex set contains only two of the edges on its boundary.

Complete geometric graph: A geometric graph G whose edge set consists of all $\binom{|V(G)|}{2}$ segments between its vertices.

Complete bipartite geometric graph: A geometric graph G with $V(G) = V_1 \cup V_2$, whose edge set consists of all segments between V_1 and V_2.

Geometric subgraph of G: A geometric graph H, for which $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Crossing: A common interior point of two edges of a geometric graph.

(k, l)-Grid: $k + l$ vertex-disjoint edges in a geometric graph such that each of the first k edges crosses all of the last l edges. It is called natural if the first k edges are pairwise disjoint segments and the last l edges are pairwise disjoint segments.

Disjoint edges: Edges of a geometric graph that do not cross and do not even share an endpoint.

Parallel edges: Edges of a geometric graph whose supporting lines are parallel or intersect at points not belonging to any of the edges (including their endpoints).

x-Monotone curve: A continuous curve that intersects every vertical line in at most one point.

Outerplanar graph: A (planar) graph that can be drawn in the plane without crossing so that all points representing its vertices lie on the outer face of the resulting subdivision of the plane. A ***maximal outerplanar graph*** is a triangulated cycle.

Hamiltonian path: A path going through all elements of a finite set S. If the elements of S are colored by two colors, and no two adjacent elements of the path have the same color, then it is called an *alternating* path.

Hamiltonian cycle: A cycle going through all elements of a finite set S.

Caterpillar: A tree consisting of a path P and of some extra edges, each of which is adjacent to a vertex of P.

CROSSING-FREE GEOMETRIC GRAPHS

1. Hanani-Tutte theorem: Any graph that can be drawn in the plane so that its edges are represented by simple Jordan arcs such that any two that do not share an endpoint properly cross an even number of times, is planar [Cho34, Tut70]. The analogous result also holds in the projective plane [PSS09].

2. Fáry's theorem: Every planar graph admits a crossing-free straight-line drawing [Fár48, Tut60, Ste22]. Moreover, every 3-connected planar graph and its dual have simultaneous straight-line drawings in the plane such that only dual pairs of edges cross and every such pair is perpendicular [BS93].

3. Koebe's theorem: The vertices of every planar graph can be represented by nonoverlapping disks in the plane such that two of them are tangent to each other if and only if the corresponding two vertices are adjacent [Koe36, Thu78]. This immediately implies Fáry's theorem.

4. Pach-Tóth theorem: Any graph that can be drawn in the plane so that its edges are represented by x-monotone curves with the property that any two of them either share an endpoint or properly cross an even number of times admits a crossing-free straight-line drawing, in which the x-coordinates of the vertices remain the same [PT04]. Fulek *et al.* [FPSŠ13] generalized this result in two directions: it is sufficient to assume that (a) any two edges that do not share an endpoint cross an even number of times, and that (b) the projection of every edge to the x-axis lies between the projections of its endpoints.

5. Grid drawings of planar graphs: Every planar graph of n vertices admits a straight-line drawing such that the vertices are represented by points belonging to an $(n-1) \times (n-1)$ grid [FPP90, Sch90]. Furthermore, such a drawing can be found in $O(n)$ time. For other small-area grid drawings, consult [DF13].

6. Straight-line drawings of outerplanar graphs: For any outerplanar graph H with n vertices and for any set P of n points in the plane in general position, there is a crossing-free geometric graph G with $V(G) = P$, whose underlying graph is isomorphic to H [GMPP91]. For any rooted tree T and for any set P of $|V(T)|$ points in the plane in general position with a specified element $p \in P$, there is a crossing-free straight-line drawing of T such that every vertex of T is represented by an element of P and the root is represented by p [IPTT94]. This theorem generalizes to any *pair* of rooted trees, T_1 and T_2: for any set P of $n = |V(T_1)| + |V(T_2)|$ points in general position in the plane, there is a crossing-free mapping of $T_1 \cup T_2$ that takes the roots to arbitrarily prespecified elements of P. Such a mapping can be found in $O(n^2 \log n)$ time [KK00]. The analogous statement for *triples* of trees is false.

7. Alternating paths: Given n red points and n blue points in general position in the plane, separated by a straight line, they always admit a noncrossing alternating Hamiltonian path [KK03].

TURÁN-TYPE PROBLEMS

By Euler's Polyhedral Formula, if a geometric graph G with $n \geq 3$ vertices has no 2 crossing edges, it cannot have more than $3n - 6$ edges. It was shown in [AAP$^+$97] and [Ack09] that under the weaker condition that no 3 (resp. 4) edges are *pairwise crossing*, the number of edges of G is still $O(n)$. It is not known whether this statement remains true even if we assume only that no 5 edges are pairwise crossing. As for the analogous problem when the forbidden configuration consists of k pairwise *disjoint edges*, the answer is linear for every k [PT94]. In particular, for $k = 2$, the number of edges of G cannot exceed the number of vertices [HP34]. The best lower and upper bounds known for the number of edges of a geometric graph with n vertices, containing no *forbidden* geometric subgraph of a certain type, are summarized in Table 10.1.1. The letter k always stands for a *fixed* positive integer parameter and n tends to infinity. Wherever k does not appear in the asymptotic bounds, it is hidden in the constants involved in the O- and Ω-notations.

Better results are known for *convex* geometric graphs, i.e., when the vertices are in convex position. The relevant bounds are listed in Table 10.1.2. For any convex geometric graph G, let $\chi_c(G)$ denote its *cyclic chromatic number*. Furthermore, let

TABLE 10.1.1 Maximum number of edges of a geometric graph of n vertices containing no forbidden subconfigurations of a certain type.

FORBIDDEN CONFIGURATION	LOWER BOUND	UPPER BOUND	SOURCE
2 crossing edges	$3n-6$	$3n-6$	Euler
3 pairwise crossing edges	$6.5n-O(1)$	$6.5n-O(1)$	[AT07]
4 pairwise crossing edges	$\Omega(n)$	$72(n-2)$	[Ack09]
$k>4$ pairwise crossing edges	$\Omega(n)$	$O(n\log n)$	[Val98]
an edge crossing 2 others	$4n-8$	$4n-8$	[PT97]
an edge crossing 3 others	$5n-12$	$5n-10$	[PT97]
an edge crossing 4 others	$5.5n+\Omega(1)$	$5.5n+O(1)$	[PRTT06]
an edge crossing 5 others	$6n-O(1)$	$6n-12$	[Ack16]
an edge crossing k others	$\Omega(\sqrt{k}n)$	$O(\sqrt{k}n)$	[PT97]
2 crossing edges crossing k others	$\Omega(n)$	$O(n)$	[PRT04]
(k,l)-grid	$\Omega(n)$	$O(n)$	[PPST05]
natural (k,l)-grid	$\Omega(n)$	$O(n\log^2 n)$	[AFP+14]
self-intersecting path of length 3	$\Omega(n\log n)$	$O(n\log n)$	[PPTT02]
self-intersecting path of length 5	$\Omega(n\log\log n)$	$O(n\log n/\log\log n)$	[Tar13], [PPTT02]
self-intersecting cycle of length 4	$\Omega(n^{3/2})$	$O(n^{3/2}\log n)$	[PR04], [MT06]
2 disjoint edges	n	n	[HP34]
noncrossing path of length k	$\Omega(kn)$	$O(k^2n)$	[Tót00]
k pairwise parallel edges	$\Omega(n)$	$O(n)$	[Val98]

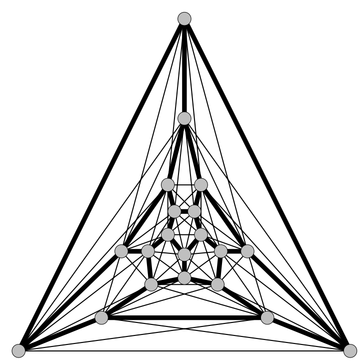

FIGURE 10.1.1
Geometric graph with $n=20$ vertices and $5n-12=88$ edges, none of which crosses 3 others.

$\mathrm{ex}(n,K_k)$ stand for the maximum number of edges of a graph with n vertices that does not have a complete subgraph with k vertices. By Turán's theorem [Tur54] mentioned above, $\mathrm{ex}(n,K_k)=\frac{k-2}{k-1}\binom{n}{2}+O(n)$ is equal to the number of edges of a complete $(k-1)$-partite graph with n vertices whose vertex classes are of size $\lfloor n/(k-1)\rfloor$ or $\lceil n/(k-1)\rceil$. Two disjoint self-intersecting paths of length 3, $xyvz$ and $x'y'v'z'$, in a convex geometric graph are said to be of the *same orientation* if the cyclic order of their vertices is x,v,x',v',y',z',y,z ( ). They are said to have *opposite orientations* if the cyclic order of their vertices is x,v,v',x',z',y',y,z (*type 1*:) or v,x,x',v',y',z',z,y (*type 2*:).

RAMSEY-TYPE PROBLEMS

In classical Ramsey theory, one wants to find large monochromatic subgraphs in a

TABLE 10.1.2 Maximum number of edges of a *convex* geometric graph of n vertices containing no forbidden subconfigurations of a certain type.

FORBIDDEN CONFIGURATION	LOWER BOUND	UPPER BOUND	SOURCE
2 crossing edges	$2n-3$	$2n-3$	Euler
self-intersecting path of length 3	$2n-3$	$2n-3$	Perles
k self-intersecting paths of length 3	$\Omega(n)$	$O(n)$	[BKV03]
2 self-intersecting paths of length 3 with opposite orientations of type 1	$\Omega(n\log n)$	$O(n\log n)$	[BKV03]
2 self-intersecting paths of length 3 with opposite orientations of type 2	$\Omega(n\log n)$	$O(n\log n)$	[BKV03]
2 adjacent edges crossing a 3rd	$\lfloor 5n/2-4\rfloor$	$\lfloor 5n/2-4\rfloor$	Perles-Pinchasi,[BKV03]
k pairwise crossing edges	$2(k-1)n-\binom{2k-1}{2}$	$2(k-1)n-\binom{2k-1}{2}$	[CP92]
noncrossing outerplanar graph of k vertices, having a Hamiltonian cycle	$\mathrm{ex}(n,K_k)$	$\mathrm{ex}(n,K_k)$	Pach [PA95], Perles
convex geometric subgraph G	$\mathrm{ex}(n,K_{\chi_c(G)})$	$\mathrm{ex}(n,K_{\chi_c(G)})+o(n^2)$	[BKV03]
convex matching of k disjoint edges	$\mathrm{ex}(n,K_k)+n-k+1$	$\mathrm{ex}(n,K_k)+n-k+1$	[KP96]
parallel matching of k disjoint edges	$(k-1)n$	$(k-1)n$	[Kup84]
noncrossing caterpillar C of k vertices	$\lfloor (k-2)n/2\rfloor$	$\lfloor (k-2)n/2\rfloor$	Perles [BKV03]

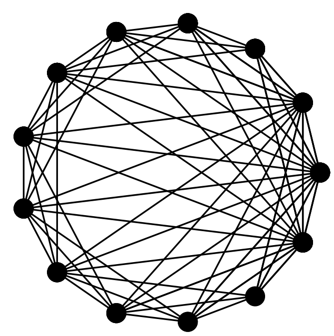

FIGURE 10.1.2
Convex geometric graph with $n=13$ *vertices and* $6n-\binom{7}{2}=57$ *edges, no* 4 *of which are pairwise crossing* [CP92].

complete graph whose edges are colored with several colors [GRS90]. Most questions of this type can be generalized to complete geometric graphs, where the monochromatic subgraphs are required to satisfy certain geometric conditions.

1. Károlyi-Pach-Tóth theorem [KPT97]: If the edges of a finite complete geometric graph are colored by two colors, there exists a noncrossing spanning tree, all of whose edges are of the same color. (This statement was conjectured by Bialostocki and Dierker [BDV04]. The analogous assertion for abstract graph follows from the fact that any graph or its complement is connected.)

2. Geometric Ramsey numbers: Let $\mathcal{G}_1,\ldots,\mathcal{G}_k$ be not necessarily different classes of geometric graphs. Let $R(\mathcal{G}_1,\ldots,\mathcal{G}_k)$ denote the smallest positive number R with the property that any complete geometric graph of R vertices whose edges are colored with k colors $(1,\ldots,k$, say) contains, for some i, an i-colored subgraph belonging to $\mathcal{G}_i$. If $\mathcal{G}_1=\ldots=\mathcal{G}_k=\mathcal{G}$, we write $R(\mathcal{G};k)$ instead of $R(\mathcal{G}_1,\ldots,\mathcal{G}_k)$. If $k=2$, for the sake of simplicity, let $R(\mathcal{G})$ stand for $R(\mathcal{G};2)$. Some known results on the numbers $R(\mathcal{G}_1,\mathcal{G}_2)$ are listed in Table 10.1.3. In line 3 of the table, we have a better result if we restrict our attention to *convex*

geometric graphs: For any 2-coloring of the edges of a complete convex geometric graph with $2k-1$ vertices, there exists a noncrossing monochromatic path of length $k \geq 2$, and this result cannot be improved. The upper bound $2(k-1)(k-2)+2$ in line 4 is tight for convex geometric graphs [BCK+15]. The bounds in line 4 also hold when $\mathcal{G}_1 = \mathcal{G}_2$ consists of all noncrossing cycles of length k, triangulated from one of their vertices. The geometric Ramsey numbers of convex geometric graphs, when $\mathcal{G}_1 = \mathcal{G}_2$ consists of all isomorphic copies of a given convex geometric graph with at most 4 vertices, can be found in [BH96]. In [CGK+15], polynomial upper bounds are established for the geometric Ramsey numbers of the "ladder graphs" L_k, consisting of two paths of length k with an edge connecting each pair of corresponding vertices.

TABLE 10.1.3 Geometric Ramsey numbers $R(\mathcal{G}_1, \mathcal{G}_2)$ from [KPT97] and [KPTV98].

$\mathcal{G}_1$	$\mathcal{G}_2$	LOWER BOUND	UPPER BOUND
all noncrossing trees of k vertices	all noncrossing trees of k vertices	k	k
k disjoint edges	l disjoint edges	$k+l+\max\{k,l\}-1$	$k+l+\max\{k,l\}-1$
noncrossing paths of length k	noncrossing paths of length k	$\Omega(k)$	$O(k^{3/2})$
noncrossing cycles of length k	noncrossing cycles of length k	$(k-1)^2$	$2(k-1)(k-2)+2$

3. Pairwise disjoint copies: For any positive integer k, let $k\mathcal{G}$ denote the class of all geometric graphs that can be obtained by taking the union of k pairwise disjoint members of $\mathcal{G}$. If k is a power of 2 then

$$R(k\mathcal{G}) \leq (R(\mathcal{G})+1)k-1.$$

In particular, if $\mathcal{G} = \mathcal{T}$ is the class of triangles, we have $R(\mathcal{T}) = 6$. Thus, the above bound yields that

$$R(k\mathcal{T}) \leq 7k-1,$$

provided that k is a power of 2. This result cannot be improved [KPTV98].

Furthermore, for any $k > 0$, we have

$$R(k\mathcal{G}) \leq \left\lceil \frac{3(R(\mathcal{G})+1)}{2} \right\rceil k - \left\lceil \frac{R(\mathcal{G})+1}{2} \right\rceil .$$

For the corresponding quantities for convex geometric graphs, we have

$$R_c(k\mathcal{G}) \leq (R_c(\mathcal{G})+1)k-1.$$

4. Constructive vertex- and edge-Ramsey numbers: Given a class of geometric graphs $\mathcal{G}$, let $R_v(\mathcal{G})$ denote the smallest number R such that *there exists* a (complete) geometric graph of R vertices that, for any 2-coloring of its edges, has a monochromatic subgraph belonging to $\mathcal{G}$. Similarly, let $R_e(\mathcal{G})$ denote the minimum number of *edges* of a geometric graph with this property. $R_v(\mathcal{G})$ and $R_e(\mathcal{G})$ are called the ***vertex-*** and ***edge-Ramsey number*** of $\mathcal{G}$, respectively. Clearly, we have

$$R_v(\mathcal{K}) \leq R(\mathcal{G}), \quad R_e(\mathcal{G}) \leq \binom{R(\mathcal{G})}{2}.$$

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(For abstract graphs, similar notions are discussed in [EFRS78, Bec83].)

For $\mathcal{P}_k$, the class of noncrossing paths of length k, we have $R_v(\mathcal{P}_k) = O(k^{3/2})$ and $R_e(\mathcal{P}_k) = O(k^2)$.

OPEN PROBLEMS

1. What is the smallest number $u = u(n)$ such that there exists a "universal" set U of u points in the plane with the property that every planar graph of n vertices admits a noncrossing straight-line drawing on a suitable subset of U [FPP90]? It follows from the existence of a small grid drawing (see above) that $u(n) \le n^2$, and it is shown in [BCD$^+$14] that $u(n) \le n^2/4$. From below we have only $u(n) > 1.235n$; [Kur04]. Certain subclasses of planar graphs admit universal sets of size $o(n^2)$; [FT15].

2. It was shown by Chalopin and Gonçalvez [CG09] that the vertices of every planar graph G be represented by straight-line segments in the plane so that two segments intersect if and only if the corresponding vertices are adjacent. If the chromatic number of G is 2 or 3, then segments of 2 resp. 3 different directions suffice [FMP94, FM07]. Does there exist a constant c such that the vertices of every planar graph can be represented by segments using at most c different directions?

3. (Erdős, Kaneko-Kano) What is the largest number $A = A(n)$ such that any set of n red and n blue points in the plane admits a noncrossing alternating path of length A? It is known that $A(n) \le (4/3 + o(1))n$; [KPT07].

4. Is it true that, for any fixed k, the maximum number of edges of a geometric graph with n vertices that does not have k pairwise crossing edges is $O(n)$?

5. (Aronov et al.) Is it true that any complete geometric graph with n vertices has at least $\Omega(n)$ pairwise crossing edges? It was shown in [AEG$^+$94] that one can always find $\sqrt{n/12}$ pairwise crossing edges. On the other hand, any complete geometric graph with n vertices has a noncrossing Hamiltonian path, hence $\lfloor n/2 \rfloor$ pairwise disjoint edges.

6. (Larman-Matoušek-Pach-Törőcsik) What is the smallest positive number $r = r(n)$ such that any family of r closed segments in general position in the plane has n members that are either pairwise disjoint or pairwise crossing? It is known [LMPT94, Kyn12] that $n^{\log 169/\log 8} \approx n^{2.466} \le r(n) \le n^5$.

10.2 CROSSING NUMBERS

The investigation of crossing numbers started during WWII with Turán's Brick Factory Problem [Tur77]: how should one redesign the routes of railroad tracks between several kilns and storage places in a brick factory so as to minimize the number of crossings? In the early eighties, it turned out that the chip area required for the realization (VLSI layout) of an electrical circuit is closely related to the crossing number of the underlying graph [Lei83]. This discovery gave an impetus to research in the subject. More recently, it has been realized that general bounds on crossing numbers can be used to solve a large variety of problems in discrete and computational geometry.

GLOSSARY

Drawing of a graph: A representation of the graph in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. In a drawing (a) no edge passes through any vertex other than its endpoints, (b) no two edges touch each other (i.e., if two edges have a common interior point, then at this point they properly cross each other), and (c) no three edges cross at the same point.

Crossing: A common interior point of two edges in a graph drawing. Two edges may have several crossings.

Crossing number of a graph: The smallest number of crossings in any drawing of G, denoted by $\text{CR}(G)$. Clearly, $\text{CR}(G) = 0$ if and only if G is planar.

Rectilinear crossing number: The minimum number of crossings in a drawing of G in which every edge is represented by a straight-line segment. It is denoted by $\text{LIN-CR}(G)$.

Pairwise crossing number: The minimum number of crossing pairs of edges over all drawings of G, denoted by $\text{PAIR-CR}(G)$. (Here the edges can be represented by arbitrary continuous curves, so that two edges may cross more than once, but every pair of edges can contribute at most one to $\text{PAIR-CR}(G)$.)

Odd crossing number: The minimum number of those pairs of edges that cross an odd number of times, over all drawings of G. It is denoted by $\text{ODD-CR}(G)$.

Biplanar crossing number: The minimum of $\text{CR}(G_1) + \text{CR}(G_2)$ over all partitions of the graph into two edge-disjoint subgraphs G_1 and G_2.

Bisection width: The minimum number $b(G)$ of edges whose removal splits the graph G into two roughly equal subgraphs. More precisely, $b(G)$ is the minimum number of edges running between V_1 and V_2 over all partitions of the vertex set of G into two disjoint parts $V_1 \cup V_2$ such that $|V_1|, |V_2| \geq |V(G)|/3$.

Cut width: The minimum number $c(G)$ such that there is a drawing of G in which no two vertices have the same x-coordinate and every vertical line crosses at most $c(G)$ edges.

Path width: The minimum number $p(G)$ such that there is a sequence of at most $(p(G)+1)$-element sets $V_1, V_2, \ldots, V_r \subseteq V(G)$ with the property that both endpoints of every edge belong to some V_i and, if a vertex occurs in V_i and V_k $(i < k)$, then it also belongs to every V_j, $i < j < k$.

GENERAL ESTIMATES

Garey and Johnson [GJ83] showed that the determination of the crossing number is an *NP-complete* problem. Analogous results hold for the rectilinear crossing number [Bie91], for the pair crossing number [SSŠ03], and for the odd crossing number [PT00b]. The exact determination of crossing numbers of relatively small graphs of a simple structure (such as complete or complete bipartite graphs) is a hopelessly difficult task, but there are several useful bounds. There is an algorithm

[EGS03] for computing a drawing of a bounded-degree graph with n vertices, for which n plus the number of crossings is $O(\log^3 n)$ times the optimum.

1. For a simple graph G with $n \geq 3$ vertices and e edges, $\textsc{cr}(G) \geq e - 3n + 6$. From this inequality, a simple probabilistic argument shows that $\textsc{cr}(G) \geq ce^3/n^2$, for a suitable positive constant c. This important bound, due to Ajtai-Chvátal-Newborn-Szemerédi [ACNS82] and, independently, to Leighton [Lei83], is often referred to as the ***crossing lemma.*** We know that $0.03 \leq c \leq 0.09$ [PT97, PRTT06, Ack16]. The lower bound follows from line 8 in Table 10.1.1. Similar statements hold for $\textsc{pair-cr}(G)$ and $\textsc{odd-cr}(G)$ [PT00b].

2. Crossing lemma for multigraphs [Szé97]: Let G be a ***multigraph*** with n vertices and e edges, i.e., the same pair of vertices can be connected by more than one edge. Let m denote the maximum multiplicity of an edge. Then
$$\textsc{cr}(G) \geq c\frac{e^3}{mn^2} - m^2 n,$$
where c denotes the same constant as in the previous paragraph.

3. Midrange crossing constant: Let $\kappa(n, e)$ denote the minimum crossing number of a graph G with n vertices and at least e edges. That is,
$$\kappa(n, e) = \min_{\substack{n(G) = n \\ e(G) \geq e}} \textsc{cr}(G).$$
It follows from the crossing lemma that, for $e \geq 4n$, $\kappa(n, e)n^2/e^3$ is bounded from below and from above by two positive constants. Erdős and Guy [EG73] conjectured that if $n \ll e \leq n^2/100$, then $\lim \kappa(n, e)n^2/e^3$ exists. (We use the notation $f(n) \gg g(n)$ to mean that $\lim_{n\to\infty} f(n)/g(n) = \infty$.) This was partially settled in [PST00]: if $n \ll e \ll n^2$, then
$$\lim_{n\to\infty} \kappa(n, e)\frac{n^2}{e^3} = C > 0$$
exists. Moreover, the same result is true with the same constant C, for drawings on every other orientable surface.

4. Graphs with monotone properties: A graph property $\mathcal{P}$ is said to be ***monotone*** if (i) for any graph G satisfying $\mathcal{P}$, every subgraph of G also satisfies $\mathcal{P}$; and (ii) if G_1 and G_2 satisfy $\mathcal{P}$, then their disjoint union also satisfies $\mathcal{P}$. For any monotone property $\mathcal{P}$, let $\mathrm{ex}(n, \mathcal{P})$ denote the maximum number of edges that a graph of n vertices can have if it satisfies $\mathcal{P}$. In the special case when $\mathcal{P}$ is the property that the graph does not contain a subgraph isomorphic to a fixed forbidden subgraph H, we write $\mathrm{ex}(n, H)$ for $\mathrm{ex}(n, \mathcal{P})$.

   Let $\mathcal{P}$ be a monotone graph property with $\mathrm{ex}(n, \mathcal{P}) = O(n^{1+\alpha})$ for some $\alpha > 0$. In [PST00], it was proved that there exist two constants $c, c' > 0$ such that the crossing number of any graph G with property $\mathcal{P}$ that has n vertices and $e \geq cn \log^2 n$ edges satisfies
$$\textsc{cr}(G) \geq c' \frac{e^{2+1/\alpha}}{n^{1+1/\alpha}}.$$

This bound is asymptotically tight, up to a constant factor. In particular, if $e > 4n$ and G has no cycle of length at most $2r$, then the crossing number of G satisfies

$$\text{CR}(G) \geq c_r \frac{e^{r+2}}{n^{r+1}},$$

where $c_r > 0$ is a suitable constant. For $r = 2, 3$, and 5, these bounds are asymptotically tight, up to a constant factor. If G does not contain a complete bipartite subgraph $K_{r,s}$ with r and s vertices in its classes, $s \geq r$, then we have

$$\text{CR}(G) \geq c_{r,s} \frac{e^{3+1/(r-1)}}{n^{2+1/(r-1)}},$$

where $c_{r,s} > 0$ is a suitable constant. These bounds are tight up to a constant factor if $r = 2, 3$, or if r is arbitrary and $s > (r-1)!$.

5. Crossing number vs. bisection width $b(G)$: For any vertex $v \in V(G)$, let $d(v)$ denote the degree of v in G. It was shown in [PSS96] and [SV94] that

$$\text{CR}(G) + \frac{1}{16} \sum_{v \in V(G)} d^2(v) \geq \frac{1}{40} b^2(G).$$

A similar statement holds with a worse constant for the cut width $c(G)$ of G [DV02]. This, in turn, implies that the same is true for $p(G)$, the path width of G, as we have $p(G) \leq c(G)$ for every G [Kin92].

6. Relations between different crossing numbers: Clearly, we have

$$\text{ODD-CR}(G) \leq \text{PAIR-CR}(G) \leq \text{CR}(G) \leq \text{LIN-CR}(G).$$

It was shown [BD93] that there are graphs with crossing number 4 whose rectilinear crossing numbers are arbitrarily large.

It was established in [PT00b] and [Mat14] that

$$\text{CR}(G) \leq 2\left(\text{ODD-CR}(G)\right)^2,$$

$$\text{CR}(G) \leq O\left(\left(\text{PAIR-CR}(G)\right)^{3/2} \log^2 \text{ODD-CR}(G)\right),$$

respectively. Pelsmajer *et al.* [PSS08] discovered a series of graphs for which $\text{ODD-CR}(G) \neq \text{PAIR-CR}(G)$. In fact, there are graphs satisfying the inequality $\text{PAIR-CR}(G) \geq 1.16\,\text{ODD-CR}(G)$; see [Tót08]. We cannot rule out the possibility that

$$\text{PAIR-CR}(G) = \text{CR}(G)$$

for every graph G.

7. Crossing numbers of random graphs: Let $G = G(n,p)$ be a ***random graph*** with n vertices, whose edges are chosen independently with probability $p = p(n)$. Let e denote the *expected number* of edges of G, i.e., $e = p \cdot \binom{n}{2}$. It is not hard to see that if $e > 10n$, then almost surely $b(G) \geq e/10$. It therefore follows from the above relation between the crossing number and the bisection width that almost surely we have $\text{LIN-CR}(G) \geq \text{CR}(G) \geq e^2/4000$. Evidently, the order of magnitude of this bound cannot be improved. A similar inequality was proved in [ST02] for the pairwise crossing number, under the stronger condition that $e > n^{1+\epsilon}$ for some $\epsilon > 0$.

8. Biplanar crossing number vs. crossing number: It is known [CSSV08] that the biplanar crossing number of every graph is at most 3/8 times its crossing number. It is conjectured that the statement remains true with 7/24 in place of 3/8.

9. Harary-Kainen-Schwenk conjecture [HKS73]: For every $n \geq m \geq 3$ and cycles C_n and C_m, $\textsc{cr}(C_n \times C_m)$ is equal to $n(m-2)$. This was proved in [GS04] for every m and for all sufficiently large n. For the crossing number of the skeleton of the n-dimensional hypercube Q_n, we have $1/20 + o(1) \leq \textsc{cr}(Q_n)/4^n \leq 163/1024$ [FF00, SV93].

OPEN PROBLEMS

1. (Pach-Tóth) Is it true that $\textsc{pair-cr}(G) = \textsc{cr}(G)$ for every graph G? Does there exist a constant γ such that $\textsc{cr}(G) \leq \gamma \textsc{odd-cr}(G)$?

2. Albertson conjectured that the crossing number of every graph whose chromatic number is at least k is at least as large as the crossing number of K_k. This conjecture was proved for $k \leq 18$, and it is also known that the crossing number of such a graph exceeds ck^4, for a suitable constant c. [ACF09, BT10, Ack16].

3. Zarankiewicz's conjecture [Guy69]: The crossing number of the complete bipartite graph $K_{n,m}$ with n and m vertices in its classes satisfies

$$\textsc{cr}(K_{n,m}) = \left\lfloor \frac{m}{2} \right\rfloor \cdot \left\lfloor \frac{m-1}{2} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor.$$

Kleitman [Kle70] verified this conjecture in the special case when $\min\{m, n\} \leq 6$ and Woodall [Woo93] for $m = 7$, $n \leq 10$.

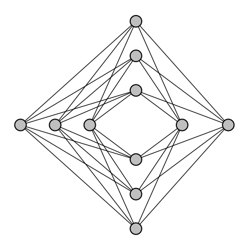


FIGURE 10.2.1
Complete bipartite graph $K_{5,6}$ with 24 crossings.

It is also conjectured that the crossing number of the complete graph K_n satisfies

$$\textsc{cr}(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n-1}{2} \right\rfloor \cdot \left\lfloor \frac{n-2}{2} \right\rfloor \cdot \left\lfloor \frac{n-3}{2} \right\rfloor.$$

This conjecture is known to be true if we restrict our attention to drawings where every edge is represented by an x-monotone curve [AAF$^+$14, BFK15]. An old construction of Blažek and Koman [BK64] shows that equality can be attained even for drawings of this type.

4. Rectilinear crossing numbers of complete graphs: Determine the value

$$\kappa = \lim_{n\to\infty} \frac{\text{LIN-CR}(K_n)}{\binom{n}{4}}.$$

The best known bounds $0.3799 < \kappa \le 0.3805$ were found by Ábrego *et al.* [ACF+10, ACF+12]; see also [LVW+04]. All known exact values of LIN-CR(K_n) are listed in Table 10.2.1. For $n < 12$, we have LIN-CR(K_n) = CR(K_n), but LIN-CR(K_{12}) > CR(K_{12}) [ACF+12].

TABLE 10.2.1 Exact values of LIN-CR(K_n).

n	LIN-CR(K_n)	n	LIN-CR(K_n)	n	LIN-CR(K_n)
4	0	13	229	22	2528
5	1	14	324	23	3077
6	3	15	447	24	3699
7	9	16	603	25	4430
8	19	17	798	26	5250
9	36	18	1029	27	6180
10	62	19	1318	30	9726
11	102	20	1657		
12	153	21	2055		

5. Let $G = G(n,p)$ be a *random* graph with n vertices, whose edges are chosen independently with probability $p = p(n)$. Let $e = p \cdot \binom{n}{2}$. Is it true that the pairwise crossing number, the odd crossing number, and the biplanar crossing number are bounded from below by a constant times e^2, provided that $e \gg n$?

10.3 GENERALIZATIONS

The concept of geometric graph can be generalized in two natural directions. Instead of straight-line drawings, we can consider curvilinear drawings. If we put them at the focus of our investigations and we wish to emphasize that they are objects of independent interest rather than planar representations of abstract graphs, we call these drawings *topological graphs*. In this sense, the results in the previous section about crossing numbers belong to the theory of topological graphs. Instead of systems of segments induced by a planar point set, we can also consider systems of simplices in the plane or in higher-dimensional spaces. Such a system is called a *geometric hypergraph*.

GLOSSARY

Topological graph: A graph drawn in the plane so that its vertices are distinct points and its edges are simple continuous arcs connecting the corresponding vertices. In a topological graph (a) no edge passes through any vertex other than its endpoints, (b) any two edges have only a finite number of interior points in common, at which they properly cross each other, and (c) no three edges cross at the same point. (Same as *drawing of a graph*.)

Weakly isomorphic topological graphs: Two topological graphs, G and H, such that there is an incidence-preserving one-to-one correspondence between $\{V(G), E(G)\}$ and $\{V(H), E(H)\}$ in which two edges of G intersect if and only if the corresponding edges of H do.

Thrackle: A topological graph in which any two nonadjacent edges cross precisely once and no two adjacent edges cross.

Generalized thrackle: A topological graph in which any two nonadjacent edges cross an odd number of times and any two adjacent edges cross an even number of times (not counting their common endpoint).

d-Dimensional geometric r-hypergraph H_r^d: A pair (V, E), where V is a set of points in general position in d-space, and E is a set of *closed* $(r-1)$-dimensional simplices induced by some r-tuples of V. The sets V and E are called the ***vertex set*** and ***(hyper)edge set*** of H_r^d, respectively. Clearly, a geometric graph is a 2-dimensional geometric 2-hypergraph.

Forbidden geometric hypergraphs: A class $\mathcal{F}$ of geometric hypergraphs not permitted to be contained in the geometric hypergraphs under consideration. Given a class $\mathcal{F}$ of forbidden geometric hypergraphs, $\mathrm{ex}_r^d(\mathcal{F}, n)$ denotes the maximum number of edges that a d-dimensional geometric r-hypergraph H_r^d of n vertices can have without containing a geometric subhypergraph belonging to $\mathcal{F}$.

Nontrivial intersection: k simplices are said to have a nontrivial intersection if their relative interiors have a point in common.

Crossing of k simplices: A common point of the relative interiors of k simplices, all of whose vertices are *distinct*. The simplices are called ***crossing simplices*** if such a point exists. A set of simplices may be ***pairwise crossing*** but not necessarily crossing. If we want to emphasize that they *all* cross, we say that they cross in the ***strong sense*** or, in brief, that they ***strongly cross***.

TOPOLOGICAL GRAPHS

The fairly extensive literature on topological graphs focuses on very few special questions, and there is no standard terminology. Most of the methods developed for the study of geometric graphs break down for topological graphs, unless we make some further structural assumptions. For example, many arguments go through for x-monotone drawings such that any two edges cross at most once. Sometimes it is sufficient to assume the latter condition.

1. An Erdős-Szekeres type theorem: A classical theorem of Erdős and Szekeres states that every complete geometric graph with n vertices has a complete geometric subgraph, weakly isomorphic to a convex complete graph C_m with $m \geq c \log n$ vertices. For complete *topological* graphs with n vertices, any two of whose edges cross at most once, one can prove the existence of a complete topological subgraph with $m \geq c \log^{1/8} n$ vertices that is weakly isomorphic either to a convex complete graph C_m or to a so-called *twisted* complete graph T_m, as depicted in Figure 10.3.1 [PT03].

2. Every topological complete graph with n vertices, any two of whose edges cross at most once, has a noncrossing subgraph isomorphic to any given tree T with at most $c \log^{1/6} n$ vertices. In particular, it contains a noncrossing path with at least $c \log^{1/6} n$ vertices [PT03].

Preliminary version (July 26, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

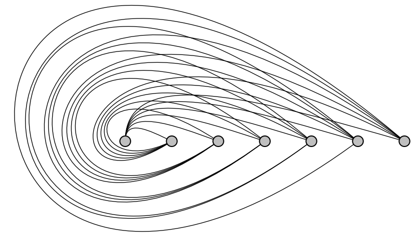


FIGURE 10.3.1
The twisted drawing T_m discovered by Harborth and Mengersen [HM92].

3. Number of topological complete graphs: Let $\overline{\Phi}(n), \Phi(n)$, and $\Phi_d(n)$ denote the number of different (i.e., pairwise weakly nonisomorphic) geometric complete graphs, topological complete graphs, and topological complete graphs in which every pair of edges cross at most d times, resp. We have $\log \overline{\Phi}(n) = \Theta(n \log n)$, $\log \Phi(n) = \Theta(n^4)$, $\Omega(n^2) \leq \log \Phi_1(n) \leq O(n^2 \alpha(n)^{O(1)})$, where $\alpha(n)$ denotes the (extremely slowly growing) inverse of the Ackermann function and $\Omega(n^2 \log n) \leq \log \Phi_d(n) \leq o(n^4)$ for every $d \geq 2$. [PT06, Kyn13].

4. Reducing the number of crossings [SŠ04]: Given an abstract graph $G = (V, E)$ and a set of pairs of edges $P \subseteq \binom{E}{2}$, we say that a topological graph K is a ***weak realization*** of G if no pair of edges not belonging to P cross each other. If G has a weak realization, then it also has a weak realization in which every edge crosses at most $2^{|E|}$ other edges. There is an almost matching lower bound for this quantity [KM91].

5. Every cycle of length different from 4 can be drawn as a thrackle [Woo69]. A bipartite graph can be drawn in the plane as a generalized thrackle if and only if it is planar [LPS97]. Every generalized thrackle with $n > 2$ vertices has at most $2n - 2$ edges, and this bound is sharp [CN00].

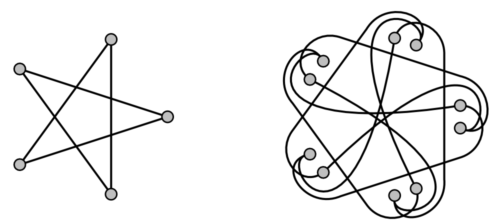


FIGURE 10.3.2
Cycles C_5 and C_{10} drawn as thrackles.

GEOMETRIC HYPERGRAPHS

If we want to generalize the results in the first two sections to higher dimensional geometric hypergraphs, we face some unexpected difficulties. Even if we restrict our attention to systems of triangles induced by 3-dimensional point sets in general position, it is not completely clear how a "crossing" should be defined. If two segments cross, they do not share an endpoint. Should this remain true for triangles? In this subsection, we describe some scattered results in this direction, but it will require further research to identify the key notions and problems.

1. Let $\mathcal{D}_k^r$ denote the class of all geometric r-hypergraphs consisting of k pairwise disjoint edges (closed $(r-1)$-dimensional simplices). Let $\mathcal{I}_k^r$ (respectively, $\mathcal{SI}_k^r$) denote the class of all geometric r-hypergraphs consisting of k simplices, any two of which have a nontrivial intersection (respectively, all of which are

strongly intersecting). Similarly, let $\mathcal{C}_k^r$ (respectively, $\mathcal{SC}_k^r$) denote the class of all geometric r-hypergraphs consisting of k pairwise crossing (respectively, strongly crossing) edges. In Table 10.3.1, we summarize the known estimates on $\mathrm{ex}_r^d(\mathcal{F}, n)$, the maximum number of hyperedges (or, simply, edges) that a d-dimensional geometric r-hypergraph of n vertices can have without containing any forbidden subconfiguration belonging to $\mathcal{F}$. We assume $d \geq 3$. In the first line of the table, the lower bound is conjectured to be tight. The upper bounds in the second line are tight for $d = 2, 3$.

TABLE 10.3.1 Estimates on $\mathrm{ex}_r^d(\mathcal{F}, n)$, the maximum number of edges of a d-dimensional geometric r-hypergraph of n vertices containing no forbidden subconfigurations belonging to $\mathcal{F}$.

r	$\mathcal{F}$	LOWER BOUND	UPPER BOUND	SOURCE
d	$\mathcal{D}_k^d$	$\Omega(n^{d-1})$	$n^{d-(1/k)^{d-1}}$	[AA89]
d	$\mathcal{I}_k^d$ $(k = 2, 3)$	?	$O(n^{d-1})$	[DP98]
d	$\mathcal{I}_k^d$ $(k > 3)$	?	$O(n^{d-1} \log n)$	[Val98]
d	$\mathcal{C}_2^d$	$\Omega(n^{d-1})$	$O(n^{d-1})$	[DP98]
d	$\mathcal{C}_k^d$ $(k > 2)$	?	$O(n^{d-(1/d)^{k-2}})$	[DP98]
$d+1$	$\mathcal{I}_k^{d+1}$	$\Omega(n^{\lceil d/2 \rceil})$	$O(n^{\lceil d/2 \rceil})$	[BF87, DP98]
$d+1$	$\mathcal{SI}_k^{d+1}$	$\Omega(n^{\lceil d/2 \rceil})$	$O(n^{\lceil d/2 \rceil})$	[BF87, DP98]
$d+1$	$\mathcal{C}_2^{d+1}$	$\Omega(n^d)$	$O(n^d)$	[DP98]

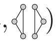
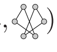
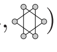
2. Akiyama-Alon theorem [AA89]: Let $V = V_1 \cup \ldots \cup V_d$ $(|V_1| = \ldots = |V_d| = n)$ be a dn-element set in general position in d-space, and let E consist of all $(d-1)$-dimensional simplices having exactly one vertex in each V_i. Then E contains n disjoint simplices. This result can be applied to deduce the upper bound in the first line of Table 10.3.1.

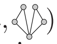
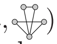
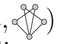
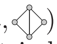
3. Assume that, for suitable constants c_1 and $0 \leq \delta \leq 1$, we have $\mathrm{ex}_r^d(\mathcal{SC}_k^r, n) < c_1\binom{n}{r}/n^\delta$ and $e \geq (c_1+1)\binom{n}{r}/n^\delta$. Then there exists $c_2 > 0$ such that the minimum number of strongly crossing k-tuples of edges in a d-dimensional r-hypergraph with n vertices and e edges is at least

$$c_2 \binom{n}{kr} e^\gamma / \binom{n}{r}^\gamma,$$

where $\gamma = 1 + (k-1)r/\delta$. This result can be used to deduce the upper bound in line 5 of Table 10.3.1.

4. A Ramsey-type result [DP98]: Let us 2-color all $(d-1)$-dimensional simplices induced by $(d+1)n - 1$ points in general position in $\mathbb{R}^d$. Then one can always find n disjoint simplices of the same color. This result cannot be improved.

5. Convex geometric hypergraphs in the plane [Bra04]: If we choose triangles from points in convex position in the plane, then the concept of isomorphism is much clearer than in the higher-dimensional cases. Thus two triangles without a common vertex can occur in three mutual positions, and we have $\mathrm{ex}(n,$  $) = \Theta(n^3)$, $\mathrm{ex}(n,$  $) = \Theta(n^2)$, $\mathrm{ex}(n,$  $) = \Theta(n^2)$. Similarly, two

triangles with one common vertex can occur again in three positions and we have $\mathrm{ex}(n,$$) = \Theta(n^3)$, $\mathrm{ex}(n,$$) = \Theta(n^2)$, $\mathrm{ex}(n,$$) = \Theta(n^2)$, which is surprising, since the underlying hypergraph has a linear Turán function. Finally, two triangles with two common vertices have two possible positions, and we have $\mathrm{ex}(n,$$) = \Theta(n^3)$, $\mathrm{ex}(n,$$) = \Theta(n^2)$. Larger sets of forbidden subconvex geometric hypergraphs occur as the combinatorial core of several combinatorial geometry problems.

OPEN PROBLEMS

1. (Ringel, Harborth) For any k, determine or estimate the smallest integer $n = n(k)$ for which there is a complete topological graph with n vertices, every pair of whose edges intersect at most once (including possibly at their common endpoints), and every edge of which crosses at least k others. It is known that $n(1) = 8$, $7 \le n(2) \le 11$, $7 \le n(3) \le 14$, $7 \le n(4) \le 16$, and $n(k) \le 4k/3 + O(\sqrt{k})$ [HT69]. Does $n(k) = o(k)$ hold?
2. (Harborth) Is it true that in every complete topological graph with n vertices, every pair of whose edges cross at most once (including possibly at their common endpoints), there are at least $2n-4$ empty triangles [Har98]? (A triangle bounded by all edges connecting three vertices is said to be ***empty***, if there is no point in its interior.) It is known that every complete topological graph with the above property has at least n empty triangles [AHP$^+$15], [Rui15].
3. Conway conjectured that the number of edges of a thrackle cannot exceed its number of vertices. It is known that every thrackle with n vertices has at most $1.4n$ edges [Xu14]. Conway's conjecture is true for thrackles drawn by x-monotone edges [PS11], and for thrackles drawn as outerplanar graphs [CN08].
4. (Kalai) What is the maximum number $\mu(n)$ of hyperedges that a 3-dimensional geometric 3-hypergraph of n vertices can have, if any pair of its hyperedges either are disjoint or share at most one vertex? Is it true that $\mu(n) = o(n^2)$? Károlyi and Solymosi [KS02] showed that $\mu(n) = \Omega(n^{3/2})$.

10.4 SOURCES AND RELATED MATERIAL

SURVEYS

All results not given an explicit reference above may be traced in these surveys.

[PA95]: Monograph devoted to combinatorial geometry. Chapter 14 is dedicated to geometric graphs.

[Pac99, Kár13]: Introduction to geometric graph theory and survey on Ramsey-type problems in geometric graph theory, respectively.

Preliminary version (July 26, 2017). To appear in the Handbook of Discrete and Computational Geometry, J.E. Goodman, J. O'Rourke, and C. D. Tóth (editors), 3rd edition, CRC Press, Boca Raton, FL, 2017.

[Pac91, DP98]: The first surveys of results in geometric graph theory and geometric hypergraph theory, respectively.

[PT00a, Pac00, Szé04, SSSV97, Sch14]: Surveys on open problems and on crossing numbers.

[DETT99]: Monograph on graph drawing algorithms.

[BMP05]: Survey of representative results and open problems in discrete geometry, originally started by the Moser brothers.

[Grü72]: Monograph containing many results and conjectures on configurations and arrangements of points and arcs.

RELATED CHAPTERS

Chapter 1: Finite point configurations
Chapter 5: Pseudoline arrangements
Chapter 11: Euclidean Ramsey theory
Chapter 28: Arrangements
Chapter 55: Graph drawing

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